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Research Article

A Note on Strongly Starlike Mappings in Several Complex Variables

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Let f be a normalized biholomorphic mapping on the Euclidean unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$ and let $\alpha \in (0,1)$. In this paper, we will show that if f is strongly starlike of order α in the sense of Liczberski and Starkov, then it is also strongly starlike of order α in the sense of Kohr and Liczberski. We also give an example which shows that the converse of the above result does not hold in dimension $n \geq 2$.

1. Introduction and Preliminaries

Let $\mathbb{C}^n$ denote the space of n complex variables $z = (z_1, \dots, z_n)$ with the Euclidean inner product $\langle z, w\rangle = \sum_{j=1}^{n} z_j \overline{w}_j$ and the norm $\| z \| = \langle z, z\rangle^{1/2}$. The open unit ball $\{z \in \mathbb{C}^n : \|z\| < 1\}$ is denoted by $\mathbb{B}^n$. In the case of one complex variable, $\mathbb{B}^1$ is denoted by U.

If Ω is a domain in $\mathbb{C}^n$, let $H(\Omega)$ be the set of holomorphic mappings from Ω to $\mathbb{C}^n$. If Ω is a domain in $\mathbb{C}^n$ which contains the origin and $f \in H(\Omega)$, we say that f is normalized if $f(0) = 0$ and $Df(0) = I_n$, where I_n is the identity matrix.

A normalized mapping $f \in H(\mathbb{B}^n)$ is said to be *starlike* if f is biholomorphic on $\mathbb{B}^n$ and $tf(\mathbb{B}^n) \subset f(\mathbb{B}^n)$ for $t \in [0,1]$, where the last condition says that the image $f(\mathbb{B}^n)$ is a starlike domain with respect to the origin. For a normalized locally biholomorphic mapping f on $\mathbb{B}^n$, f is starlike if and only if

$$\Re \left\langle [Df(z)]^{-1} f(z), z \right\rangle > 0, \quad z \in \mathbb{B}^n \setminus \{0\} \tag{1}$$

(see [1–4] and the references therein, cf. [5]).

Let $\alpha \in (0,1]$. A function $f \in H(U)$, normalized by $f(0) = 0$ and $f'(0) = 1$, is said to be *strongly starlike of order* α if

$$\left| \arg \frac{zf'(z)}{f(z)} \right| < \alpha \frac{\pi}{2}, \quad z \in U. \tag{2}$$

If f is strongly starlike of order α, then f is also starlike and thus univalent on U. Stankiewicz [6] proved that if $\alpha \in (0,1)$, then a domain $\Omega \neq \mathbb{C}$ which contains the origin is α-accessible if and only if $\Omega = f(U)$, where U is the unit disc in $\mathbb{C}$ and f is a strongly starlike function of order $1-\alpha$ on U. For strongly starlike functions on U, see also Brannan and Kirwan [7], Ma and Minda [8], and Sugawa [9].

Kohr and Liczberski [10] introduced the following definition of strongly starlike mappings of order α on $\mathbb{B}^n$.

Definition 1. Let $0 < \alpha \leq 1$. A normalized locally biholomorphic mapping $f \in H(\mathbb{B}^n)$ is said to be strongly starlike of order α if

$$\left| \arg \left\langle [Df(z)]^{-1} f(z), z \right\rangle \right| < \alpha \frac{\pi}{2}, \quad z \in \mathbb{B}^n \setminus \{0\}. \tag{3}$$

Obviously, if f is strongly starlike of order α, then f is also starlike, and if $\alpha = 1$ in (3), one obtains the usual notion of starlikeness on the unit ball $\mathbb{B}^n$.

Using this definition, Hamada and Honda [11], Hamada and Kohr [12], Liczberski [13], and Liu and Li [14] obtained various results for strongly starlike mappings of order α in several complex variables.

Recently, Liczberski and Starkov [15] gave another definition of strongly starlike mappings of order α on the Euclidean unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$, where $\alpha \in (0,1]$, and proved that a normalized biholomorphic mapping f on $\mathbb{B}^n$ is strongly starlike of order $1-\alpha$ if and only if $f(\mathbb{B}^n)$ is an α-accessible domain in $\mathbb{C}^n$ for $\alpha \in (0,1)$. Their definition is as follows.

Definition 2. Let $0 < \alpha \le 1$. A normalized locally biholomorphic mapping $f \in H(\mathbb{B}^n)$ is said to be strongly starlike of order α (in the sense of Liczberski and Starkov) if

$$\Re \left\langle [Df(z)]^{-1} f(z), z \right\rangle \ge \left\| \left([Df(z)]^{-1}\right)^{*} z \right\| \cdot \|f(z)\| \sin\left((1-\alpha)\frac{\pi}{2}\right), \quad z \in \mathbb{B}^n \setminus \{0\}. \tag{4}$$

In the case $n = 1$, it is obvious that both notions of strong starlikeness of order α are equivalent. Thus, the following natural question arises in dimension $n \ge 2$.

Question 1. Let $\alpha \in (0,1)$. Is there any relation between the above two definitions of strong starlikeness of order α?

Let f be a normalized biholomorphic mapping on the Euclidean unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$ and let $\alpha \in (0,1)$. In this paper, we will show that if f is strongly starlike of order α in the sense of Definition 2, then it is also strongly starlike of order α in the sense of Definition 1. As a corollary, the results obtained in [11–14] for strongly starlike mappings of order α in the sense of Definition 1 also hold for strongly starlike mappings of order α in the sense of Definition 2. We also give an example which shows that the converse of the above result does not hold in dimension $n \ge 2$.

2. Main Results

Let $\angle(a,b)$ denote the angle between $a, b \in \mathbb{C}^n \setminus \{0\}$ regarding a, b as real vectors in $\mathbb{R}^{2n}$.

Lemma 3. *Let $a, b \in \mathbb{C}^n \setminus \{0\}$ be such that $\langle a,b\rangle \ne 0$. If $|\arg\langle a,b\rangle| \le \pi$ and $0 \le \angle(a,b) < \pi/2$, then*

$$\left|\arg\langle a,b\rangle\right| \le \angle(a,b). \tag{5}$$

Proof. Let $\theta = \arg\langle a,b\rangle$, $\varphi = \angle(a,b)$. Then we have $\langle a,b\rangle = re^{i\theta}$ for some $r \ge 0$ and

$$\Re\langle a,b\rangle = \|a\| \cdot \|b\| \cos\varphi = r\cos\theta. \tag{6}$$

Since $\cos\varphi > 0$ and $r = |\langle a,b\rangle| \le \|a\| \cdot \|b\|$, we have

$$\cos\varphi \le \cos\theta. \tag{7}$$

Therefore, we have $|\theta| \le \varphi$, as desired. □

Theorem 4. *Let f be a normalized biholomorphic mapping on the Euclidean unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$ and let $\alpha \in (0,1)$. If f is strongly starlike of order α in the sense of Definition 2, then it is also strongly starlike of order α in the sense of Definition 1.*

Proof. Assume that f is strongly starlike of order α in the sense of Definition 2. Then by (4), we have $\langle [Df(z)]^{-1} f(z), z\rangle \ne 0$ and

$$\angle\left(\left([Df(z)]^{-1}\right)^{*} z, f(z)\right) \le \alpha\frac{\pi}{2}, \quad z \in \mathbb{B}^n \setminus \{0\}. \tag{8}$$

Using Lemma 3, we have

$$\begin{aligned}\left|\arg\left\langle [Df(z)]^{-1} f(z), z\right\rangle\right| &= \left|\arg\left\langle f(z), \left([Df(z)]^{-1}\right)^{*} z\right\rangle\right| \\ &\le \angle\left(\left([Df(z)]^{-1}\right)^{*} z, f(z)\right) \\ &\le \alpha\frac{\pi}{2}, \quad z \in \mathbb{B}^n \setminus \{0\}.\end{aligned} \tag{9}$$

For fixed $z \in \mathbb{B}^n \setminus \{0\}$, let $w = z/\|z\|$ and

$$p(\zeta) = \begin{cases} \dfrac{1}{\zeta}\left\langle [Df(\zeta w)]^{-1} f(\zeta w), w\right\rangle, & \text{for } \zeta \in U \setminus \{0\}, \\ 1, & \text{for } \zeta = 0. \end{cases} \tag{10}$$

Then p is a holomorphic function on U with $|\arg p(\zeta)| \le \pi\alpha/2$ for $\zeta \in U$. Since $\arg p$ is a harmonic function on U and $\arg p(0) = 0$, by applying the maximum and minimum principles for harmonic functions, we obtain $|\arg p(\zeta)| < \pi\alpha/2$ for $\zeta \in U$. Thus, we have

$$\left|\arg\left\langle [Df(z)]^{-1} f(z), z\right\rangle\right| < \alpha\frac{\pi}{2}, \quad z \in \mathbb{B}^n \setminus \{0\}. \tag{11}$$

Hence f is strongly starlike of order α in the sense of Definition 1, as desired. □

The following example shows that the converse of the above theorem does not hold in dimension $n \ge 2$.

Example 5. For $\alpha \in (0,1)$, let

$$f(z) = f_\alpha(z) = \left(z_1 + bz_2^2, z_2\right), \quad z = (z_1, z_2) \in \mathbb{B}^2, \tag{12}$$

where

$$b = \frac{3\sqrt{3}}{2}\sin\left(\alpha\frac{\pi}{2}\right). \tag{13}$$

Then

$$Df(z) = \begin{bmatrix} 1 & 2bz_2 \\ 0 & 1 \end{bmatrix}, \qquad [Df(z)]^{-1} = \begin{bmatrix} 1 & -2bz_2 \\ 0 & 1 \end{bmatrix}. \tag{14}$$

Therefore,

$$\begin{aligned}\left\langle [Df(z)]^{-1} f(z), z\right\rangle &= \left(z_1 + bz_2^2 - 2bz_2^2\right)\overline{z_1} \\ &\quad + |z_2|^2 = |z_1|^2 + |z_2|^2 - b\overline{z_1}z_2^2.\end{aligned} \tag{15}$$

Since $|z_1 z_2^2| \le 2/(3\sqrt{3})$, for $z \in \partial\mathbb{B}^2$, we obtain that $|bz_1 z_2^2| \le \sin(\alpha\pi/2)\|z\|^3$ for $z \in \mathbb{B}^2$. This implies that $\langle [Df(z)]^{-1} f(z), z\rangle$ lies in the disc of center $\|z\|^2$ and radius $\sin(\alpha\pi/2)\|z\|^2$ for each $z \in \mathbb{B}^2 \setminus \{0\}$ and thus

$$\left|\arg \left\langle [Df(z)]^{-1} f(z), z\right\rangle\right| < \alpha\frac{\pi}{2}, \quad z \in \mathbb{B}^2 \setminus \{0\}. \tag{16}$$

Therefore, $f = f_\alpha$ is strongly starlike of order α in the sense of Definition 1.

On the other hand,

$$\left([Df(z)]^{-1}\right)^* z = (z_1, z_2 - 2b\bar{z}_2 z_1). \tag{17}$$

So, for $z^0 = (1/\sqrt{3}, \sqrt{2}/\sqrt{3})$, we have

$$\begin{gathered} \left\langle \left[Df\left(z^0\right)\right]^{-1} f\left(z^0\right), z^0\right\rangle = 1 - m, \\ \left\|\left(\left[Df\left(z^0\right)\right]^{-1}\right)^* z^0\right\|^2 = \frac{1}{3} + \frac{2}{3}(1-3m)^2, \\ \left\|f\left(z^0\right)\right\|^2 = \frac{1}{3}(1+3m)^2 + \frac{2}{3}, \\ \sin\left((1-\alpha)\frac{\pi}{2}\right) = \sqrt{1-m^2}, \end{gathered} \tag{18}$$

where

$$m = \sin\left(\alpha\frac{\pi}{2}\right). \tag{19}$$

Then, we obtain

$$\begin{aligned} &\left\|\left(\left[Df\left(z^0\right)\right]^{-1}\right)^* z^0\right\|^2 \left\|f\left(z^0\right)\right\|^2 \sin^2\left((1-\alpha)\frac{\pi}{2}\right) \\ &\quad - \left(\Re \left\langle \left[Df\left(z^0\right)\right]^{-1} f\left(z^0\right), z^0\right\rangle\right)^2 \\ &= (1-m)\left\{\left[\frac{1}{3} + \frac{2}{3}(1-3m)^2\right]\left[\frac{1}{3}(1+3m)^2 + \frac{2}{3}\right]\right. \\ &\quad \left. \times (1+m) - (1-m)\right\}. \end{aligned} \tag{20}$$

Since

$$\left[\frac{1}{3} + \frac{2}{3}(1-3m)^2\right]\left[\frac{1}{3}(1+3m)^2 + \frac{2}{3}\right](1+m) - (1-m) \tag{21}$$

is increasing on $[1/3, 1]$ and positive for $m = 1/3$, we have

$$\begin{aligned} \Re \left\langle \left[Df\left(z^0\right)\right]^{-1} f\left(z^0\right), z^0\right\rangle &< \left\|\left(\left[Df\left(z^0\right)\right]^{-1}\right)^* z^0\right\| \\ &\quad \times \left\|f\left(z^0\right)\right\| \sin\left((1-\alpha)\frac{\pi}{2}\right) \end{aligned} \tag{22}$$

for $m \in [1/3, 1)$.

On the other hand, for $\tilde{z}^0 = (i/\sqrt{3}, \sqrt{2}/\sqrt{3})$, we have

$$\begin{gathered} \left\langle \left[Df\left(\tilde{z}^0\right)\right]^{-1} f\left(\tilde{z}^0\right), \tilde{z}^0\right\rangle = 1 + mi, \\ \left\|\left(\left[Df\left(\tilde{z}^0\right)\right]^{-1}\right)^* \tilde{z}^0\right\|^2 = \frac{1}{3} + \frac{2}{3}|1-3mi|^2 = 6m^2 + 1, \\ \left\|f\left(\tilde{z}^0\right)\right\|^2 = \frac{1}{3}|i+3m|^2 + \frac{2}{3} = 3m^2 + 1. \end{gathered} \tag{23}$$

Then, we obtain

$$\begin{aligned} &\left\|\left(\left[Df\left(\tilde{z}^0\right)\right]^{-1}\right)^* \tilde{z}^0\right\|^2 \left\|f\left(\tilde{z}^0\right)\right\|^2 \sin^2\left((1-\alpha)\frac{\pi}{2}\right) \\ &\quad - \left(\Re \left\langle \left[Df\left(\tilde{z}^0\right)\right]^{-1} f\left(\tilde{z}^0\right), \tilde{z}^0\right\rangle\right)^2 \\ &= \left(6m^2+1\right)\left(3m^2+1\right)\left(1-m^2\right) - 1 \\ &= m^2\left(-18m^4 + 9m^2 + 8\right). \end{aligned} \tag{24}$$

Since $-18m^4 + 9m^2 + 8$ is positive for $m \in [0, 1/3]$, we have

$$\begin{aligned} \Re \left\langle \left[Df\left(\tilde{z}^0\right)\right]^{-1} f\left(\tilde{z}^0\right), \tilde{z}^0\right\rangle &< \left\|\left(\left[Df\left(\tilde{z}^0\right)\right]^{-1}\right)^* \tilde{z}^0\right\| \\ &\quad \times \left\|f\left(\tilde{z}^0\right)\right\| \sin\left((1-\alpha)\frac{\pi}{2}\right) \end{aligned} \tag{25}$$

for $m \in (0, 1/3]$.

Thus, $f = f_\alpha$ is not strongly starlike of order α in the sense of Definition 2 for $\alpha \in (0, 1)$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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