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Research Article

New Inequalities for Gamma and Digamma Functions

M. R. Farhangdoost and M. Kargar Dolatabadi

Department of Mathematics, College of Science, Shiraz University, Shiraz 71475-44776, Iran

Correspondence should be addressed to M. R. Farhangdoost; farhang@shirazu.ac.ir

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By using the mean value theorem and logarithmic convexity, we obtain some new inequalities for gamma and digamma functions.

1. Introduction

Let $\Gamma(x)$, $\psi(x)$, $\psi^n(x)$, and $\zeta(x)$ denote the Euler gamma function, digamma function, polygamma functions, and Riemann zeta function, respectively, which are defined by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, \quad \text{for } x > 0, \qquad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \quad \text{for } x > 0, \tag{1}$$

$$\psi^{(n)}(x) = (-1)^{n+1} \int_0^\infty \frac{t^n e^{-xt}}{1-e^{-t}} dt, \quad \text{for } x > 0;\ n = 1, 2, 3, \ldots, \tag{2}$$

$$\zeta(x) = \sum_{n=1}^\infty \frac{1}{n^x}, \quad \text{for } x > 1. \tag{3}$$

In the past different papers appeared providing inequalities for the gamma, digamma, and polygamma functions (see [1–18]).

By using the mean value theorem to the function $\log\Gamma(x)$ on $[u, u+1]$, with $x > 0$ and $u > 0$, Batir [19] presented the following inequalities for the gamma and digamma functions:

$$\psi(x) \leqslant \log\left(x - 1 + e^{-\gamma}\right), \quad \text{for } x > 0,$$
$$\log(x) - \psi(x) < \frac{1}{2}\psi'(x), \quad \text{for } x > 1, \tag{4}$$
$$\psi'(x) \geqslant \frac{\pi^2}{6e^\gamma} e^{-\psi(x)}, \quad \text{for } x \geqslant 1.$$

In Section 2, by applying the mean value theorem on

$$\left(\log\Gamma(x)\right)' = \psi(x), \quad \text{for } x > 0, \tag{5}$$

we obtain some new inequalities on gamma and digamma functions.

Section 3 is devoted to some new inequalities on digamma function, by using convex properties of logarithm of this function.

Note that in this paper by $\gamma = \lim_{n\to\infty}(\sum_{k=1}^n (1/k) - \log(n)) = 0.5772156\cdots$ we mean Euler's constant [5].

2. Inequalities for Gamma and Digamma Functions by the Mean Value Theorem

Lemma 1. *For $t > 0$, one has*

$$\frac{-\psi''(t)}{\psi'(t)^2} < 1. \tag{6}$$

Proof. By [6, Proposition 1], we have

$$\psi'(t)\psi'''(t) - 2\left[\psi''(t)\right]^2 < 0, \quad \text{for } t > 0. \tag{7}$$

Thus the function $\psi''(t)/\psi'(t)^2$ is strictly decreasing on $(0, \infty)$.

By using asymptotic expansions [20, pages 253–256 and 364],

$$\psi'(t) = \frac{1}{t} + \frac{1}{2t^2} + \frac{1}{6t^3} + \frac{\theta'}{30t^5}, \quad \left(0 \leqslant \theta' \leqslant 1\right), \tag{8}$$

$$\psi''(t) = -\frac{1}{t^2} - \frac{1}{t^3} - \frac{1}{2t^4} + \frac{1}{6t^6} - \frac{\theta''}{6t^8}, \quad \left(0 \leqslant \theta'' \leqslant 1\right). \tag{9}$$

For $t > 0$, we get

$$\lim_{t\to\infty} \frac{\psi''(t)}{\psi'(t)^2} = -1. \tag{10}$$

Now, the proof follows from the monotonicity of $\psi''(t)/\psi'(t)^2$ on $(0, \infty)$ and

$$\lim_{t\to\infty} \frac{\psi''(t)}{\psi'(t)^2} = -1. \tag{11}$$

□

Theorem 2. *One has the following:*

(a) $x - (1/2) < 1/\psi'(x) \leqslant x + (6/\pi^2) - 1$ *for* $x \geqslant 1$;

(b) $1/x^2 < \psi'(x)\psi'(x+1) < 2/x^2$ *for* $x > 0$;

(c) $[\psi'(x)]^2/\psi''(x) \geqslant -\pi^4/72\zeta(3)$ *for* $x \geqslant 1$ *and* $x^2\psi'(x+1)\psi'(x) < \pi^4/72\zeta(3)$ *for* $x > 2$;

(d) $([\psi'(x+h)]^2 - \psi'(x)\psi'(x+h))/h\psi'(x) > \psi''(x+h)$ *for* $x > 0$ *and* $h > 0$;

(e) $(\psi'(x+h)\psi'(x) - [\psi'(x)]^2)/h\psi'(x+h) < \psi''(x)$ *for* $x > 0$ *and* $h > 0$;

(f) $-x^2\psi''(x) < \psi'(x)/\psi'(x+1)$ *and* $\psi'(x+1)/\psi'(x) < -x^2\psi''(x+1)$ *for* $x > 0$;

(g) $((\pi^2 x/6) + 1)^{(x+(6/\pi^2))} e^{-x(\gamma+1)} \leqslant \Gamma(x+1) < (2x+1)^{(x+(1/2))} e^{-x(1+\gamma)}$ *for* $x \geqslant 1$;

(h) $(1/x) - \psi'(x) < (1/2)\psi''(x + (1/2))$ *for* $x > 0$ *and* $(1/x) - \psi'(x) > ((\psi')^{-1}(1) - 1)\psi''(x)$ *for* $x > 1$;

(i) $\psi(x+1) > \log(x + (1/2)) + \psi((\psi')^{-1}(1))$ *for* $x \geqslant 1/2$;

(j) $(\pi^4/72\zeta(3)) \log(x - (\psi')^{-1}(1) + 2) + \psi((\psi')^{-1}(1)) \geqslant \psi(x+1)$ *for* $x > (\psi')^{-1}(1) - 1$.

Proof. Let u be a positive real number and $\psi(x)$ defined on the closed interval $[u, u+1]$. By using the mean value theorem for the function $\psi(x)$ on $[u, u+1]$ with $u > 0$ and since ψ' is a decreasing function, there is a unique θ depending on u such that $0 \leqslant \theta = \theta(u) < 1$, for all $u \geqslant 0$; then

$$\psi(u+1) - \psi(u) = \psi'(u + \theta(u)), \tag{12}$$

Since $\psi(x+1) - \psi(x) = 1/x$ and $\psi'(x+1) - \psi'(x) = -1/x^2$, we have

$$\psi'(u + \theta(u)) = \frac{1}{u}, \quad \text{for } u > 0. \tag{13}$$

We show that the function $\theta(u)$ has the following properties:

(1) $\theta(u)$ is strictly increasing on $(0, \infty)$;

(2) $\lim_{u\to\infty} \theta(u) = 1/2$;

(3) $\theta'(u)$ is strictly decreasing on $(0, \infty)$;

(4) $\lim_{u\to\infty} \theta'(u) = 0$.

To prove these four properties, since ψ' is a decreasing function on $(0, \infty)$, we put $u = 1/\psi'(t)$, where $t > 0$; by formula (13) we have

$$\psi'\left(\frac{1}{\psi'(t)} + \theta\left(\frac{1}{\psi'(t)}\right)\right) = \psi'(t). \tag{$*$}$$

Since by formula (8) we have $\psi''(t) < 0$ and $\psi'(t) > 0$, for all $t > 0$, then the mapping $t \to \psi'(t)$ from $(0, \infty)$ into $(0, \infty)$ is injective since also $\psi'(t) \to 0$ and $\psi'(t) \to \infty$ when $t \to \infty$ and $t \to 0^+$, respectively, then the mapping $t \to \psi'(t)$ from $(0,\infty)$ into $(0,\infty)$ is a bijective map. Clearly, by injectivity of ψ', we find that

$$\theta\left(\frac{1}{\psi'(t)}\right) = t - \frac{1}{\psi'(t)}, \quad \text{for } t > 0. \tag{14}$$

Differentiating between both sides of this equation, we get

$$\theta'\left(\frac{1}{\psi'(t)}\right) = \frac{-\left[\left(\psi'(t)\right)^2 + \psi''(t)\right]}{\psi''(t)}. \tag{15}$$

Since by formula (8), $\psi''(t) < 0$, where $t > 0$, hence formula (15) gives $\theta'(1/\psi'(t)) > 0$, for all $t > 0$. Since the mapping $t \to 1/\psi'(t)$ from $(0, \infty)$ to $(0, \infty)$ is also bijective, then $\theta'(t) > 0$ for all $t > 0$, and the proof of (1) is completed.

From (8) we have

$$\begin{aligned} &\lim_{u\to\infty} \theta(u) \\ &\quad = \lim_{t\to\infty} \theta\left(\frac{1}{\psi'(t)}\right) = \lim_{t\to\infty}\left(t - \frac{1}{\psi'(t)}\right) \\ &\quad = \lim_{t\to\infty}\left(t - \frac{1}{(1/t) + (1/2t^2) + (1/6t^3) + (1/3t^5)}\right) \\ &\quad = \frac{1}{2}. \end{aligned} \tag{16}$$

Differentiating between both sides of (15), we obtain

$$\begin{aligned} &\theta''\left(\frac{1}{\psi'(t)}\right) \\ &\quad = \frac{\left[\psi'(t)\right]^3}{\psi''(t)} \left[2\left(\psi''(t)\right)^2 - \psi'(t)\psi''(t)\right]. \end{aligned} \tag{$**$}$$

Since $\psi'(t) > 0$ and $\psi''(t) < 0$, where $t > 0$, then $\theta''(1/\psi'(t)) < 0$ for all $t > 0$. Proceeding as above we conclude that $\theta''(t) < 0$, for $t > 0$. This proves (3).

For (4), from (8), (9), we conclude that

$$\lim_{u\to\infty}\theta'(u)=\lim_{t\to\infty}\theta'\left(\frac{1}{\psi'(t)}\right)=\lim_{t\to\infty}-\frac{\left[\left(\psi'(t)\right)^2+\psi''(t)\right]}{\psi''(t)}$$
$$=-1-\lim_{t\to\infty}\frac{\left[\psi'(t)\right]^2}{\psi''(t)}=0. \tag{17}$$

Now, we prove the theorem. To prove (a), let $1/\psi'(1)=6/\pi^2\leqslant t<\infty$; then by (1) and (2) we have

$$\theta\left(\frac{1}{\psi'(1)}\right)\leqslant\theta(t)<\lim_{t\to\infty}\theta(t). \tag{18}$$

Equation (13) and $\psi''(t)<0$ for all $t>0$ give

$$\theta(t)=\left(\psi'\right)^{-1}\left(\frac{1}{t}\right)-t. \tag{19}$$

By substituting the value of $\theta(t)$ into (18), we get

$$1-\frac{1}{\psi'(1)}\leqslant\left(\psi'\right)^{-1}\left(\frac{1}{t}\right)-t<\lim_{t\to\infty}\theta(t)=\frac{1}{2}. \tag{20}$$

By substituting the value $t=1/\psi'(u)$ into this inequality, we get

$$u-\frac{1}{2}<\frac{1}{\psi'(u)}\leqslant u+\frac{6}{\pi^2}-1, \tag{21}$$

where $u\geqslant 1$.

In order to prove (b), by using the mean value theorem on the interval $[1/\psi'(t),1/\psi'(t+1)]$, and since θ is a decreasing function, there exists a unique δ such that

$$0<\delta(t)<1, \tag{22}$$

for $t>0$ and

$$\theta\left(\frac{1}{\psi'(t+1)}\right)-\theta\left(\frac{1}{\psi'(t)}\right)$$
$$=\left(\frac{1}{\psi'(t+1)}-\frac{1}{\psi'(t)}\right)\theta'\left(\frac{1}{\psi'(t+\delta(t))}\right). \tag{23}$$

Now, by (14), we have

$$1-\frac{1}{\psi'(t+1)}+\frac{1}{\psi'(t)}$$
$$=\left(\frac{1}{\psi'(t+1)}-\frac{1}{\psi'(t)}\right)\theta'\left(\frac{1}{\psi'(t+\delta(t))}\right). \tag{24}$$

Since θ is strictly increasing on $(0,\infty)$, by (1), we have

$$1+\frac{\psi'(t+1)-\psi'(t)}{\psi'(t+1)\psi'(t)}$$
$$=\theta\left(\frac{1}{\psi'(t+1)}\right)-\theta\left(\frac{1}{\psi'(t)}\right)>0. \tag{25}$$

By using this inequality and the fact that $\psi(x+1)-\psi(x)=1/x$ and

$$\psi'(x+1)-\psi'(x)=-\frac{1}{x^2}, \tag{26}$$

we obtain

$$\psi'(t+1)\psi'(t)>\frac{1}{t^2},\quad t>0. \tag{27}$$

Since θ is strictly increasing on $(0,\infty)$, by (1), it is clear that

$$\theta\left(\frac{1}{\psi'(t+1)}\right)-\theta\left(\frac{1}{\psi'(t)}\right)$$
$$<\lim_{t\to\infty}\theta(t)-\theta\left(0^+\right)=\frac{1}{2},\quad t>0. \tag{28}$$

and then it is clear that (b) holds.

For (c), since $t>2$, $t+\delta(t)>1+\delta(1)$, and θ' is strictly decreasing on $(0,\infty)$ by (3), then

$$\theta'\left(\frac{1}{\psi'(t+\delta(t))}\right)<\theta'\left(\frac{1}{\psi'(1)}\right)=-1-\frac{\left[\psi'(1)\right]^2}{\psi''(1)},\quad\forall t>2. \tag{29}$$

Since $\psi(x+1)-\psi(x)=1/x$ and $\psi'(x+1)-\psi'(x)=-1/x^2$, by using (24), we obtain

$$t^2\psi'(t+1)\psi'(t)<\frac{\pi^4}{72\zeta(3)}, \tag{30}$$

where $t>2$.

Since θ' is strictly decreasing on $(0,\infty)$ by (3) and $\psi''(t)<0$, for all $t>0$, we have

$$\theta'\left(\frac{1}{\psi'(t)}\right)\leqslant\theta'\left(\frac{1}{\psi'(1)}\right), \tag{31}$$

where $t\geqslant 1$.

Then it is clear that (c) is true.

Now we prove (d) and (e) by using the mean value theorem on $[1/\psi'(t),1/\psi'(t+h)]$ $(t>0,h>0)$, for θ, we conclude

$$\theta\left(\frac{1}{\psi'(t+h)}\right)-\theta\left(\frac{1}{\psi'(t)}\right)$$
$$=\left(\frac{1}{\psi'(t+h)}-\frac{1}{\psi'(t)}\right)\theta'\left(\frac{1}{\psi'(t+a)}\right), \tag{32}$$

where $0<a<h$.

After brief computation we have

$$\theta'\left(\frac{1}{\psi'(t+a)}\right)=\frac{h\psi'(t+h)\psi'(t)}{\psi'(t)-\psi'(t+h)}-1,\quad t>0. \tag{33}$$

Since $t+a>t$ for all $a>0$, $t>0$, and by the monotonicity of θ' and ψ' we have $\theta'(1/\psi'(t+a))<\theta'(1/\psi'(t))$; then

$$\frac{\psi'(t+h)\psi'(t)-\left[\psi'(t)\right]^2}{h\psi'(t+h)}<\psi''(t),\quad t>0,\ h>0. \tag{34}$$

By monotonicity of θ' and ψ', we have

$$\theta'\left(\frac{1}{\psi'(t+a)}\right) > \theta'\left(\frac{1}{\psi'(t+h)}\right). \tag{35}$$

After some simplification of this inequality (d) is proved.

For (f), we put $h = 1$ in (e) and (d).

For (g), we integrate (a) on $[1, t]$ for $t > 0$; then we have

$$\log\left(\frac{(t-1)\pi^2}{6} + 1\right) - \gamma \leqslant \psi(t) < \log(2t-1) - \gamma, \quad \text{for } t \geqslant 1; \tag{36}$$

the proof is completed when we integrate these inequalities on $[1, s]$, for $s > 0$.

By using the mean value theorem for the $\psi'(t)$ on $[t, t+\theta(t)]$, there is a $\alpha(t)$ depending on t such that $0 < \alpha(t) < \theta(t)$ for all $t > 0$, and so

$$\psi'(t+\theta(t)) = \theta(t)\psi''(t+\alpha(t)) + \psi'(t). \tag{37}$$

By formula (13) and (2), since ψ'' is strictly increasing on $(0, \infty)$, we have

$$\psi''(t+\alpha(t))\theta(t) = \frac{1}{t} - \psi'(t) < \lim_{t\to\infty}\theta(t)\psi''\left(t + \lim_{t\to\infty}\theta(t)\right), \quad \text{for } t > 0, \tag{38}$$

or

$$\frac{1}{t} - \psi'(t) < \frac{1}{2}\psi''\left(t + \frac{1}{2}\right), \quad \text{for } t > 0; \tag{39}$$

since ψ'' is strictly increasing on $(0, \infty)$, by (1), we have

$$\theta(t)\psi''(t+\alpha(t)) = \frac{1}{t} - \psi'(t) > \theta(1)\psi''(t), \quad \text{for } t > 1, \tag{40}$$

or

$$\frac{1}{t} - \psi'(t) > \left(\left(\psi'\right)^{-1}(1) - 1\right)\psi''(t), \quad \text{for } t > 1. \tag{41}$$

In order to prove (i) and (j), we integrate both sides of (13) over $1 \leqslant u \leqslant x$ to obtain

$$\int_1^x \psi'(u+\theta(u))\,du = \int_1^x \frac{1}{u}du. \tag{42}$$

Making the change of variable $u = 1/\psi'(t)$ on the left-hand side, by (14), we have

$$\int_{(\psi')^{-1}(1)}^{x+\theta(x)} \psi'(t)\frac{-\psi''(t)}{\psi'(t)^2}dt = \log(x); \tag{43}$$

since $\psi'(t) > 0$ for all $t > 0$ and $\psi'(x)\psi''(x) - 2[\psi''(x)]^2 < 0$, we find that, for $x > 1$,

$$\log(x) < \int_{(\psi')^{-1}(1)}^{x+\theta(x)} \psi'(t)\,dt = \psi(x+\theta(x)) - \psi\left(\left(\psi'\right)^{-1}(1)\right) \tag{44}$$

or

$$\log(x) + \psi\left(\left(\psi'\right)^{-1}(1)\right) < \psi(x+\theta(x)). \tag{45}$$

Again using the monotonicity of θ and ψ, after some simplifications as for $x \geqslant 1/2$, we can rewrite

$$\log\left(x + \frac{1}{2}\right) + \psi\left(\left(\psi'\right)^{-1}(1)\right) < \psi(x+1). \tag{46}$$

This proves (i). By inequality (c) for $x \geqslant 1$, we have

$$\log(x) \geqslant \frac{72\zeta(3)}{\pi^4}\int_{(\psi')^{-1}(1)}^{x+\theta(x)} \psi'(t)\,dt = \frac{72\zeta(3)}{\pi^4}\left(\psi(x+\theta(x)) - \psi\left(\left(\psi'\right)^{-1}(1)\right)\right); \tag{47}$$

since for $x \geqslant 1, \theta(x) \geqslant \theta(1) = ((\psi')^{-1}(1) - 1)) = (\psi')^{-1}(1) - 1$, from this inequality we find that

$$\frac{\pi^4}{72\zeta(3)}\log(x) + \psi\left(\left(\psi'\right)^{-1}(1)\right) \geqslant \psi\left(x + \left(\psi'\right)^{-1}(1) - 1\right); \tag{48}$$

replacing x by $x - (\psi')^{-1}(1) + 2$, we get for $x \geqslant (\psi')^{-1}(1) - 1$

$$\frac{\pi^4}{72\zeta(3)}\log\left(x - \left(\psi'\right)^{-1}(1) + 2\right) + \psi\left(\left(\psi'\right)^{-1}(1)\right) \geqslant \psi(x+1), \tag{49}$$

which proves (j). Then the proof is completed. □

Example 3. Consider the matrix

$$A_n = \begin{bmatrix} 3 & 1 & 1 & \cdots & 1 \\ 1 & 4 & 1 & \cdots & 1 \\ & & \vdots & & \\ 1 & 1 & \cdots & 1 & n+1 \end{bmatrix}. \tag{50}$$

By using inequalities (a), we obtain

$$\frac{\pi^2}{\pi^2 x + 6 - \pi^2} \leqslant \psi'(x) < \frac{2}{2x-1}, \quad x \geqslant 1. \tag{51}$$

Now, we integrate on $[1, t]$ (for $t > 0$) from both sides of (51) to obtain

$$\log\left(\frac{(t-1)\pi^2}{6} + 1\right) - \gamma \leqslant \psi(t) < \log(2t-1) - \gamma; \tag{52}$$

replacing t by $n+1$ (n is an integer number) and using the identity $\psi(n+1) = H_n - \gamma$ [6] and $\det A_n = n!H_n$ [21], where $H_n = \sum_{k=1}^{n}(1/k)$ is the nth harmonic number, then we have

$$\log\left(\frac{n\pi^2}{6} + 1\right)^{n!} \leqslant n!H_n < \log(2n+1)^{n!}. \tag{53}$$

3. New Inequalities for Digamma Function by Properties of Strictly Logarithmically Convex Functions

Definition 4. A positive function f is said to be logarithmically convex on an interval I if f has derivative of order two on I and

$$(\log f(x))'' \geqslant 0 \tag{54}$$

for all $x \in I$.

If inequality (54) is strict, for all $x \in I$, then f is said to be strictly logarithmically convex [22].

Lemma 5. *The function Γ is increasing on $[c, \infty)$, where $c = 1/46163\cdots$ is the only positive zero of ψ [1, 19].*

Lemma 6. *If $x \geqslant c$ and $k(x) = 1/\psi(x)$, then k is strictly logarithmically convex on $[c, \infty)$.*

Proof. By differentiation we have

$$[\log k(x)]'' = \left[\frac{-\psi'(x)}{\psi(x)}\right]' = \frac{-\psi''(x)\psi(x) + [\psi'(x)]^2}{[\psi(x)]^2}; \tag{55}$$

by Lemma 5, we obtain $\psi(x) = \Gamma'(x)/\Gamma(x) > 0$, for every $x \in [c, \infty)$ and since $\psi''(x) < 0$ on $(0, \infty)$, then we have $(\log k(x))'' > 0$, for $x \geqslant c$.

This implies that $1/\psi(x)$ is strictly logarithmically convex on $[c, \infty)$. □

Theorem 7. *One has the following:*

(a) $[\psi(x+3)]^a/\psi(ax+3) > ((3/2)-\gamma)^{a-1}$, *for $a > 1$ and $x > -3/a$;*

(b) $[\psi(x+3)]^a/\psi(ax+3) < ((11/6)-\gamma)^a/\psi(3+a)$, *for $a > 1$ and $x \in (0,1)$;*

(c) $[\psi(x+3)]^a/\psi(ax+3) > ((11/6)-\gamma)^a/\psi(3+a)$, *for $a > 1$ and $x > 1$;*

(d) $[\psi(x+3)]^a/\psi(ax+3) > ((11/6)-\gamma)^a/\psi(3+a)$, *for $a \in (0,1)$ and $x \in (0,1)$;*

(e) $[\psi(x+3)]^a/\psi(ax+3) < ((11/6)-\gamma)^a/\psi(3+a)$, *for $a \in (0,1)$ and $x > 1$.*

Proof. By Lemma 6 we have, for $a > 1$,

$$\psi\left[\frac{u}{p} + \frac{v}{q}\right] > [\psi(u)]^{1/p}[\psi(v)]^{1/q}, \tag{56}$$

where $p > 1$, $q > 1$, $(1/p) + (1/q) = 1$, $u \geqslant c$, and $v \geqslant c$.

If $p = a$ and $q = a/(a-1)$, then

$$\psi\left[\frac{1}{a}u + \left(1 - \frac{1}{a}\right)v\right] > [\psi(u)]^{1/a}[\psi(v)]^{1-(1/a)} \tag{57}$$

for $u \geqslant c$ and $v \geqslant c$.

Let $v = 3$ and $u = ax + 3$. Note that $\psi(3) = (3/2) - \gamma$ and $(1/a)u + (1 - (1/a))v = x + 3$; also we obtain

$$\frac{[\psi(x+3)]^a}{\psi(ax+3)} > \left(\frac{3}{2} - \gamma\right)^{a-1} \quad \text{for } x = \frac{u-3}{a} > -\frac{3}{a}. \tag{58}$$

In order to prove (b), let

$$\begin{aligned} f(x) = \log\psi(ax+3) - \log\psi(3+a) \\ - a\log\psi(x+3); \end{aligned} \tag{59}$$

since $\psi(4) = (11/6) - \gamma$, we have $f(1) = \log((11/6)-\gamma)^{-a}$. Also

$$f'(x) = a\left[\frac{\psi'(ax+3)}{\psi(ax+3)} - \frac{\psi'(x+3)}{\psi(x+3)}\right]. \tag{60}$$

By Lemma 6, $\log(1/\psi(t))$ is strictly convex on $[c, \infty)$; then $(\log\psi(t))'' < 0$ and so $(\psi'(t)/\psi(t))' < 0$; this implies that $(\psi'(t)/\psi(t))$ is strictly decreasing on $[c, \infty)$. Since $a > 1$ and $x \in (0,1)$, we have $ax + 3 > x + 3$. Then

$$\frac{\psi'(ax+3)}{\psi(ax+3)} < \frac{\psi'(x+3)}{\psi(x+3)}. \tag{61}$$

And then $f'(x) < 0$; also $f(1) = \log((11/6)-\gamma)^{-a}$. Then

$$f(x) > f(1) = \log\left(\frac{11}{6} - \gamma\right)^{-a} \tag{62}$$

for $a > 1$ and $x \in (0,1)$ or

$$\frac{[\psi(x+3)]^a}{\psi(ax+3)} < \frac{((11/6)-\gamma)^a}{\psi(3+a)}. \tag{63}$$

So (b) is proved.

By

$$\begin{aligned} &ax + 3 > x + 3, \quad \text{for } a > 1,\ x > 1, \\ &ax + 3 < x + 3, \quad \text{for } a \in (0,1),\ x \in (0,1), \\ &ax + 3 < x + 3, \quad \text{for } a \in (0,1),\ x > 1, \end{aligned} \tag{64}$$

(c), (d), and (e) are clear. □

Corollary 8. *For all $x \in (0,1)$ and all integers $n > 1$, one has*

$$\left(\frac{3}{2} - \gamma\right)^{n-1} < \frac{[\psi(x+3)]^n}{\psi(nx+3)} < \frac{((11/6)-\gamma)^n}{H_{n+2} - \gamma}, \tag{65}$$

where $H_n = \sum_{k=1}^{n}(1/k)$ is the nth harmonic number.

Proof. By [6], for all integers $n \geqslant 1$, we have

$$\psi(n+1) = H_n - \gamma, \tag{66}$$

and replacing a by n in Theorem 7, the proof is completed. □

Theorem 9. *Let f be a function defined by*

$$f(x) = \frac{[\psi(3+bx)]^a}{[\psi(3+ax)]^b}; \quad \forall x > 0, \tag{67}$$

where $3+ax \geqslant c$ and $3+bx \geqslant c$; then for all $a > b > 0$ or $0 > a > b$ ($a > 0$ and $b < 0$), f is strictly increasing (strictly decreasing) on $(0,\infty)$.

Proof. Let g be a function defined by

$$g(x) = \log f(x) = a\log\psi(3+bx) - b\log\psi(3+ax); \tag{68}$$

then

$$g'(x) = ab\left[\frac{\psi'(3+bx)}{\psi(3+bx)} - \frac{\psi'(3+ax)}{\psi(3+ax)}\right]. \tag{69}$$

By proof of Theorem 7, we have

$$(\log\psi(x))'' < 0, \quad \text{for } x \in [c,\infty); \tag{70}$$

this implies that $g'(x) > 0$ if $a > b > 0$ or $0 > a > b$ ($g'(x) < 0$ if $a > 0$ and $b < 0$); that is, g is strictly increasing on $(0,\infty)$ (strictly decreasing on $(0,\infty)$). Hence f is strictly increasing on $(0,\infty)$, if $a > b > 0$ or $0 > a > b$ (strictly decreasing if $a > 0$ and $b < 0$). □

Corollary 10. *For all $x \in (0,1)$ and all $a > b > 0$ or $0 > a > b$, one has*

$$\left(\frac{3}{2}-\gamma\right)^{a-b} < \frac{[\psi(3+bx)]^a}{[\psi(3+ax)]^b} < \frac{[\psi(3+b)]^a}{[\psi(3+a)]^b}, \tag{71}$$

where $3+bx \geqslant c$, $3+ax \geqslant c$, $3+b \geqslant c$, and $3+a \geqslant c$.

Proof. To prove (71), applying Theorem 9 and taking account of $\psi(3) = (3/2) - \gamma$, we get $f(0) < f(x) < f(1)$ for all $x \in (0,1)$, and we obtain (71). □

Corollary 11. *For all $x \in (0,1)$ and all $a > 0$ and $b < 0$, one has*

$$\frac{[\psi(3+b)]^a}{[\psi(3+a)]^b} < \frac{[\psi(3+bx)]^a}{[\psi(3+ax)]^b} < \left(\frac{3}{2}-\gamma\right)^{a-b}, \tag{72}$$

where $3+ax \geqslant c$, $3+bx \geqslant c$, $3+b \geqslant c$, and $3+a \geqslant c$.

Proof. Applying Theorem 9, we get $f(1) < f(x) < f(0)$ for all $x \in (0,1)$, and we obtain (72). □

Corollary 12. *For all $x \in (0,1)$ and all $a > b > 0$ or $0 > a > b$, one has*

$$\frac{[\psi(3+by)]^a}{[\psi(3+ay)]^b} < \frac{[\psi(3+bx)]^a}{[\psi(3+ax)]^b}, \tag{73}$$

where $3+ax \geqslant c$, $3+bx \geqslant c$, $3+ay \geqslant c$, $3+by \geqslant c$, and $0 < y < x < 1$.

Corollary 13. *For all $x \in (0,1)$ and all $a > 0$ and $b < 0$, one has*

$$\frac{[\psi(3+bx)]^a}{[\psi(3+ax)]^b} < \frac{[\psi(3+by)]^a}{[\psi(3+ay)]^b}, \tag{74}$$

where $3+ax \geqslant c$, $3+bx \geqslant c$, $3+ay \geqslant c$, $3+by \geqslant c$, and $0 < y < x < 1$.

Remark 14. Taking $a = n$ and $b = 1$ in Corollary 10, we obtain inequalities of Corollary 8.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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