Research Article

Dual Numbers Approach in Multiaxis Machines Error Modeling

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Multiaxis machines error modeling is set in the context of modern differential geometry and linear algebra. We apply special classes of matrices over dual numbers and propose a generalization of such concept by means of general Weil algebras. We show that the classification of the geometric errors follows directly from the algebraic properties of the matrices over dual numbers and thus the calculus over the dual numbers is the proper tool for the methodology of multiaxis machines error modeling.

1. Introduction

The concepts of multiaxis machines error modeling can be found in classical literature [1, 2], including the appropriate methodology. For the process description, the homogeneous transformation matrices (HTM) are used as the crucial mathematical tool for the error models; see [3, 4].

The complex multiaxis machine positioning is represented by a kinematic chain. Thus, by means of the product of the transformations between successive coordinate systems associated to the mechanisms elements, from the absolute reference system to the tool reference system, the global transformation matrix $T \in \mathrm{Mat}_4(\mathbb{R})$ is obtained. The basic setting takes place in the affine extension of vector space $\mathbb{R}^3$. This approach appears quite often in modern literature with minor modifications; see [5–8]; for a rare attempt to use modern advanced mathematical structures such as the algebra of quaternions, see [9].

The main uncertainty sources in the design and construction of machine tools are geometric and kinematic errors, thermal errors, stiffness error, and errors addressed to the deflection of cutting tools. Those mentioned above are the known sources. Their consequences are complex, but techniques to evaluate them or compensate their effects have being developed; see [10, 11]. In our paper we work with geometric and kinematic errors in any machine component which can be considered in the kinematic model as a new parameter.

For example, the coordinate transformation matrices and the corresponding error matrices for three-axis machines are

$$
T_1 = \begin{pmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

$$
E_1 = \begin{pmatrix} 1 & -\epsilon_{zx} & \epsilon_{yx} & \delta_{xx} \\ \epsilon_{zx} & 1 & -\epsilon_{xx} & \delta_{yx} \\ -\epsilon_{yx} & \epsilon_{xx} & 1 & \delta_{zx} \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

$$
T_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

$$
E_2 = \begin{pmatrix} 1 & -\epsilon_{zy} & \epsilon_{yy} & \delta_{xy} - S_{xy}y \\ \epsilon_{zy} & 1 & -\epsilon_{xy} & \delta_{yy} \\ -\epsilon_{yy} & \epsilon_{xy} & 1 & \delta_{zy} \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

$$
T_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

$$
E_3 = \begin{pmatrix} 1 & -\epsilon_{zz} & \epsilon_{yz} & \delta_{xz} - S_{xz}z \\ \epsilon_{zz} & 1 & -\epsilon_{xz} & \delta_{yz} - S_{yz}z \\ -\epsilon_{yz} & \epsilon_{xz} & 1 & \delta_{zz} \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

(1)

where δ_{ii} denotes the linear errors along the ith axis, $\delta_{ij}, i \neq j$ is the straightness errors in the ith axis direction when moving along the jth axis, ϵ_{ij} is the angular errors around the ith axis when moving along the jth axis, and S_{ij} is the squareness errors between the corresponding axes.

Our goal is to set the methodology of multiaxis machines geometric error in the context of modern theory of Weil algebras [12]. In this paper, we essentially use Weil algebra $\mathbb{D}_1^1$ and in the final section we present the directions of further

research and discuss the advantages of employing more general Weil algebras into the error analysis. When assembling the kinematic chain containing geometric errors, we embed the error matrix corresponding to any kinematic joint, that is, the errors of joint translation or rotation. In particular, for the translation in the vector (x, y, z) direction or for the rotation around the z axis by the angle γ, the following error matrices apply, respectively:

$$\begin{pmatrix} 1 & -\alpha & \beta & x \\ \alpha & 1 & \gamma & y \\ -\beta & -\gamma & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\alpha\cos(\theta+\gamma) - \beta\sin(\theta+\gamma) & \alpha\sin(\theta+\gamma) - \beta\cos(\theta+\gamma) & 0 \\ \alpha & \cos(\theta+\gamma) & -\sin(\theta+\gamma) & 0 \\ -\beta & \sin(\theta+\gamma) & \cos(\theta+\gamma) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2}$$

The parameters α, β, and γ represent the error rotations around the axes z, y, and x, respectively, and θ gives the proper rotation around axis z. The error matrices were derived from the rotation matrices around particular axes by approximation. More precisely, for the translation of the rotation around axis x, error matrix is approximated as follows:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\alpha & 0 \\ 0 & \alpha & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3}$$

Thus $\cos x \rightsquigarrow 1$ and $\sin x \rightsquigarrow x$. In case two approximations are multiplied, the whole term vanishes. This is caused by the assumption that the errors are by order smaller than the proper rotation parameters. The above mentioned representation is a standard description of the error matrices to be embedded into the kinematic chain.

For example, in case of two-axis machines with two translation joints, we obtain the following kinematic chain:

$$\begin{pmatrix} 1 & -\alpha_1 & \beta_1 & x_1 \\ \alpha_1 & 1 & \gamma_1 & y_1 \\ -\beta_1 & -\gamma_1 & 1 & z_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \overline{x} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$\times \begin{pmatrix} 1 & -\alpha_2 & \beta_2 & x_2 \\ \alpha_2 & 1 & \gamma_2 & y_2 \\ -\beta_2 & -\gamma_2 & 1 & z_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \overline{y} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4}$$

If the above mentioned identities $\alpha_i\beta_j = 0$, $\alpha_i\gamma_j = 0$, and $\beta_i\gamma_j = 0$ for all $i, j \in \{1, 2\}$ are applied, we obtain the matix

$$\rightsquigarrow \begin{pmatrix} 1 & -\alpha_1 & \beta_1 & \overline{x} + x_1 \\ \alpha_1 & 1 & \gamma_1 & \alpha_1\overline{x} + y_1 \\ -\beta_1 & -\gamma_1 & 1 & -\beta\overline{x} + z_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\alpha_2 & \beta_2 & -\alpha_2\overline{y} + x_2 \\ \alpha_2 & 1 & \gamma_2 & \overline{y} + y_2 \\ -\beta_2 & -\gamma_2 & 1 & \gamma_2\overline{y} + z_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & -\alpha_1 - \alpha_2 & \beta_1 + \beta_2 & -\alpha_2\overline{y} + \overline{x} + x_1 \\ \alpha_1 + \alpha_2 & 1 & \gamma_1 + \gamma_2 & \overline{y} + y_2 + \alpha_1\overline{x} + y_1 \\ -\beta_2 - \beta_2 & -\gamma_1 - \gamma_2 & 1 & \gamma_2\overline{y} + z_2 - \beta_1\overline{x} + z_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$\tag{5}$$

and the corresponding kinematic equations which are to be solved within the error analysis. Generally, in case of the system of linear equations, we proceed by Gauss elimination; for nonlinear systems we use Gröbner bases. In our setting, we compute with the matrices using the identities $\alpha_i\beta_j = 0$, $\alpha_i\gamma_j = 0$ and $\beta_i\gamma_j = 0$ for all $i, j \in \{1, 2\}$, which resemble the identities for the imaginary parts of the dual numbers. Thus it makes sense for the whole theory to use the homogeneous transformation matrices over the dual numbers. Our further approach to error calculations will thus be based on the dual numbers calculus; more generally we use a Weil algebra. This gives us a formal setting for the geometric errors modeling.

2. Matrices over Dual Numbers

As we are going to calculate with matrices over a structure different from real or complex numbers, moreover a structure which is not a field but a ring only, we have to guarantee that the calculations within the inverse kinematics make sense. In mathematical language, we need the dual numbers to form the so-called Euclidean domain.

By an Euclidean domain R we understand an integral domain which is endowed with at least one Euclidean function, that is, function of the form $f : R - \{0_R\} \rightarrow \mathbb{Z}_0^+$ satisfying the following property: if $a, b \in R$ and $b \neq 0_R$, then there are q and $r \in R$ such that $a = bq + r$ and either $r = 0_R$ or $f(r) < f(b)$.

Let us recall that for any field we shall define $f(x) = 1$ for any nonzero x and thus any field is Euclidean. The most important property of the Euclidean domain is that the Euclidean algorithm can be used to find the greatest common divisor of its two elements (i.e., one can easily see that any Euclidean domain is a principal ideal domain—PID). This leads to the fact that in Euclidean domains the Gauss elimination method for solving systems of linear equations can be applied.

As an example of Euclidean domain, let us recall three well-known extensions of real numbers.

Consider

$$\mathbb{C} = \left\{ a + bia, b \in \mathbb{R}, i^2 = -1 \right\},$$

$$\mathbb{D} = \left\{ a + bia, b \in \mathbb{R}, i^2 = 0 \right\}, \qquad (6)$$

$$\mathbb{P} = \left\{ a + bia, b \in \mathbb{R}, i^2 = 1 \right\}.$$

Rings $\mathbb{C}$, $\mathbb{D}$, and $\mathbb{P}$ can be obtained as factor rings of $\mathbb{R}[x]$: $\mathbb{C} = \mathbb{R}[x]/(x^2 + 1)$, $\mathbb{D} = \mathbb{R}[x]/(x^2)$, and $\mathbb{P} = R[x]/(x^2 - 1)$. Only $x^2 + 1$ is irreducible over $\mathbb{R}$; thereby $\mathbb{C}$ is a field. Neither $\mathbb{D}$ nor $\mathbb{P}$ are integral domains: we have $ai \times bi = 0$ for $\mathbb{D}$ and $(a + ai) \times (b - bi) = 0$ for $\mathbb{P}$ (by equations also presented all zero divisors of these rings).

In particular, the dual numbers extend the real numbers by adjoining one new element ϵ with the property $\epsilon^2 = 0$ (ϵ is nilpotent). The collection of dual numbers forms a particular two-dimensional commutative unital associative algebra over the real numbers. Every dual number has the form

$$z = a + b\epsilon \qquad (7)$$

with a and b uniquely determined real numbers. Division of dual numbers is defined when the real part of the denominator is nonzero. The division process is analogous to complex division in that the denominator is multiplied by its conjugate in order to cancel the nonreal parts.

We define the following class of matrices over dual numbers:

$$\mathrm{Mat}_n \mathbb{D} = \left\{ A + \epsilon B \mid A, B \in \mathrm{Mat}_n \mathbb{R} \right\} \qquad (8)$$

and the group of orthogonal matrices over dual numbers:

$$O(n, \mathbb{D}) = \left\{ C \in \mathrm{Mat}_n \mathbb{D} \mid C^T C = E \right\}. \qquad (9)$$

Indeed, the set $O(n, \mathbb{D})$ with standard matrix multiplication is a group as it is closed under the operation, more precisely $(AB)^T AB = B^T A^T AB = B^T B = E$; the inverse to the matrix A is the matrix A^T and the unitary matrix is of the form $E + \epsilon 0$.

Theorem 1. *An element $A + \epsilon B \in \mathrm{Mat}_n \mathbb{D}$ lies in $O(n, \mathbb{D})$ if $A \in O(n) = O(n, \mathbb{R})$ and B satisfies the identity*

$$A^T B = -B^T A. \qquad (10)$$

Proof. $(A + \epsilon B)^T (A + \epsilon B) = (A^T + \epsilon B^T)(A + \epsilon B) = A^T A + \epsilon(A^T B + B^T A)$ and thus $A^T A = E$ a $A^T B + B^T A$. $\qquad \square$

Definition 2. Let $A \in O(n)$. Then the set

$$O^A = \left\{ A + B\epsilon \mid A^T B + AB^T = 0 \right\} \qquad (11)$$

is called the *dual extension of the matrix A* and the set

$$I^A = \left\{ B \mid A^T B + AB^T = 0 \right\} \qquad (12)$$

of the imaginary parts of the matrices within the class O^A is called the *A-admissible error class* or shortly the *error class*.

Let us note that in the special case $A = E$, the error class I^E is a Lie algebra $\mathfrak{so}(n)$.

Lemma 3. *The error class I^A is a Lie algebra with respect to the commutator operation $[F, G] = FG - GF$ if and only if A is symmetric, that is, $A^T = A$.*

Proof. For $A \in O(n)$, $B, C \in I^A$ holds the following: $A^T[B, C] = A^T(BC - CB) = A^T BC - A^T CB = -B^T AC + C^T AB = B^T C^T A^T - C^T B^T A^T = ((CB)^T - (BC)^T) A^T = (CB - BC)^T A^T = [C, B]^T A^T = -[B, C]^T A^T$. $\qquad \square$

Example. Another possible choice of the matrix A such that I^A is an algebra is $A = \overline{E}$, where

$$\overline{E} = \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix} \in \mathrm{Mat}_{2n} \mathbb{R}, \qquad (13)$$

where E is an identity $n \times n$ matrix. Indeed, for another matrix $\begin{pmatrix} M & N \\ O & P \end{pmatrix} \in I^{\overline{E}}$ with $M, N, O, P \in \mathrm{Mat}_n \mathbb{R}$ we obtain

$$\begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix} \begin{pmatrix} M & N \\ O & P \end{pmatrix} = -\begin{pmatrix} M & O \\ N & P \end{pmatrix} \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix} \qquad (14)$$

and thus

$$I^{\overline{E}} = \begin{pmatrix} M & 0 \\ 0 & -M \end{pmatrix} \cong \mathrm{Mat}_n \mathbb{R}. \qquad (15)$$

Furthermore, the set

$$SO(n, \mathbb{D}) = \{ A + \epsilon B \in O(n, \mathbb{D}) \mid A \in SO(n) \} \qquad (16)$$

will be called the class of the *special orthogonal matrices over* $\mathbb{D}$.

Lemma 4. *Let $A + B\epsilon \in SO(2, \mathbb{D})$. Then $A + B\epsilon$ is of the form*

$$\begin{pmatrix} \cos(\varphi) - k\mathrm{tg}(\varphi)\epsilon & -\sin(\varphi) - k\epsilon \\ \sin(\varphi) + k\epsilon & \cos(\varphi) - k\mathrm{tg}(\varphi)\epsilon \end{pmatrix}, \qquad (17)$$

where $\varphi \in \langle 0, 2\pi \rangle$, and $k \in \mathbb{R}$.

Proof. Because $A \in SO(2)$, it has to be of the form

$$A = \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix}. \qquad (18)$$

Furthermore, let us consider a matrix B in the general form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. As $A + B\epsilon \in SO(2, \mathbb{D})$, the identity (10) has to be fulfilled:

$$\begin{pmatrix} \cos(\varphi) & \sin(\varphi) \\ -\sin(\varphi) & \cos(\varphi) \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
$$= -\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix}. \qquad (19)$$

This leads to the following equations:

$$a\cos(\varphi) + c\sin(\varphi) = -a\cos(\varphi) - c\sin(\varphi),$$
$$b\cos(\varphi) + d\sin(\varphi) = a\sin(\varphi) - c\cos(\varphi),$$
$$-a\sin(\varphi) + c\cos(\varphi) = -b\cos(\varphi) - d\sin(\varphi),$$
$$-b\sin(\varphi) + d\cos(\varphi) = b\sin(\varphi) - d\cos(\varphi). \qquad (20)$$

From the first and the last one we obtain identities $a = -ctg(\varphi)$ and $d = btg(\varphi)$. If we substitute these into the rest of the equalities, we obtain the following calculations:

$$b\cos(\varphi) + btg(\varphi)\sin(\varphi) = -ctg(\varphi)\sin(\varphi) - c\cos(\varphi),$$

$$b(\cos(\varphi))^2 + b\left(\sin(\varphi)^2\right) = -c(\sin(\varphi))^2 - c(\cos(\varphi))^2,$$

$$b = -c. \tag{21}$$

Thus for the matrix A we conclude that

$$I^A = \begin{pmatrix} -ktg(\varphi)\,\epsilon & -k\epsilon \\ k\epsilon & -ktg(\varphi)\,\epsilon \end{pmatrix}, \quad k \in \mathbb{R}, \tag{22}$$

which completes the proof. $\qquad\square$

Lemma 5. *Let $A, B \in O(n)$ then*

$$\left(A + I^A\epsilon\right)\left(B + I^B\epsilon\right) = AB + I^{AB}\epsilon. \tag{23}$$

Proof. $(A+I^A\epsilon)(B+I^B\epsilon) = AB+(AI^B+I^AB)\epsilon$ and it is easy to see that $(AI^B + I^AB) \in I^{AB}$:

$$(AB)^T\left(AI^B + I^AB\right) + \left(AI^B + I^AB\right)^T AB$$

$$= B^T A^T\left(AI^B + I^AB\right) + \left(\left(I^B\right)^T A^T + B^T\left(I^A\right)^T\right)AB$$

$$= B^T I^B + B^T A^T I^A B + \left(I^B\right)^T B + B^T\left(I^A\right)^T AB$$

$$= B^T I^B + \left(I^B\right)^T B + B^T\left(A^T I^A + \left(I^A\right)^T A\right)B = 0. \tag{24}$$

On the other hand, if $C \in I^{AB}$ then one can find $C_1 \in I^B$ and $C_2 \in I^A$ such that $AC_1 + C_2B = C$; for any $C_2 \in I^A$ we can define $C_1 = A^T C - A^T C_2 B$, such that $C_1 \in I^B$:

$$B^T\left(A^T C - A^T C_2 B\right) + \left(A^T C - A^T C_2 B\right)^T B$$

$$= B^T A^T C - B^T A^T C_2 B + \left(C^T A - B^T C_2^T A\right)B \tag{25}$$

$$= B^T A^T C - B^T A^T C_2 B + C^T AB - B^T C_2^T AB = 0.$$

$\qquad\square$

Theorem 6. *Let $A + B\epsilon \in SO(3, \mathbb{D})$. Then $A + B\epsilon$ is of the form*

$$R_1 R_2 R_3 + \left(R_1 R_2 I^{R_3} + R_1 I^{R_2} R_3 + I^{R_1} R_2 R_3\right)\epsilon, \tag{26}$$

where R_1, R_2, and R_3 are the matrices of rotations around the axes x, y, and z, respectively, and

$$I^{R_1} = \begin{pmatrix} 0 & -\alpha\cos(\varphi) - \beta\sin(\varphi) & \alpha\sin(\varphi) - \beta\cos(\varphi) \\ \alpha & -ktg(\varphi) & -k \\ \beta & k & -ktg(\varphi) \end{pmatrix},$$

$$I^{R_2} = \begin{pmatrix} -ktg(\varphi) & \alpha & -k \\ -\alpha\cos(\varphi) - \beta\sin(\varphi) & 0 & \alpha\sin(\varphi) - \beta\cos(\varphi) \\ k & \beta & -ktg(\varphi) \end{pmatrix},$$

$$I^{R_3} = \begin{pmatrix} -ktg(\varphi) & -k & \alpha \\ k & -ktg(\varphi) & \beta \\ -\alpha\cos(\varphi) - \beta\sin(\varphi) & \alpha\sin(\varphi) - \beta\cos(\varphi) & 0 \end{pmatrix}. \tag{27}$$

Proof. It is known that any element $A \in SO(3)$ can be represented as a composition of three rotation matrices R_1, R_2, R_3 (around axes x, y, and z, resp.). Thus if the extension to $SO(3, \mathbb{D})$ is considered, we obtain

$$\left(R_1 + I^{R_1}\epsilon\right)\left(R_2 + I^{R_2}\epsilon\right)\left(R_3 + I^{R_3}\epsilon\right)$$

$$= R_1 R_2 R_3 + \left(R_1 R_2 I^{R_3} + R_1 I^{R_2} R_3 + I^{R_1} R_2 R_3\right)\epsilon \tag{28}$$

and $A = R_1 R_2 R_3 \in SO(3)$, according to Lemma 5.

Let us now consider the rotation around the axis x in particular. The matrix A will be of the form $\begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix}$, where $M \in SO(2)$. If we consider a general error matrix of such type in the form $\begin{pmatrix} a & x \\ y & N \end{pmatrix}$, where $a \in \mathbb{R}$ and x, y are the dimension two row and column vectors, respectively, from the identity (10) we obtain

$$\begin{pmatrix} 1 & 0 \\ 0 & M^T \end{pmatrix}\begin{pmatrix} a & x \\ y & N \end{pmatrix} = -\begin{pmatrix} a & y^T \\ x^T & N^T \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix},$$

$$\begin{pmatrix} a & x \\ M^T y & M^T N \end{pmatrix} = -\begin{pmatrix} a & y^T M \\ x^T & N^T M \end{pmatrix}. \tag{29}$$

Thus $a = 0$ and $x = -y^T M$ which is equal to $M^T y = x^T$ and the identity $M^T N = -N^T M$ for the dimension two matrices holds from Lemma 4. If, in addition,

$$y = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \tag{30}$$

then

$$x = \left(\left(-\alpha\cos(\varphi) - \beta\sin(\varphi)\right)\ \left(\alpha\sin(\varphi) - \beta\cos(\varphi)\right)\right) \tag{31}$$

and we obtain that

$$I^{R_1} = \begin{pmatrix} 0 & -\alpha\cos(\varphi) - \beta\sin(\varphi) & \alpha\sin(\varphi) - \beta\cos(\varphi) \\ \alpha & -ktg(\varphi) & -k \\ \beta & k & -ktg(\varphi) \end{pmatrix}. \tag{32}$$

The remaining rotation represented by R_2 and R_3 can be computed similarly. $\qquad\square$

Let us note that the matrices $I^{R_i}, i \in \{1, 2, 3\}$ contain the errors α, β and if φ is understood as the sum $\theta + \gamma$ of the rotation angle θ and the rotation error angle γ, then all classical rotation errors are involved and the appropriate error matrix with $k = 0$ in addition corresponds to the classical error matrix. The geometric role of the parameter k within the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -ktg(\varphi) & -k \\ 0 & k & -ktg(\varphi) \end{pmatrix} \tag{33}$$

is unknown as it does not appear in the geometric error modeling.

3. Example

The following elementary example of two-axis machine will show the methodology of geometric errors modeling. The kinematic chain is described by means of the moving frame method, where the rotation matrices $R_i \in SO(n)$ are replaced by the matrices $R_i + I^{R_i} \epsilon \in SO(n, \mathbb{D})$. It is crucial that $\mathbb{D}$ is the Euclidean domain and thus the methods of Gauss elimination and Gröbner bases can be used when the inverse kinematics is solved. We will demonstrate the process in the case of two-axis machine with one rotation axis x and one translation in the direction of axis y. Thus, in the following, we shall work in the affine extension of the vector space $\mathbb{R}^3$, where vectors $\mathbb{X} = (x, y, z)^T \in \mathbb{R}^3$ are represented as the elements $(x, y, z, 1)^T \in \mathbb{R}^4$. The matrix $A \in \mathrm{Mat}_n(3, \mathbb{D})$ is then represented by the matrix $\begin{pmatrix} A & \mathbb{X} \\ 0 & 1 \end{pmatrix} \in \mathrm{Mat}_n(3, \mathbb{D}) \rtimes \mathbb{R}^3 \subset \mathrm{Mat}_n(4, \mathbb{D})$. For $T = (A, \mathbb{X}) \in \mathrm{Mat}_n(3, \mathbb{D}) \rtimes \mathbb{R}^3$ we write $I^T := (I^A, \mathbb{X})$.

The transformation matrices

$$T_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad T_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\varphi) & -\sin(\varphi) & 0 \\ 0 & \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{34}$$

are the elements of $O(3) \rtimes \mathbb{R}^3$. When the errors are added, we obtain

$$\begin{aligned} T_1 + I^{T_1}\epsilon &= \begin{pmatrix} 1 & -\alpha_1\epsilon & -\beta_1\epsilon & x_1\epsilon \\ \alpha_1\epsilon & 1 & -\gamma_1\epsilon & y_1\epsilon \\ \beta_1\epsilon & \gamma_1\epsilon & 1 & z_1\epsilon \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\[2mm] T_2 + I^{T2} &= \begin{pmatrix} 1 + a\epsilon & -\alpha_2\cos(\varphi)\epsilon - \beta_2\sin(\varphi)\epsilon & \alpha_2\sin(\varphi)\epsilon - \beta_2\cos(\varphi)\epsilon & x_2\epsilon \\ \alpha_2\epsilon & \cos(\varphi) - ktg(\varphi)\epsilon & \sin(\varphi) - k\epsilon & y_2\epsilon \\ \beta_2\epsilon & \sin(\varphi) + k\epsilon & \cos(\varphi) - ktg(\varphi)\epsilon & z_2\epsilon \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned} \tag{35}$$

Now the transformation matrices became the elements of $O(3, \mathbb{D}) \rtimes \mathbb{R}^3$ (note that it would make sense to consider the calculations in the algebra $O(3, \mathbb{D}) \otimes \mathbb{D}$ instead, but this is not the topic considered in this paper). The resulting matrix is then of the form

$$\begin{aligned} &\left(T_1 + I^E\right)\left(T_2 + I^{T2}\right) \\ &= \begin{pmatrix} a\epsilon & -\cos(\varphi)(\alpha_1 + \alpha_2)\epsilon - \sin(\varphi)(\beta_1 + \beta_2)\epsilon & -\sin(\varphi)(\alpha_1 - \alpha_2)\epsilon - \cos(\varphi)(\beta_1 + \beta_2)\epsilon & x_1 + x_2 \\ (\alpha_1 + \alpha_2)\epsilon & \cos(\varphi) + (-ktg(\varphi) + \gamma_1\sin(\varphi))\epsilon & \sin(\varphi) + (-k - \gamma_1\cos(\varphi))\epsilon & y_1 + y_2 + y \\ (\beta_1 + be_2)\epsilon & \sin(\varphi) + (k + \gamma_1\sin(\varphi))\epsilon & \cos(\varphi) + (ktg(\varphi) + \gamma_1\cos(\varphi))\epsilon & \overline{z}_1 + z_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned} \tag{36}$$

This describes the case of two-axis machine completely. To apply this approach on the three-axis machine, it is enough to extend the kinematic chain by the term $(T_3 + I^E)$. If, in addition, rotation machine elements are considered, it is necessary to employ the rotation error matrices $(R_i + I^{R_i})$.

4. Notes on Weil Algebras

Let $r, k \in \mathbb{N}$ and let us denote $\mathbb{D}_k^r$ the $\mathbb{R}$-algebra

$$\mathbb{R}\frac{[x_1, \ldots, x_k]}{\langle x_1, \ldots, x_n \rangle^{r+1}}, \tag{37}$$

where $\mathbb{R}[x_1, \ldots, x_k]$ is the $\mathbb{R}$-algebra of real polynomials in k indeterminates and $\langle x_1, \ldots, x_n \rangle^{r+1}$ is the $(r+1)$th power of its maximal ideals.

Definition 7. Weil algebra *is an arbitrary nontrivial quotient* $\mathbb{R}$-*algebra of* $\mathbb{D}_k^r$. For example, for $\mathbb{D}_1^1$ we obtain the dual numbers $\mathbb{D}$ and for $\mathbb{D}_2^2$ we have the set of polynomials in the form $ax + by + cx^2 + dx^2 + exy$.

Furthermore, the Weil algebra $\mathbb{W}_1 = \mathbb{D}_2^2 / \langle xy \rangle$ can be defined and the set of polynomials

$$ax + by + cx^2 + dx^2 \tag{38}$$

is obtained.

For the sake of the error analysis, the matrix class $R_1 + I^{R_1}$ can be represented by the element of $\mathbb{D}_4^n$ (where n determines the level of accuracy of the error analysis):

$$
R_1 + \begin{pmatrix} a & -\cos(\varphi) & \sin(\varphi) \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} x_1
$$
$$
+ \begin{pmatrix} 0 & -\sin(\varphi) & -\cos(\varphi) \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} x_2 + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} x_3 \tag{39}
$$
$$
+ \begin{pmatrix} 0 & 0 & 0 \\ 0 & -tg(\varphi) & -1 \\ 0 & 1 & -tg(\varphi) \end{pmatrix} x_4.
$$

The choice $n = 1$ neglects the interference of any two errors and the calculations will be similar to those over the dual numbers. In case $n = 3$, the actual interference of three errors for the term to be neglected is needed, but with the additional choice of the Weil algebra one can determine those error combinations which can be neglected or eventually replaced.

For instance, the choice

$$\mathbb{W} = \frac{\mathbb{D}_4^3}{\langle x_1^2, x_2^2, x_3^2, x_4^2, x_3 x_2, x_3 x_4, x_3 x_1, \rangle} \tag{40}$$

works similarly to the classical calculations over the dual numbers, but the interferences of two different rotation errors are not neglected, that is, $x_i x_j \neq 0$ for $i \neq j$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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