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Research Article

The Interval-Valued Trapezoidal Approximation of Interval-Valued Fuzzy Numbers and Its Application in Fuzzy Risk Analysis

Zengtai Gong[1] and Shexiang Hai[1,2]

[1] *College of Mathematics and Statistics, Northwest Normal University, Lanzhou 730070, China*
[2] *School of Science, Lanzhou University of Technology, Lanzhou 730050, China*

Correspondence should be addressed to Zengtai Gong; zt-gong@163.com

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Taking into account that interval-valued fuzzy numbers can provide more flexibility to represent the imprecise information and interval-valued trapezoidal fuzzy numbers are widely used in practice, this paper devotes to seek an approximation operator that produces an interval-valued trapezoidal fuzzy number which is the nearest one to the given interval-valued fuzzy number, and the approximation operator preserves the core of the original interval-valued fuzzy number with respect to the weighted distance. As an application, we use the interval-valued trapezoidal approximation to handle fuzzy risk analysis problems, which overcome the drawback of existing fuzzy risk analysis methods.

1. Introduction

The theory of fuzzy set, proposed by Zadeh [1], has received a great deal of attention due to its capability of handling uncertainty. Uncertainty exists almost everywhere, except in the most idealized situations; it is not only an inevitable and ubiquitous phenomenon, but also a fundamental scientific principle. As a generalization of an ordinary Zadeh's fuzzy set, the notion of interval-valued fuzzy sets was suggested for the first time by Gorzalczany [2] and Turksen [3]. It was introduced to alleviate some drawbacks of fuzzy set theory and has been applied to the fields of approximate inference, signal transmission and control, and so forth.

In 1998, Wang and Li [4] defined interval-valued fuzzy numbers and gave their extended operations. In practice, interval-valued trapezoidal fuzzy numbers are widely used in decision making, risk analysis, sensitivity analysis, and other fields [5–7]. In this paper, we are interested in approximating interval-valued fuzzy numbers by means of interval-valued trapezoidal fuzzy numbers to simplify calculations. The interval-valued trapezoidal approximation must preserve some parameters of the given interval-valued fuzzy number, such as α-level set invariance, translation invariance, scale invariance, identity, nearness criterion, ranking invariance, and continuity. Considering that the core (α-level set, where $\alpha = 1$) of an interval-valued fuzzy number is an important parameter in practical problems, we use the Karush-Kuhn-Tucher Theorem to investigate the interval-valued trapezoidal approximation of an interval-valued fuzzy number, which preserves its core.

The plan of this paper goes as follows. Section 2 contains some basic notations of interval-valued fuzzy numbers and the α-level set of interval-valued fuzzy numbers is presented, which differs from [8]. Some results related to interval-valued fuzzy numbers are investigated, these results will be frequently referred to in the subsequent sections. Section 3 is devoted to seek an approximation operator $T : \mathrm{IF}(R) \rightarrow \mathrm{IF}^T(R)$ that produces an interval-valued trapezoidal fuzzy number which is the nearest one to the given interval-valued fuzzy number among all interval-valued trapezoidal fuzzy numbers, and it preserves the core of the original interval-valued fuzzy number with respect to the weighted distance D_I. In Section 4, some properties of the approximation operator such as translation invariance, scale invariance, identity, nearness criterion, ranking invariance, and distance property are discussed. As an application we also use the approximation operator to handle fuzzy risk analysis problems, which provides us with a useful way to deal with fuzzy risk analysis problems in Section 5.

2. Preliminaries

2.1. Fuzzy Numbers. In 1972, Chang and Zadeh [9] introduced the conception of fuzzy numbers with the consideration of the properties of probability functions. Since then, the theory of fuzzy numbers and its applications have expansively been developed in data analysis, artificial intelligence, and decision making. This section will remind us of the basic notations of fuzzy numbers and give readers a better understanding of the paper.

Definition 1 (see [11–13]). A fuzzy number A is a subset of the real line R, with the membership function $\mu : R \to [0,1]$ such that the following holds.

(i) A is normal; that is, there is an $x_0 \in R$ with $\mu(x_0) = 1$.

(ii) A is fuzzy convex; that is, $\mu(\lambda x + (1-\lambda)y) \geq \min\{\mu(x), \mu(y)\}$, for any $x, y \in R$ and $\lambda \in [0,1]$.

(iii) μ is upper semicontinuous; that is, $\mu^{-1}([\alpha, 1])$ is closed for any $\alpha \in [0,1]$.

(iv) The support of μ is bounded; that is, the closure of $\{x \in R : \mu(x) > 0\}$ is bounded.

We denote by $F(R)$ the set of all fuzzy numbers on R.

Let $A \in F(R)$, whose membership function $\mu(x)$ can generally be defined as [14]

$$\mu(x) = \begin{cases} l_A(x), & a \leq x < b, \\ 1, & b \leq x \leq c, \\ r_A(x), & c < x \leq d, \\ 0, & \text{otherwise}, \end{cases} \quad (1)$$

where $a, b, c, d \in R$, $l_A : [a, b) \to [0,1]$ is a nondecreasing upper semicontinuous function such that $l_A(a) = 0$, $l_A(b) = 1$. $r_A : (c, d] \to [0,1]$ is a nonincreasing upper semicontinuous function satisfying $r_A(c) = 1$, $r_A(d) = 0$. l_A and r_A are called the left and the right side of A, respectively.

For any $\alpha \in (0,1]$, the α-level set of a fuzzy number A is a crisp set defined as [15]

$$A_\alpha = \{x \in R : \mu(x) \geq \alpha\}. \quad (2)$$

The support or 0-level set A_0 of a fuzzy number is defined as

$$A_0 = \overline{\{x \in R : \mu(x) \geq 0\}}. \quad (3)$$

It is well known that every α-level set of a fuzzy number A is a closed interval, denoted as

$$A_\alpha = [A_-(\alpha), A_+(\alpha)], \quad (4)$$

where

$$A_-(\alpha) = \inf\{x \in R : \mu(x) \geq \alpha\}, \quad A_+(\alpha) = \sup\{x \in R : \mu(x) \geq \alpha\}. \quad (5)$$

It is obvious that $A_-(\alpha)$ and $A_+(\alpha)$ are the inverse functions of l_A and r_A, respectively.

An often used fuzzy number is the trapezoidal fuzzy number, which is completely characterized by four real numbers $t_1 \leq t_2 \leq t_3 \leq t_4$, denoted by $T = \{t_1, t_2, t_3, t_4\}$ and with the membership function

$$\mu(x) = \begin{cases} \dfrac{x - t_1}{t_2 - t_1}, & t_1 \leq x < t_2, \\ 1, & t_2 \leq x \leq t_3, \\ \dfrac{t_4 - x}{t_4 - t_3}, & t_3 < x \leq t_4, \\ 0, & \text{otherwise}. \end{cases} \quad (6)$$

We write $F^T(R)$ as the family of all trapezoidal fuzzy numbers on R.

2.2. Interval-Valued Fuzzy Numbers. This section is devoted to review basic concept of interval-valued fuzzy numbers, which will be used extensively throughout this paper.

Let I be a closed unit interval; that is, $I = [0,1]$ and $[I] = \{\overline{a} = [a^-, a^+] : a^- \leq a^+, a^-, a^+ \in I\}$.

Definition 2 (see [16]). Let X be an ordinary nonempty set. Then the mapping $A : X \to [I]$ is called an interval-valued fuzzy set on X. All interval-valued fuzzy sets on X are denoted by IF(X).

An interval-valued fuzzy set A defined on X is given by

$$A = \{(x, [A^L(x), A^U(x)]) : x \in X\}, \quad (7)$$

where $0 \leq A^L(x) \leq A^U(x) \leq 1$. The interval-valued fuzzy set A can be represented by an interval $A(x) = [A^L(x), A^U(x)]$, and the ordinary fuzzy sets $A^L : X \to I$ and $A^U : X \to I$ are called a lower and an upper fuzzy set of A, respectively.

Definition 3 (see [17]). If an interval-valued fuzzy set $A(x) = [A^L(x), A^U(x)]$ satisfies the following conditions:

(i) A is normal, that is, there is an $x_0 \in R$ with $A(x_0) = [1, 1]$,

(ii) A is convex, that is, $A^L(\lambda x + (1-\lambda)y) \geq \min(A^L(x), A^L(y))$ and $A^U(\lambda x + (1-\lambda)y) \geq \min(A^U(x), A^U(y))$ for any $x, y \in R$ and $\lambda \in [0,1]$,

(iii) $A^L(x)$ and $A^U(x)$ are upper semicontinuous,

(iv) the support of $A^L(x)$ and $A^U(x)$ are bounded, that is, the closure of $\{x \in R : A^L(x) > 0\}$ and $\{x \in R : A^U(x) > 0\}$ are bounded,

then A is called an interval-valued fuzzy number on R. All interval-valued fuzzy numbers on R are denoted by IF(R).

For any $A = [A^L, A^U] \in \mathrm{IF}(R)$, the lower fuzzy number A^L and the upper fuzzy number A^U can be represented as

$$A^L(x) = \begin{cases} l_{A^L}(x), & a^L \le x < b^L, \\ 1, & b^L \le x \le c^L, \\ r_{A^L}(x), & c^L < x \le d^L, \\ 0, & \text{otherwise}, \end{cases} \tag{8}$$

$$A^U(x) = \begin{cases} l_{A^U}(x), & a^U \le x < b^U, \\ 1, & b^U \le x \le c^U, \\ r_{A^U}(x), & c^U < x \le d^U, \\ 0, & \text{otherwise}, \end{cases} \tag{9}$$

respectively, where $a^L, b^L, c^L, d^L, a^U, b^U, c^U, d^U \in R$. $l_{A^L} : [a^L, b^L) \to [0,1]$, and $l_{A^U} : [a^U, b^U) \to [0,1]$ are nondecreasing upper semicontinuous functions, such that $l_{A^L}(a^L) = 0$, $l_{A^L}(b^L) = 1$, $l_{A^U}(a^U) = 0$, and $l_{A^U}(b^U) = 1$. $r_{A^L} : (c^L, d^L] \to [0,1]$, and $r_{A^U} : (c^U, d^U] \to [0,1]$ are nonincreasing upper semicontinuous functions fulfilling $r_{A^L}(c^L) = 1$, $r_{A^L}(d^L) = 0$, $r_{A^U}(c^U) = 1$, and $r_{A^U}(d^U) = 0$.

If $a^L = a^U$, $b^L = b^U$, $c^L = c^U$, $d^L = d^U$, $l_{A^L}(x) = l_{A^U}(x)$, and $r_{A^L}(x) = r_{A^U}(x)$, that is, $A^L(x) = A^U(x)$, then the interval-valued fuzzy number $A = [A^L, A^U]$ is a fuzzy number.

For any $\alpha \in [0,1]$, the α-level set of an interval-valued fuzzy number A is defined as

$$\begin{aligned} A_\alpha &= \{(x,y) \in R^2 : A^L(x) \ge \alpha, A^U(y) \ge \alpha\} \\ &= \{(x,y) \in R^2 : x \in [A^L_-(\alpha), A^L_+(\alpha)], \\ &\qquad y \in [A^U_-(\alpha), A^U_+(\alpha)]\}, \end{aligned} \tag{10}$$

where $A^L_-(\alpha)$, $A^L_+(\alpha)$, $A^U_-(\alpha)$ and $A^U_+(\alpha)$ are the inverse functions of l_{A^L}, r_{A^L}, l_{A^U}, and r_{A^U}, respectively. If $A^L = A^U$, then this definition coincides with (4). The core of A is presented as

$$\begin{aligned} \mathrm{core}A = \{(x,y) \in R^2 : x \in [A^L_-(1), A^L_+(1)], \\ y \in [A^U_-(1), A^U_+(1)]\}. \end{aligned} \tag{11}$$

Theorem 4. *Let $A^L, A^U \in F(R)$. $A = [A^L, A^U] \in IF(R)$ if and only if $A^U_-(\alpha) \le A^L_-(\alpha)$, $A^U_+(\alpha) \ge A^L_+(\alpha)$ for any $\alpha \in [0,1]$.*

Proof. If: If $x \in [a^U, b^U)$, then there exist $\alpha_1, \alpha_2 \in [0,1]$, such that

$$\alpha_1 = l_{A^L}(x), \qquad \alpha_2 = l_{A^U}(x). \tag{12}$$

Since $A^U_-(\alpha) \le A^L_-(\alpha)$ for any $\alpha \in [0,1]$, this implies that

$$x = A^U_-(\alpha_2) \le A^L_-(\alpha_2) \triangleq x', \tag{13}$$

where $x' \in [a^U, b^L]$. By the monotonicity of l_{A^L}, we have

$$l_{A^L}(x) \le l_{A^L}(x') = \alpha_2 = l_{A^U}(x). \tag{14}$$

Similarly, we can prove that $r_{A^U}(x) \ge r_{A^L}(x)$ for any $x \in (c^U, d^U]$. If $x \in [b^U, c^U]$, then $A^U(x) = 1 \ge A^L(x)$. Therefore, $A^L(x) \le A^U(x)$ for any $x \in [a^U, d^U]$; that is, $A = [A^L, A^U] \in \mathrm{IF}(R)$.

Only if: If $\alpha \in [0,1]$, then there exist $x_1 \in [a^U, b^U]$, $x_2 \in [a^L, b^L]$, such that

$$x_1 = A^U_-(\alpha), \qquad x_2 = A^L_-(\alpha). \tag{15}$$

Since $l_{A^U}(x) \ge l_{A^L}(x)$ for any $x \in [a^U, b^U]$, this implies that

$$\alpha = l_{A^U}(x_1) \ge l_{A^L}(x_1) \triangleq \alpha', \tag{16}$$

where $\alpha' \in [0,1]$. By the monotonicity of A^L_-, we have

$$A^L_-(\alpha) \ge A^L_-(\alpha') = x_1 = A^U_-(\alpha). \tag{17}$$

Similarly, we can prove that $A^U_+(\alpha) \ge A^L_+(\alpha)$ for any $\alpha \in [0,1]$.

This concludes the proof. □

It is well known, interval-valued fuzzy numbers with simple membership functions are preferred in practice. However, as a particular of interval-valued fuzzy numbers, interval-valued trapezoidal fuzzy numbers could be wide applied in real mathematical modeling. Thus, the properties of the interval-valued trapezoidal fuzzy number are discussed as follows.

Definition 5 (see [6, 18–20]). Let $A = [A^L, A^U] \in \mathrm{IF}(R)$. If $A^L, A^U \in F^T(R)$, then A is called an interval-valued trapezoidal fuzzy number. The lower trapezoidal fuzzy number A^L is expressed as

$$A^L(x) = \begin{cases} \dfrac{x - t_1^L}{t_2^L - t_1^L}, & t_1^L \le x < t_2^L, \\ 1, & t_2^L \le x \le t_3^L, \\ \dfrac{t_4^L - x}{t_4^L - t_3^L}, & t_3^L < x \le t_4^L, \\ 0, & \text{otherwise}, \end{cases} \tag{18}$$

and the upper trapezoidal fuzzy number A^U is expressed as

$$A^U(x) = \begin{cases} \dfrac{x - t_1^U}{t_2^U - t_1^U}, & t_1^U \le x < t_2^U, \\ 1, & t_2^U \le x \le t_3^U, \\ \dfrac{t_4^U - x}{t_4^U - t_3^U}, & t_3^U < x \le t_4^U, \\ 0, & \text{otherwise}. \end{cases} \tag{19}$$

An interval-valued trapezoidal fuzzy number A can be represented as $A = [(t_1^L, t_2^L, t_3^L, t_4^L), (t_1^U, t_2^U, t_3^U, t_4^U)]$. The family of all interval-valued trapezoidal fuzzy numbers on R is denoted as $\mathrm{IF}^T(R)$.

Theorem 6. *Let $A^L, A^U \in F^T(R)$. $A = [A^L, A^U] \in IF^T(R)$ if and only if $t_1^U \le t_1^L$, $t_2^U \le t_2^L$, $t_3^U \ge t_3^L$ and $t_4^U \ge t_4^L$.*

2.3. The Weighted Distance of Interval-Valued Fuzzy Numbers. In 2007, Zeng and Li [21] introduced the weighted distance of fuzzy numbers A and B as follows:

$$d_f^2(A,B) = \int_0^1 f(\alpha)\left(A_-(\alpha) - B_-(\alpha)\right)^2 d\alpha + \int_0^1 f(\alpha)\left(A_+(\alpha) - B_+(\alpha)\right)^2 d\alpha, \tag{20}$$

where the function $f(\alpha)$ is nonnegative and increasing on $[0,1]$ with $f(0) = 0$ and $\int_0^1 f(\alpha)d\alpha = 1/2$. The function $f(\alpha)$ is also called the weighting function. The property of monotone increasing of function $f(\alpha)$ means that the higher the cut level, the more important its weight in determining the distance of fuzzy numbers A and B. Both conditions $f(0) = 0$ and $\int_0^1 f(\alpha)d\alpha = 1/2$ ensure that the distance defined by (20) is the extension of the ordinary distance in R defined by its absolute value. That means, this distance becomes an absolute value in R when a fuzzy number reduces to a real number. In applications, the function $f(\alpha)$ can be chosen according to the actual situation.

We will define the weighted distance of interval-valued fuzzy numbers as follows. It can be considered as a natural extension of the weighted distance $d_f(A,B)$ of fuzzy numbers.

Definition 7. Let $A, B \in \mathrm{IF}(R)$. The weighted distance of A and B is defined as

$$\begin{aligned} D_I(A,B) = \frac{1}{2}\Bigg[&\left(\int_0^1 f(\alpha)\left(A_-^L(\alpha) - B_-^L(\alpha)\right)^2 d\alpha + \int_0^1 f(\alpha)\left(A_+^L(\alpha) - B_+^L(\alpha)\right)^2 d\alpha\right)^{1/2} \\ &+ \left(\int_0^1 f(\alpha)\left(A_-^U(\alpha) - B_-^U(\alpha)\right)^2 d\alpha + \int_0^1 f(\alpha)\left(A_+^U(\alpha) - B_+^U(\alpha)\right)^2 d\alpha\right)^{1/2}\Bigg] \\ = \frac{1}{2}&\left[d_f\left(A^L, B^L\right) + d_f\left(A^U, B^U\right)\right]. \end{aligned} \tag{21}$$

If $A^L = A^U$ and $B^L = B^U$, then $D_I(A,B) = d_f(A,B)$.

Property 1. Let $A, B \in \mathrm{IF}(R)$. Then $D_I(A,B) = 0$ if and only if $D(A^L, B^L) = 0$ and $D(A^U, B^U) = 0$.

Theorem 8. *$(IF(R), D_I)$ is a metric space.*

By the completeness of metric space $(F(R), d_f)$, we can obtain the following conclusion.

Theorem 9. *The metric space $(IF(R), D_I)$ is complete.*

2.4. The Ranking of Interval-Valued Fuzzy Numbers. The ranking of fuzzy numbers was studied by many researchers and it was extended to interval-valued fuzzy numbers because of its attraction and applicability. We will propose a ranking of interval-valued fuzzy numbers, which embodies the importance of the core of interval-valued fuzzy numbers.

Definition 10. Let $A, B \in \mathrm{IF}(R)$. The ranking of A, B can be defined by the following formula:

$$\begin{aligned} A \succeq B \Longleftrightarrow A_-^L(1) + A_+^L(1) \geq B_-^L(1) + B_+^L(1), \\ A_-^U(1) + A_+^U(1) \geq B_-^U(1) + B_+^U(1). \end{aligned} \tag{22}$$

Example 11. Let

$$\begin{aligned} A^L(x) = B^L(x) &= \begin{cases} 1-(x-3)^2, & x \in [2,4], \\ 0, & \text{otherwise}, \end{cases} \\ A^U(x) &= \begin{cases} 1-(x-3)^2, & x \in [2,3), \\ 1, & x \in [3,5], \\ -x+6, & x \in (5,6], \\ 0, & \text{otherwise}, \end{cases} \\ B^U(x) &= \begin{cases} 1-(x-3)^2, & x \in [2,3), \\ 1-\frac{1}{9}(x-3)^2, & x \in [3,6], \\ 0, & \text{otherwise}. \end{cases} \end{aligned} \tag{23}$$

We obtain $\mathrm{core}A = \{(x,y) \in R^2 : x = 3, y \in [3,5]\}$ and $\mathrm{core}B = \{(x,y) \in R^2 : x = 3, y = 3\}$. By a direct calculation, we have $A \succeq B$.

3. Weighted Interval-Valued Trapezoidal Approximation

3.1. Criteria for Interval-Valued Trapezoidal Approximation. If we want to approximate an interval-valued fuzzy number by an interval-valued trapezoidal fuzzy number, we must use an approximate operator $T : \mathrm{IF}(R) \to \mathrm{IF}^T(R)$ which transforms a family of all interval-valued fuzzy numbers A into a family of interval-valued trapezoidal fuzzy numbers $T(A)$; that is, $T : A \to T(A)$. Since interval-valued trapezoidal approximation could also be performed in many ways, we propose a number of criteria which the approximation operator should possess at least one. Reference [22] has given some criteria for the fuzzy number approximation, similarly we give some criteria for interval-valued trapezoidal approximation as follows.

3.1.1. α-Level Set Invariance. An approximation operator T is α-level set invariant if

$$(T(A))_\alpha = A_\alpha. \tag{24}$$

Remark 12. For any two different levels α_1 and α_2 ($\alpha_1 \neq \alpha_2$), we obtain one and only one approximation operator which is invariant both in α_1- and α_2-level set.

Proof. Let $A = [A^L, A^U] \in \mathrm{IF}(R)$, $A_\alpha = \{(x,y) \in R^2 : x \in [A_-^L(\alpha), A_+^L(\alpha)],\ y \in [A_-^U(\alpha), A_+^U(\alpha)]\}$, $\alpha \in [0,1]$. Then we can obtain one and only one interval-valued trapezoidal fuzzy number $T(A) = [(T(A))^L, (T(A))^U]$, where

$$(T(A))^L_\alpha = \left[\frac{A^L_-(\alpha_2) - A^L_-(\alpha_1)}{\alpha_2 - \alpha_1}(\alpha - \alpha_1) + A^L_-(\alpha_1), \frac{A^L_+(\alpha_2) - A^L_+(\alpha_1)}{\alpha_2 - \alpha_1}(\alpha - \alpha_1) + A^L_+(\alpha_1)\right],$$
$$(T(A))^U_\alpha = \left[\frac{A^U_-(\alpha_2) - A^U_-(\alpha_1)}{\alpha_2 - \alpha_1}(\alpha - \alpha_1) + A^U_-(\alpha_1), \frac{A^U_+(\alpha_2) - A^U_+(\alpha_1)}{\alpha_2 - \alpha_1}(\alpha - \alpha_1) + A^U_+(\alpha_1)\right]. \quad (25)$$

It is obvious that

$$(T(A))_{\alpha_1} = \left\{(x, y) \in R^2 : x \in \left[A^L_-(\alpha_1), A^L_+(\alpha_1)\right], y \in \left[A^U_-(\alpha_1), A^U_+(\alpha_1)\right]\right\},$$
$$(T(A))_{\alpha_2} = \left\{(x, y) \in R^2 : x \in \left[A^L_-(\alpha_2), A^L_+(\alpha_2)\right], y \in \left[A^U_-(\alpha_2), A^U_+(\alpha_2)\right]\right\}. \quad (26)$$

Hence $(T(A))_{\alpha_1} = A_{\alpha_1}$ and $(T(A))_{\alpha_2} = A_{\alpha_2}$. □

3.1.2. Translation Invariance. For $A \in \mathrm{IF}(R)$ and $z \in R$, we define

$$A + z = \left[(A + z)^L, (A + z)^U\right], \quad (27)$$

where $(A+z)_\alpha = \{(x, y) \in R^2 : x \in [A^L_-(\alpha) + z, A^L_+(\alpha) + z], y \in [A^U_-(\alpha) + z, A^U_+(\alpha) + z]\}$, $\alpha \in [0, 1]$; that is,

$$(A + z)^L_-(\alpha) = A^L_-(\alpha) + z,$$
$$(A + z)^L_+(\alpha) = A^L_+(\alpha) + z, \quad (28)$$
$$(A + z)^U_-(\alpha) = A^U_-(\alpha) + z,$$
$$(A + z)^U_+(\alpha) = A^U_+(\alpha) + z. \quad (29)$$

An approximation operator T is invariant to translation if

$$T(A + z) = T(A) + z, \quad z \in R. \quad (30)$$

Translation invariance means that the relative position of the interval-valued trapezoidal approximation remains constant when the membership function is moved to the left or to the right.

3.1.3. Scale Invariance. For $A \in \mathrm{IF}(R)$ and $\lambda \in R \setminus \{0\}$, we define

$$\lambda A = \left[\lambda A^L, \lambda A^U\right]. \quad (31)$$

When $\lambda > 0$, $(\lambda A)_\alpha = \{(x, y) \in R^2 : x \in [\lambda A^L_-(\alpha), \lambda A^L_+(\alpha)], y \in [\lambda A^U_-(\alpha), \lambda A^U_+(\alpha)]\}$, $\alpha \in [0, 1]$; that is,

$$(\lambda A)^L_-(\alpha) = \lambda A^L_-(\alpha),$$
$$(\lambda A)^L_+(\alpha) = \lambda A^L_+(\alpha),$$
$$(\lambda A)^U_-(\alpha) = \lambda A^U_-(\alpha),$$
$$(\lambda A)^U_+(\alpha) = \lambda A^U_+(\alpha). \quad (32)$$

When $\lambda < 0$, $(\lambda A)_\alpha = \{(x, y) \in R^2 : x \in [\lambda A^L_+(\alpha), \lambda A^L_-(\alpha)], y \in [\lambda A^U_+(\alpha), \lambda A^U_-(\alpha)]\}$, $\alpha \in [0, 1]$; that is,

$$(\lambda A)^L_-(\alpha) = \lambda A^L_+(\alpha),$$
$$(\lambda A)^L_+(\alpha) = \lambda A^L_-(\alpha), \quad (33)$$
$$(\lambda A)^U_-(\alpha) = \lambda A^U_+(\alpha),$$
$$(\lambda A)^U_+(\alpha) = \lambda A^U_-(\alpha). \quad (34)$$

We say that an approximation operator T is scale invariant if

$$T(\lambda A) = \lambda T(A), \quad \lambda \in R \setminus \{0\}. \quad (35)$$

3.1.4. Identity. This criterion states that the interval-valued trapezoidal approximation of an interval-valued trapezoidal fuzzy number is equivalent to that number; that is, if $A \in \mathrm{IF}^T(R)$, then

$$T(A) = A. \quad (36)$$

3.1.5. Nearness Criterion. An approximation operator T fulfills the nearness criterion if for any interval-valued fuzzy number A its output value $T(A)$ is the nearest interval-valued trapezoidal fuzzy number to A with respect to the weighted distance D_I defined by (21). In other words, for any $B \in \mathrm{IF}^T(R)$, we have

$$D_I(A, T(A)) \le D_I(A, B). \quad (37)$$

Remark 13. We can verify that $\mathrm{IF}(R)$ is closed and convex, so $T(A)$ exists and is unique.

3.1.6. Ranking Invariance. A reasonable approximation operator should preserve the accepted ranking. We say that an approximation operator T is ranking invariant if for any $A, B \in \mathrm{IF}(R)$,

$$A \succeq B \Longleftrightarrow T(A) \succeq T(B). \quad (38)$$

3.1.7. Continuity. Let $A, B \in \mathrm{IF}(R)$. An approximation operator T is continuous if for any $\varepsilon > 0$, there is $\delta > 0$; when $D_I(A, B) < \delta$, we have

$$D_I(T(A), T(B)) < \varepsilon. \quad (39)$$

The continuity constraint means that if two interval-valued fuzzy numbers are close, then their interval-valued trapezoidal approximations also should be close.

3.2. Interval-Valued Trapezoidal Approximation Based on the Weighted Distance. In this section, we are looking for an approximation operator $T : \mathrm{IF}(R) \to \mathrm{IF}^T(R)$ which produces an interval-valued trapezoidal fuzzy number, that is, the nearest one to the given interval-valued fuzzy number and preserves its core with respect to the weighted distance D_I defined by (21).

Lemma 14. *Let $A \in F(R)$, $A_\alpha = [A_-(\alpha), A_+(\alpha)]$, $\alpha \in [0,1]$. If function $f(\alpha)$ is nonnegative and increasing on $[0,1]$ with $f(0) = 0$ and $\int_0^1 f(\alpha)d\alpha = 1/2$, then we have*

(i)

$$-\frac{\int_0^1 f(\alpha)(1-\alpha)[\alpha \cdot A_-(1) - A_-(\alpha)]\,d\alpha}{\int_0^1 (\alpha-1)^2 f(\alpha)\,d\alpha} \le A_-(1), \quad (40)$$

(ii)

$$-\frac{\int_0^1 f(\alpha)(1-\alpha)[\alpha \cdot A_+(1) - A_+(\alpha)]\,d\alpha}{\int_0^1 (\alpha-1)^2 f(\alpha)\,d\alpha} \ge A_+(1). \quad (41)$$

Proof. (i) See [23] the proof of Theorem 3.1.

(ii) Since $A_+(\alpha)$ is a nonincreasing function, we have $A_+(\alpha) \ge A_+(1)$ for any $\alpha \in [0,1]$. By $f(\alpha) \ge 0$, we can prove that

$$\begin{aligned}(1-\alpha)A_+(\alpha)f(\alpha) &\ge (1-\alpha)A_+(1)f(\alpha)\\ &= [(\alpha-1)^2 - \alpha(\alpha-1)]A_+(1)f(\alpha).\end{aligned} \quad (42)$$

According to the monotonicity of integration, we have

$$\begin{aligned}&\int_0^1 (1-\alpha)A_+(\alpha)f(\alpha)\,d\alpha\\ &\quad\ge \int_0^1 [(\alpha-1)^2 - \alpha(\alpha-1)]A_+(1)f(\alpha)\,d\alpha.\end{aligned} \quad (43)$$

That is

$$\begin{aligned}&\int_0^1 (1-\alpha)A_+(\alpha)f(\alpha)\,d\alpha\\ &\quad - A_+(1)\int_0^1 \alpha(1-\alpha)f(\alpha)\,d\alpha\\ &\quad\ge A_+(1)\int_0^1 (\alpha-1)^2 f(\alpha)\,d\alpha.\end{aligned} \quad (44)$$

Because $\int_0^1 (\alpha-1)^2 f(\alpha)d\alpha > 0$, it follows that

$$-\frac{\int_0^1 f(\alpha)(1-\alpha)[\alpha \cdot A_+(1) - A_+(\alpha)]\,d\alpha}{\int_0^1 (\alpha-1)^2 f(\alpha)\,d\alpha} \ge A_+(1). \quad (45)$$

□

Theorem 15 (see [24]). *Let $f, g_1, g_2, \ldots, g_m : R^n \to R$ be convex and differentiable functions. Then $\overline{x}$ solves the convex programming problem:*

$$\begin{aligned}\min\quad & f(x),\\ s.t.\quad & g_i(x) \le b_i \quad i \in \{1,2,\ldots,m\},\end{aligned} \quad (46)$$

if and only if there exist $\mu_i, i \in \{1,2,\ldots,m\}$, such that

(i) $\nabla f(\overline{x}) + \Sigma_{i=1}^m \mu_i \nabla g_i(\overline{x}) = 0$;

(ii) $g_i(\overline{x}) - b_i \le 0$;

(iii) $\mu_i \ge 0$;

(iv) $\mu_i(b_i - g_i(\overline{x})) = 0$.

Suppose that $A = [A^L, A^U] \in IF(R)$, $A_\alpha = \{(x,y) \in R^2 : x \in [A_-^L(\alpha), A_+^L(\alpha)], y \in [A_-^U(\alpha), A_+^U(\alpha)]\}$, $\alpha \in [0,1]$. We will try to find an interval-valued trapezoidal fuzzy number $T(A) = [(t_1^L, t_2^L, t_3^L, t_4^L), (t_1^U, t_2^U, t_3^U, t_4^U)]$, which is the nearest interval-valued trapezoidal fuzzy number of A and preserves its core with respect to the weighted distance D_I. Thus we have to find such real numbers t_1^L, t_2^L, t_3^L, t_4^L, t_1^U, t_2^U, t_3^U and t_4^U that minimize

$$\begin{aligned}&D_I(A, T(A))\\ &= \frac{1}{2}\Bigg[\Bigg(\int_0^1 f(\alpha)\left(A_-^L(\alpha) - \left(t_1^L + \left(t_2^L - t_1^L\right)\alpha\right)\right)^2 d\alpha\\ &\qquad + \int_0^1 f(\alpha)\left(A_+^L(\alpha) - \left(t_4^L - \left(t_4^L - t_3^L\right)\alpha\right)\right)^2 d\alpha\Bigg)^{1/2}\\ &\qquad + \Bigg(\int_0^1 f(\alpha)\left(A_-^U(\alpha) - \left(t_1^U + \left(t_2^U - t_1^U\right)\alpha\right)\right)^2 d\alpha\\ &\qquad + \int_0^1 f(\alpha)\left(A_+^U(\alpha) - \left(t_4^U - \left(t_4^U - t_3^U\right)\alpha\right)\right)^2 d\alpha\Bigg)^{1/2}\Bigg]\end{aligned} \quad (47)$$

with respect to condition $coreA = coreT(A)$; that is,

$$\begin{aligned}t_2^L &= A_-^L(1), \qquad t_3^L = A_+^L(1),\\ t_2^U &= A_-^U(1), \qquad t_3^U = A_+^U(1).\end{aligned} \quad (48)$$

It follows that

$$t_2^L \le t_3^L, \qquad t_2^U \le t_3^U. \quad (49)$$

Making use of Theorem 4, we have

$$t_2^U \le t_2^L, \qquad t_3^L \le t_3^U. \quad (50)$$

Using (47) *and* (50)*, together with Theorem 6, we only need to minimize the function*

$$
\begin{aligned}
F&\left(t_1^L, t_4^L, t_1^U, t_4^U\right)\\
&= \int_0^1 f(\alpha)\left[A_-^L(\alpha) - \left(t_1^L + \left(A_-^L(1) - t_1^L\right)\alpha\right)\right]^2 d\alpha\\
&\quad + \int_0^1 f(\alpha)\left[A_+^L(\alpha) - \left(t_4^L - \left(t_4^L - A_+^L(1)\right)\alpha\right)\right]^2 d\alpha\\
&\quad + \int_0^1 f(\alpha)\left[A_-^U(\alpha) - \left(t_1^U + \left(A_-^U(1) - t_1^L\right)\alpha\right)\right]^2 d\alpha\\
&\quad + \int_0^1 f(\alpha)\left[A_+^U(\alpha) - \left(t_4^U - \left(t_4^U - A_+^U(1)\right)\alpha\right)\right]^2 d\alpha,
\end{aligned} \tag{51}
$$

subject to

$$
t_1^U - t_1^L \le 0, \qquad t_4^L - t_4^U \le 0. \tag{52}
$$

After simple calculations we obtain

$$
\begin{aligned}
F&\left(t_1^L, t_4^L, t_1^U, t_4^U\right)\\
&= \int_0^1 f(\alpha)(1-\alpha)^2 d\alpha \cdot \left(t_1^L\right)^2\\
&\quad + \int_0^1 f(\alpha)(1-\alpha)^2 d\alpha \cdot \left(t_4^L\right)^2\\
&\quad + \int_0^1 f(\alpha)(1-\alpha)^2 d\alpha \cdot \left(t_1^U\right)^2\\
&\quad + \int_0^1 f(\alpha)(1-\alpha)^2 d\alpha \cdot \left(t_4^U\right)^2\\
&\quad + 2\int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_-^L(1) - A_-^L(\alpha)\right] d\alpha \cdot t_1^L\\
&\quad + 2\int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_+^L(1) - A_+^L(\alpha)\right] d\alpha \cdot t_4^L\\
&\quad + 2\int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_-^U(1) - A_-^U(\alpha)\right] d\alpha \cdot t_1^U\\
&\quad + 2\int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_+^U(1) - A_+^U(\alpha)\right] d\alpha \cdot t_4^U\\
&\quad + \int_0^1 f(\alpha)\left(A_-^L(\alpha) - \alpha \cdot A_-^L(1)\right)^2 d\alpha\\
&\quad + \int_0^1 f(\alpha)\left(A_+^L(\alpha) - \alpha \cdot A_+^L(1)\right)^2 d\alpha\\
&\quad + \int_0^1 f(\alpha)\left(A_-^U(\alpha) - \alpha \cdot A_-^U(1)\right)^2 d\alpha\\
&\quad + \int_0^1 f(\alpha)\left(A_+^U(\alpha) - \alpha \cdot A_+^U(1)\right)^2 d\alpha,
\end{aligned} \tag{53}
$$

subject to

$$
\begin{aligned}
t_1^U - t_1^L &\le 0,\\
t_4^L - t_4^U &\le 0.
\end{aligned} \tag{54}
$$

We present the main result of the paper as follows.

Theorem 16. *Let* $A = [A^L, A^U] \in IF(R)$, $A_\alpha = \{(x, y) \in R^2 : x \in [A_-^L(\alpha), A_+^L(\alpha)], y \in [A_-^U(\alpha), A_+^U(\alpha)]\}$, $\alpha \in [0, 1]$. $T(A) = [(t_1^L, t_2^L, t_3^L, t_4^L), (t_1^U, t_2^U, t_3^U, t_4^U)]$ *is the nearest interval-valued trapezoidal fuzzy number to A and preserves its core with respect to the weighted distance* D_I*. Consider the following.*

(i) *If*

$$
\begin{aligned}
&\int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_-^U(1) - A_-^U(\alpha)\right] d\alpha\\
&\quad - \int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_-^L(1) - A_-^L(\alpha)\right] d\alpha < 0,
\end{aligned} \tag{55}
$$

$$
\begin{aligned}
&\int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_+^U(1) - A_+^U(\alpha)\right] d\alpha\\
&\quad - \int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_+^L(1) - A_+^L(\alpha)\right] d\alpha \le 0,
\end{aligned} \tag{56}
$$

then we have

$$
\begin{aligned}
t_1^L = t_1^U &= -\Bigg(\Bigg(\int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_-^L(1) - A_-^L(\alpha)\right] d\alpha\\
&\qquad + \int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_-^U(1) - A_-^U(\alpha)\right] d\alpha\Bigg)\\
&\qquad \times \left(2\int_0^1 f(\alpha)(1-\alpha)^2 d\alpha\right)^{-1}\Bigg),\\
t_4^L &= -\frac{\int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_+^L(1) - A_+^L(\alpha)\right] d\alpha}{\int_0^1 f(\alpha)(1-\alpha)^2 d\alpha},\\
t_4^U &= -\frac{\int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_+^U(1) - A_+^U(\alpha)\right] d\alpha}{\int_0^1 f(\alpha)(1-\alpha)^2 d\alpha}.
\end{aligned} \tag{57}
$$

(ii) *If*

$$
\begin{aligned}
&\int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_-^U(1) - A_-^U(\alpha)\right] d\alpha\\
&\quad - \int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_-^L(1) - A_-^L(\alpha)\right] d\alpha < 0,
\end{aligned} \tag{58}
$$

$$
\begin{aligned}
&\int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_+^U(1) - A_+^U(\alpha)\right] d\alpha\\
&\quad - \int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_+^L(1) - A_+^L(\alpha)\right] d\alpha > 0,
\end{aligned} \tag{59}
$$

then we have

$$t_1^L = t_1^U = -\left(\left(\int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_-^L(1) - A_-^L(\alpha)\right] d\alpha + \int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_-^U(1) - A_-^U(\alpha)\right] d\alpha\right) \times \left(2\int_0^1 f(\alpha)(1-\alpha)^2 d\alpha\right)^{-1}\right),$$
$$t_4^L = t_4^U = -\left(\left(\int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_+^L(1) - A_+^L(\alpha)\right] d\alpha + \int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_+^U(1) - A_+^U(\alpha)\right] d\alpha\right) \times \left(2\int_0^1 f(\alpha)(1-\alpha)^2 d\alpha\right)^{-1}\right). \quad (60)$$

(iii) *If*

$$\int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_-^U(1) - A_-^U(\alpha)\right] d\alpha - \int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_-^L(1) - A_-^L(\alpha)\right] d\alpha \geq 0, \quad (61)$$

$$\int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_+^U(1) - A_+^U(\alpha)\right] d\alpha - \int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_+^L(1) - A_+^L(\alpha)\right] d\alpha > 0, \quad (62)$$

then we have

$$t_1^L = -\frac{\int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_-^L(1) - A_-^L(\alpha)\right] d\alpha}{\int_0^1 f(\alpha)(1-\alpha)^2 d\alpha},$$
$$t_1^U = -\frac{\int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_-^U(1) - A_-^U(\alpha)\right] d\alpha}{\int_0^1 f(\alpha)(1-\alpha)^2 d\alpha},$$
$$t_4^L = t_4^U = -\left(\left(\int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_+^L(1) - A_+^L(\alpha)\right] d\alpha + \int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_+^U(1) - A_+^U(\alpha)\right] d\alpha\right) \times \left(2\int_0^1 f(\alpha)(1-\alpha)^{\ 2} d\alpha\right)^{-1}\right). \quad (63)$$

(iv) *If*

$$\int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_-^U(1) - A_-^U(\alpha)\right] d\alpha - \int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_-^L(1) - A_-^L(\alpha)\right] d\alpha \geq 0,$$
$$\int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_+^U(1) - A_+^U(\alpha)\right] d\alpha - \int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_+^L(1) - A_+^L(\alpha)\right] d\alpha \leq 0, \quad (64)$$

then we have

$$t_1^L = -\frac{\int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_-^L(1) - A_-^L(\alpha)\right] d\alpha}{\int_0^1 f(\alpha)(1-\alpha)^2 d\alpha},$$
$$t_1^U = -\frac{\int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_-^U(1) - A_-^U(\alpha)\right] d\alpha}{\int_0^1 f(\alpha)(1-\alpha)^2 d\alpha},$$
$$t_4^L = -\frac{\int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_+^L(1) - A_+^L(\alpha)\right] d\alpha}{\int_0^1 f(\alpha)(1-\alpha)^2 d\alpha},$$
$$t_4^U = -\frac{\int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_+^U(1) - A_+^U(\alpha)\right] d\alpha}{\int_0^1 f(\alpha)(1-\alpha)^2 d\alpha}. \quad (65)$$

Proof. Because the function F in (53) and conditions (54) satisfy the hypothesis of convexity and differentiability in Theorem 15, after some simple calculations, conditions (i)–(iv) in Theorem 15 with respect to the minimization problem (53) in conditions (54) can be shown as follows:

$$2t_1^L \int_0^1 f(\alpha)(1-\alpha)^2 d\alpha + 2\int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_-^L(1) - A_-^L(\alpha)\right] d\alpha - \mu_1 = 0, \quad (66)$$

$$2t_4^L \int_0^1 f(\alpha)(1-\alpha)^2 d\alpha + 2\int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_+^L(1) - A_+^L(\alpha)\right] d\alpha + \mu_2 = 0, \quad (67)$$

$$2t_1^U \int_0^1 f(\alpha)(1-\alpha)^2 d\alpha + 2\int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_-^U(1) - A_-^U(\alpha)\right] d\alpha + \mu_1 = 0, \quad (68)$$

$$2t_4^U \int_0^1 f(\alpha)(1-\alpha)^2 d\alpha + 2\int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_+^U(1) - A_+^U(\alpha)\right] d\alpha - \mu_2 = 0, \quad (69)$$

$$\mu_1 \left(t_1^U - t_1^L\right) = 0, \quad (70)$$

$$\mu_2 \left(t_4^L - t_4^U\right) = 0, \quad (71)$$

$$\mu_1 \geq 0, \quad (72)$$

$$\mu_2 \geq 0, \quad (73)$$

$$t_1^U - t_1^L \leq 0, \quad (74)$$

$$t_4^L - t_4^U \leq 0. \quad (75)$$

(i) In the case $\mu_1 > 0$ and $\mu_2 = 0$, the solution of the system (66)–(75) is

$$t_1^L = t_1^U = -\left(\left(\int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_-^L(1) - A_-^L(\alpha)\right] d\alpha + \int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_-^U(1) - A_-^U(\alpha)\right] d\alpha\right) \times \left(2\int_0^1 f(\alpha)(1-\alpha)^2 d\alpha\right)^{-1}\right),$$
$$t_4^L = -\frac{\int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_+^L(1) - A_+^L(\alpha)\right] d\alpha}{\int_0^1 f(\alpha)(1-\alpha)^2 d\alpha},$$
$$t_4^U = -\frac{\int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_+^U(1) - A_+^U(\alpha)\right] d\alpha}{\int_0^1 f(\alpha)(1-\alpha)^2 d\alpha}. \quad (76)$$

Firstly, we have from (55) that $\mu_1 > 0$, and it follows from (56) that

$$\begin{aligned} t_4^U - t_4^L &= \frac{\int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_+^L(1) - A_+^L(\alpha)\right] d\alpha}{\int_0^1 f(\alpha)(1-\alpha)^2 d\alpha} - \frac{\int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_+^U(1) - A_+^U(\alpha)\right] d\alpha}{\int_0^1 f(\alpha)(1-\alpha)^2 d\alpha} \\ &= \left(\int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_+^L(1) - A_+^L(\alpha)\right] d\alpha - \int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_+^U(1) - A_+^U(\alpha)\right] d\alpha\right) \times \left(\int_0^1 f(\alpha)(1-\alpha)^2 d\alpha\right)^{-1} \\ &\geq 0. \end{aligned} \quad (77)$$

Then conditions (72), (73), (74), and (75) are verified.

Secondly, combing with (48), (55), and Lemma 14 (i), we can prove that

$$\begin{aligned} t_2^U - t_1^U &= A_-^U(1) + \left(\left(\int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_-^L(1) - A_-^L(\alpha)\right] d\alpha + \int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_-^U(1) - A_-^U(\alpha)\right] d\alpha\right) \times \left(2\int_0^1 f(\alpha)(1-\alpha)^2 d\alpha\right)^{-1}\right) \\ &> A_-^U(1) + \frac{\int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_-^U(1) - A_-^U(\alpha)\right] d\alpha}{\int_0^1 f(\alpha)(1-\alpha)^2 d\alpha} \\ &\geq 0. \end{aligned} \quad (78)$$

And on the basis of (50), we have

$$t_2^L - t_1^L \geq t_2^U - t_1^U > 0. \quad (79)$$

By making use of (48) and Lemma 14 (ii), we get

$$\begin{aligned} t_4^U - t_3^U &= -\frac{\int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_+^U(1) - A_+^U(\alpha)\right] d\alpha}{\int_0^1 f(\alpha)(1-\alpha)^2 d\alpha} - A_+^U(1) \geq 0, \\ t_4^L - t_3^L &= -\frac{\int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_+^L(1) - A_+^L(\alpha)\right] d\alpha}{\int_0^1 f(\alpha)(1-\alpha)^2 d\alpha} - A_+^L(1) \geq 0. \end{aligned} \quad (80)$$

It follows from (49) that $(t_1^L, t_2^L, t_3^L, t_4^L)$ and $(t_1^U, t_2^U, t_3^U, t_4^U)$ are two trapezoidal fuzzy numbers.

Therefore by Theorem 6 and (50), $T(A) = [(t_1^L, t_2^L, t_3^L, t_4^L), (t_1^U, t_2^U, t_3^U, t_4^U)] \in \mathrm{IF}^T(R)$ is the nearest interval-valued trapezoidal approximation of A in this case.

(ii) In the case $\mu_1 > 0$ and $\mu_2 > 0$, the solution of the system (66)–(75) is

$$t_1^L = t_1^U = -\left(\left(\int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_-^L(1) - A_-^L(\alpha)\right] d\alpha + \int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_-^U(1) - A_-^U(\alpha)\right] d\alpha\right) \times \left(2\int_0^1 f(\alpha)(1-\alpha)^2 d\alpha\right)^{-1}\right),$$

$$t_4^L = t_4^U = -\left(\left(\int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_+^L(1) - A_+^L(\alpha)\right] d\alpha\right.\right.$$

$$+\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_+^U(1)-A_+^U(\alpha)\right]d\alpha\Bigg)\times\left(2\int_0^1 f(\alpha)(1-\alpha)^2 d\alpha\right)^{-1}\Bigg).$$

$$\begin{aligned}\mu_1 &= \int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_-^L(1)-A_-^L(\alpha)\right]d\alpha\\&\quad-\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_-^U(1)-A_-^U(\alpha)\right]d\alpha,\\\mu_2 &= -\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_+^L(1)-A_+^L(\alpha)\right]d\alpha\\&\quad+\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_+^U(1)-A_+^U(\alpha)\right]d\alpha.\end{aligned}\tag{81}$$

We have from (58) and (59) that $\mu_1 > 0$ and $\mu_2 > 0$. Then conditions (72), (73), (74), and (75) are verified.

Furthermore, by making use of (48), (50), and (58) similar to (i), we can prove that

$$t_2^L - t_1^L \ge t_2^U - t_1^U > 0. \tag{82}$$

According to (48), (59), and Lemma 14 (ii), we obtain

$$\begin{aligned}&t_4^U - t_3^U\\&= -A_+^U(1) - \Bigg(\Bigg(\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_+^L(1)-A_+^L(\alpha)\right]d\alpha\\&\qquad+\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_+^U(1)-A_+^U(\alpha)\right]d\alpha\Bigg)\\&\qquad\times\left(2\int_0^1 f(\alpha)(1-\alpha)^2 d\alpha\right)^{-1}\Bigg)\\&> -A_+^U(1) - \frac{\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_+^U(1)-A_+^U(\alpha)\right]d\alpha}{\int_0^1 f(\alpha)(1-\alpha)^2 d\alpha}\\&\ge 0.\end{aligned}\tag{83}$$

This implies that

$$t_4^L - t_3^L \ge t_4^U - t_3^U > 0. \tag{84}$$

It follows from (49) that $(t_1^L, t_2^L, t_3^L, t_4^L)$ and $(t_1^U, t_2^U, t_3^U, t_4^U)$ are two trapezoidal fuzzy numbers.

Therefore by Theorem 6 and (50), $T(A) = [(t_1^L, t_2^L, t_3^L, t_4^L), (t_1^U, t_2^U, t_3^U, t_4^U)] \in \mathrm{IF}^T(R)$ is the nearest interval-valued trapezoidal approximation of A in this case.

(iii) In the case $\mu_1 = 0$ and $\mu_2 > 0$, the solution of the system (66)–(75) is

$$\begin{aligned}t_1^L &= -\frac{\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_-^L(1)-A_-^L(\alpha)\right]d\alpha}{\int_0^1 f(\alpha)(1-\alpha)^2 d\alpha},\\t_1^U &= -\frac{\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_-^U(1)-A_-^U(\alpha)\right]d\alpha}{\int_0^1 f(\alpha)(1-\alpha)^2 d\alpha},\\t_4^L = t_4^U &= -\Bigg(\Bigg(\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_+^L(1)-A_+^L(\alpha)\right]d\alpha\\&\qquad+\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_+^U(1)-A_+^U(\alpha)\right]d\alpha\Bigg)\\&\qquad\times\left(2\int_0^1 f(\alpha)(1-\alpha)^2 d\alpha\right)^{-1}\Bigg),\\\mu_1 &= 0,\\\mu_2 &= -\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_+^L(1)-A_+^L(\alpha)\right]d\alpha\\&\quad+\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_+^U(1)-A_+^U(\alpha)\right]d\alpha.\end{aligned}\tag{85}$$

First, we have from (62) that $\mu_2 > 0$. Also, it follows from (61), we can prove that

$$\begin{aligned}t_1^L - t_1^U &= \frac{\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_-^U(1)-A_-^U(\alpha)\right]d\alpha}{\int_0^1 f(\alpha)(1-\alpha)^2 d\alpha}\\&\quad-\frac{\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_-^L(1)-A_-^L(\alpha)\right]d\alpha}{\int_0^1 f(\alpha)(1-\alpha)^2 d\alpha}\\&= \Bigg(\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_-^U(1)-A_-^U(\alpha)\right]d\alpha\\&\qquad-\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_-^L(1)-A_-^L(\alpha)\right]d\alpha\Bigg)\\&\qquad\times\left(\int_0^1 f(\alpha)(1-\alpha)^2 d\alpha\right)^{-1}\\&\ge 0.\end{aligned}\tag{86}$$

Then conditions (72), (73), (74), and (75) are verified.

Secondly, combing with (48) and Lemma 14 (i), we have

$$\begin{aligned}&t_2^L - t_1^L\\&= A_-^L(1) + \frac{\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_-^L(1)-A_-^L(\alpha)\right]d\alpha}{\int_0^1 f(\alpha)(1-\alpha)^2 d\alpha} \ge 0,\\&t_2^U - t_1^U\\&= A_-^U(1) + \frac{\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_-^U(1)-A_-^U(\alpha)\right]d\alpha}{\int_0^1 f(\alpha)(1-\alpha)^2 d\alpha} \ge 0.\end{aligned}\tag{87}$$

According to (48), (50), (62), and the second result of Lemma 14 (ii), similar to (ii), we can prove that $t_4^L - t_3^L \ge t_4^U - t_3^U > 0$. It follows from (49) that $(t_1^L, t_2^L, t_3^L, t_4^L)$ and $(t_1^U, t_2^U, t_3^U, t_4^U)$ are two trapezoidal fuzzy numbers.

Therefore by Theorem 6 and (50), $T(A) = [(t_1^L, t_2^L, t_3^L, t_4^L), (t_1^U, t_2^U, t_3^U, t_4^U)] \in \mathrm{IF}^{\mathrm{T}}(R)$ is the nearest interval-valued trapezoidal approximation of A in this case.

(iv) In the case $\mu_1 = 0$ and $\mu_2 = 0$, the solution of the system (66)–(75) is

$$\begin{aligned}
t_1^L &= -\frac{\int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_-^L(1) - A_-^L(\alpha)\right] d\alpha}{\int_0^1 f(\alpha)(1-\alpha)^2 d\alpha}, \\
t_1^U &= -\frac{\int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_-^U(1) - A_-^U(\alpha)\right] d\alpha}{\int_0^1 f(\alpha)(1-\alpha)^2 d\alpha}, \\
t_4^L &= -\frac{\int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_+^L(1) - A_+^L(\alpha)\right] d\alpha}{\int_0^1 f(\alpha)(1-\alpha)^2 d\alpha}, \\
t_4^U &= -\frac{\int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_+^U(1) - A_+^U(\alpha)\right] d\alpha}{\int_0^1 f(\alpha)(1-\alpha)^2 d\alpha}. \\
\mu_1 &= 0, \qquad \mu_2 = 0.
\end{aligned} \tag{88}$$

By (64), similar to (i) and (iii) we have $t_1^L - t_1^U \geq 0$ and $t_4^U - t_4^L \geq 0$; then conditions (72), (73), (74), and (75) are verified.

Furthermore, similar to (i) and (iii), we can prove that $(t_1^L, t_2^L, t_3^L, t_4^L)$ and $(t_1^U, t_2^U, t_3^U, t_4^U)$ are two trapezoidal fuzzy numbers.

Therefore by Theorem 6 and (50), $T(A) = [(t_1^L, t_2^L, t_3^L, t_4^L), (t_1^U, t_2^U, t_3^U, t_4^U)] \in \mathrm{IF}^T(R)$ is the nearest interval-valued trapezoidal approximation of A in this case.

For any interval-valued fuzzy number, we can apply one and only one of the above situations (i)–(iv) to calculate the interval-valued trapezoidal approximation of it. We denote

$$\begin{aligned}
\Omega_1 = \Big\{ & A = \left[A^L, A^U\right] \in \mathrm{IF}(R) : \\
& \int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_-^U(1) - A_-^U(\alpha)\right] d\alpha \\
& - \int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_-^L(1) - A_-^L(\alpha)\right] d\alpha < 0, \\
& \int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_+^U(1) - A_+^U(\alpha)\right] d\alpha \\
& - \int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_+^L(1) - A_+^L(\alpha)\right] d\alpha \leq 0 \Big\}, \\
\Omega_2 = \Big\{ & A = \left[A^L, A^U\right] \in \mathrm{IF}(R) : \\
& \int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_-^U(1) - A_-^U(\alpha)\right] d\alpha \\
& - \int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_-^L(1) - A_-^L(\alpha)\right] d\alpha < 0, \\
& \int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_+^U(1) - A_+^U(\alpha)\right] d\alpha \\
& - \int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_+^L(1) - A_+^L(\alpha)\right] d\alpha > 0 \Big\}, \\
\Omega_3 = \Big\{ & A = \left[A^L, A^U\right] \in \mathrm{IF}(R) : \\
& \int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_-^U(1) - A_-^U(\alpha)\right] d\alpha \\
& - \int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_-^L(1) - A_-^L(\alpha)\right] d\alpha \geq 0, \\
& \int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_+^U(1) - A_+^U(\alpha)\right] d\alpha \\
& - \int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_+^L(1) - A_+^L(\alpha)\right] d\alpha > 0 \Big\}, \\
\Omega_4 = \Big\{ & A = \left[A^L, A^U\right] \in \mathrm{IF}(R) : \\
& \int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_-^U(1) - A_-^U(\alpha)\right] d\alpha \\
& - \int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_-^L(1) - A_-^L(\alpha)\right] d\alpha \geq 0, \\
& \int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_+^U(1) - A_+^U(\alpha)\right] d\alpha \\
& - \int_0^1 f(\alpha)(1-\alpha)\left[\alpha \cdot A_+^L(1) - A_+^L(\alpha)\right] d\alpha \leq 0 \Big\}.
\end{aligned} \tag{89}$$

It is obvious that the cases (i)–(iv) cover the set of all interval-valued fuzzy numbers and Ω_1, Ω_2, Ω_3, and Ω_4 are disjoint sets. So the approximation operator always gives an interval-valued trapezoidal fuzzy number. □

By the discussion of Theorem 16, we could find the nearest interval-valued trapezoidal fuzzy number for a given interval-valued fuzzy number. Furthermore, it preserves the core of the given interval-valued fuzzy number with respect to the weighted distance D_I.

Remark 17. If $A = [A^L, A^U] \in F(R)$, that is, $A^L = A^U$, then our conclusion is consisten with [23].

Corollary 18. *Let* $A = [A^L, A^U] = [(a_1^L, a_2^L, a_3^L, a_4^L)_r, (a_1^U, a_2^U, a_3^U, a_4^U)_r] \in IF(R)$, *where* $r > 0$ *and*

$$A^L(x) = \begin{cases} \left(\dfrac{x - a_1^L}{a_2^L - a_1^L}\right)^r, & a_1^L \leq x < a_2^L, \\ 1, & a_2^L \leq x \leq a_3^L, \\ \left(\dfrac{a_4^L - x}{a_4^L - a_3^L}\right)^r, & a_3^L < x \leq a_4^L, \\ 0, & \text{otherwise}, \end{cases}$$

$$A^U(x)=\begin{cases}\left(\dfrac{x-a_1^U}{a_2^U-a_1^U}\right)^r, & a_1^U\le x<a_2^U,\\ 1, & a_2^U\le x\le a_3^U,\\ \left(\dfrac{a_4^U-x}{a_4^U-a_3^U}\right)^r, & a_3^U<x\le a_4^U,\\ 0, & \text{otherwise}.\end{cases} \tag{90}$$

(i) *If* $f(\alpha)=\alpha$ *and*

$$\begin{aligned}&\left(-6r^2+5r+1\right)\left(a_2^U-a_2^L\right)-2(5r+1)\left(a_1^U-a_1^L\right)<0,\\&\left(-6r^2+5r+1\right)\left(a_3^U-a_3^L\right)-2(5r+1)\left(a_4^U-a_4^L\right)\le 0,\end{aligned} \tag{91}$$

then

$$\begin{aligned}t_1^L=t_1^U&=-\frac{\left(-6r^2+5r+1\right)\left(a_2^U+a_2^L\right)-2(5r+1)\left(a_1^U+a_1^L\right)}{2(1+2r)(1+3r)},\\ t_4^L&=-\frac{\left(-6r^2+5r+1\right)a_3^L-2(5r+1)a_4^L}{(1+2r)(1+3r)},\\ t_4^U&=-\frac{\left(-6r^2+5r+1\right)a_3^U-2(5r+1)a_4^U}{(1+2r)(1+3r)}.\end{aligned} \tag{92}$$

(ii) *If* $f(\alpha)=\alpha$ *and*

$$\begin{aligned}&\left(-6r^2+5r+1\right)\left(a_2^U-a_2^L\right)-2(5r+1)\left(a_1^U-a_1^L\right)<0,\\&\left(-6r^2+5r+1\right)\left(a_3^U-a_3^L\right)-2(5r+1)\left(a_4^U-a_4^L\right)>0,\end{aligned} \tag{93}$$

then

$$\begin{aligned}t_1^L&=t_1^U\\&=-\frac{\left(-6r^2+5r+1\right)\left(a_2^U+a_2^L\right)-2(5r+1)\left(a_1^U+a_1^L\right)}{2(1+2r)(1+3r)},\\ t_4^L&=t_4^U\\&=-\frac{\left(-6r^2+5r+1\right)\left(a_3^U+a_3^L\right)-2(5r+1)\left(a_4^U+a_4^L\right)}{2(1+2r)(1+3r)}.\end{aligned} \tag{94}$$

(iii) *If* $f(\alpha)=\alpha$ *and*

$$\begin{aligned}&\left(-6r^2+5r+1\right)\left(a_2^U-a_2^L\right)-2(5r+1)\left(a_1^U-a_1^L\right)\ge 0,\\&\left(-6r^2+5r+1\right)\left(a_3^U-a_3^L\right)-2(5r+1)\left(a_4^U-a_4^L\right)>0,\end{aligned} \tag{95}$$

then

$$\begin{aligned}t_1^L&=-\frac{\left(-6r^2+5r+1\right)a_2^L-2(5r+1)a_1^L}{(1+2r)(1+3r)},\\ t_1^U&=-\frac{\left(-6r^2+5r+1\right)a_2^U-2(5r+1)a_1^U}{(1+2r)(1+3r)},\\ t_4^L&=t_4^U\\&=-\frac{\left(-6r^2+5r+1\right)\left(a_3^U+a_3^L\right)-2(5r+1)\left(a_4^U+a_4^L\right)}{2(1+2r)(1+3r)}.\end{aligned} \tag{96}$$

(iv) *If* $f(\alpha)=\alpha$ *and*

$$\begin{aligned}&\left(-6r^2+5r+1\right)\left(a_2^U-a_2^L\right)-2(5r+1)\left(a_1^U-a_1^L\right)\ge 0,\\&\left(-6r^2+5r+1\right)\left(a_3^U-a_3^L\right)-2(5r+1)\left(a_4^U-a_4^L\right)\le 0,\end{aligned} \tag{97}$$

then

$$\begin{aligned}t_1^L&=-\frac{\left(-6r^2+5r+1\right)a_2^L-2(5r+1)a_1^L}{(1+2r)(1+3r)},\\ t_1^U&=-\frac{\left(-6r^2+5r+1\right)a_2^U-2(5r+1)a_1^U}{(1+2r)(1+3r)},\\ t_4^L&=-\frac{\left(-6r^2+5r+1\right)a_3^L-2(5r+1)a_4^L}{(1+2r)(1+3r)},\\ t_4^U&=-\frac{\left(-6r^2+5r+1\right)a_3^U-2(5r+1)a_4^U}{(1+2r)(1+3r)}.\end{aligned} \tag{98}$$

Proof. Let $A=[A^L,A^U]=[(a_1^L,a_2^L,a_3^L,a_4^L)_r,(a_1^U,a_2^U,a_3^U,a_4^U)_r]\in \mathrm{IF}(R)$. We have $A_-^L(\alpha)=a_1^L+(a_2^L-a_1^L)\cdot\alpha^{1/r}$, $A_-^U(\alpha)=a_1^U+(a_2^U-a_1^U)\cdot\alpha^{1/r}$, $A_+^L(\alpha)=a_4^L-(a_4^L-a_3^L)\cdot\alpha^{1/r}$, and $A_+^U(\alpha)=a_4^U-(a_4^U-a_3^U)\cdot\alpha^{1/r}$. It is obvious that $A_-^L(1)=a_2^L$, $A_-^U(1)=a_2^U$, $A_+^L(1)=a_3^L$, and $A_+^U(1)=a_3^U$. Then by $f(\alpha)=\alpha$, we can prove that

$$\begin{aligned}&\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_-^L(1)-A_-^L(\alpha)\right]d\alpha\\&\quad=\frac{\left(-6r^2+5r+1\right)a_2^L-2(5r+1)a_1^L}{12(1+2r)(1+3r)},\\&\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_-^U(1)-A_-^U(\alpha)\right]d\alpha\\&\quad=\frac{\left(-6r^2+5r+1\right)a_2^U-2(5r+1)a_1^U}{12(1+2r)(1+3r)},\\&\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_+^L(1)-A_+^L(\alpha)\right]d\alpha\\&\quad=\frac{\left(-6r^2+5r+1\right)a_3^L-2(5r+1)a_4^L}{12(1+2r)(1+3r)},\\&\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_+^U(1)-A_+^U(\alpha)\right]d\alpha\\&\quad=\frac{\left(-6r^2+5r+1\right)a_3^U-2(5r+1)a_4^U}{12(1+2r)(1+3r)},\\&\int_0^1 f(\alpha)(1-\alpha)^2d\alpha=\frac{1}{12}.\end{aligned} \tag{99}$$

According to Theorem 16 the results can be easily obtained. □

Example 19. Let $A = [A^L, A^U] = [(2,4,5,7)_{1/2}, (2,3,6,8)_{1/2}] \in \mathrm{IF}(R)$ and $f(\alpha) = \alpha$. Since

$$\left(-6r^2 + 5r + 1\right)\left(a_2^U - a_2^L\right) - 2(5r+1)\left(a_1^U - a_1^L\right) = -2 < 0,$$
$$\left(-6r^2 + 5r + 1\right)\left(a_3^U - a_3^L\right) - 2(5r+1)\left(a_4^U - a_4^L\right) = -5 < 0, \quad (100)$$

that is, $A = [A^L, A^U]$ satisfies condition (i) of Corollary 18, we have

$$t_1^L = t_1^U = -\frac{\left(-6r^2 + 5r + 1\right)\left(a_2^U + a_2^L\right) - 2(5r+1)\left(a_1^U + a_1^L\right)}{2(1+2r)(1+3r)} = \frac{7}{5},$$
$$t_4^L = -\frac{\left(-6r^2 + 5r + 1\right)a_3^L - 2(5r+1)a_4^L}{(1+2r)(1+3r)} = \frac{39}{5},$$
$$t_4^U = -\frac{\left(-6r^2 + 5r + 1\right)a_3^U - 2(5r+1)a_4^U}{(1+2r)(1+3r)} = \frac{44}{5}. \quad (101)$$

Therefore, $T(A) = [(7/5, 4, 5, 39/5), (7/5, 3, 6, 44/5)] \in \mathrm{IF}^T(R)$ is the nearest interval-valued trapezoidal fuzzy number to A, which preserves the core of A.

Remark 20. Let $A = [A^L, A^U] \in \mathrm{IF}(R)$. $T(A) = [T(A^L), T(A^U)]$ be not true in general.

Example 21. Let $A^L = (2,4,5,7)_{1/2} \in F(R)$, $A^U = (2,3,6,8)_{1/2} \in F(R)$, and $f(\alpha) = \alpha$. Then, according to condition (iv) of Corollary 18, we have

$$t_1^L = -\frac{\left(-6r^2 + 5r + 1\right)a_2^L - 2(5r+1)a_1^L}{(1+2r)(1+3r)} = \frac{6}{5},$$
$$t_1^U = -\frac{\left(-6r^2 + 5r + 1\right)a_2^U - 2(5r+1)a_1^U}{(1+2r)(1+3r)} = \frac{8}{5},$$
$$t_4^L = -\frac{\left(-6r^2 + 5r + 1\right)a_3^L - 2(5r+1)a_4^L}{(1+2r)(1+3r)} = \frac{39}{5},$$
$$t_4^U = -\frac{\left(-6r^2 + 5r + 1\right)a_3^U - 2(5r+1)a_4^U}{(1+2r)(1+3r)} = \frac{44}{5}. \quad (102)$$

Therefore, $T(A^L) = (6/5, 4, 5, 39/5)$ and $T(A^U) = (8/5, 3, 6, 44/5)$. Based on Example 19, we known that $T(A) \neq [T(A^L), T(A^U)]$. Further, we have from Theorem 6 that $[T(A^L), T(A^U)]$ is not a trapezoidal interval-valued fuzzy number.

4. Properties of the Interval-Valued Trapezoidal Approximation Operator

In this section we consider some properties of the approximation operator suggested in Section 3.2. With respect to the criteria, translation invariance, scale invariance, identity, nearness criterion, and ranking invariance, we present the following results.

Theorem 22. *The approximation operator $T : IF(R) \rightarrow IF^T(R)$ which preserves the core of the initial interval-valued fuzzy number has the following properties.*

(i) *The operator T is invariant to translations; that is, for any $A \in IF(R)$ and $z \in R$,*

$$T(A+z) = T(A) + z. \quad (103)$$

(ii) *The operator T is scale invariance; that is, for any $A \in IF(R)$ and $\lambda \in R \setminus \{0\}$,*

$$T(\lambda A) = \lambda T(A). \quad (104)$$

(iii) *The operator T fulfills the identity criterion; that is, for any $A \in IF^T(R)$,*

$$T(A) = A. \quad (105)$$

(iv) *The operator T fulfills the nearness criterion with respect to the weighted distance D_I; that is,*

$$D_I(A, T(A)) \leq D_I(A, B), \quad (106)$$

for every $A \in IF(R)$ and $B \in IF^T(R)$ such that coreB = coreT(A).

(v) *The operator T is core invariance; that is, for any $A \in IF(R)$,*

$$\mathrm{core}T(A) = \mathrm{core}A. \quad (107)$$

(vi) *The operator T is ranking invariance; that is,*

$$A \succeq B \Longleftrightarrow T(A) \succeq T(B), \quad (108)$$

for every $A, B \in IF(R)$.

Proof. If $A \in \mathrm{IF}(R)$, according to (28) and (48), we have

$$t_2^L(A+z) = (A+z)_-^L(1) = A_-^L(1) + z = t_2^L(A) + z. \quad (109)$$

Similarly, we can prove that

$$t_3^L(A+z) = t_3^L(A) + z,$$
$$t_2^U(A+z) = t_2^U(A) + z, \quad (110)$$
$$t_3^U(A+z) = t_3^U(A) + z.$$

Furthermore, we have from (28) and (29) that

$$\begin{aligned}
&\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot(A+z)_-^L(1)-(A+z)_-^L(\alpha)\right]d\alpha\\
&\quad=\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_-^L(1)-A_-^L(\alpha)\right]d\alpha\\
&\qquad-z\int_0^1 f(\alpha)(1-\alpha)^2 d\alpha,\\
&\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot(A+z)_-^U(1)-(A+z)_-^U(\alpha)\right]d\alpha\\
&\quad=\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_-^U(1)-A_-^U(\alpha)\right]d\alpha\\
&\qquad-z\int_0^1 f(\alpha)(1-\alpha)^2 d\alpha,\\
&\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot(A+z)_+^L(1)-(A+z)_+^L(\alpha)\right]d\alpha\\
&\quad=\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_+^L(1)-A_+^L(\alpha)\right]d\alpha\\
&\qquad-z\int_0^1 f(\alpha)(1-\alpha)^2 d\alpha,\\
&\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot(A+z)_+^U(1)-(A+z)_+^U(\alpha)\right]d\alpha\\
&\quad=\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_+^U(1)-A_+^U(\alpha)\right]d\alpha\\
&\qquad-z\int_0^1 f(\alpha)(1-\alpha)^2 d\alpha.
\end{aligned}\tag{111}$$

Then, one can easily prove that the interval-valued fuzzy number A satisfies one of conditions (i)–(iv) of Theorem 16 if and only if the interval-valued fuzzy number $A+z$ satisfies the same condition. In any case of Theorem 16, by making use of (111), we obtain

$$\begin{aligned}
t_k^L(A+z)&=t_k^L(A)+z,\\
t_k^U(A+z)&=t_k^U(A)+z,
\end{aligned}\tag{112}$$

for every $k\in\{1,4\}$. Therefore, combine the above results with (109) and (110) and we have $T(A+z)=T(A)+z$.

(ii) Let $A\in\mathrm{IF}(R)$. If $\lambda>0$, combing with (32), similar to (i), we can prove that $T(\lambda A)=\lambda T(A)$.

If $\lambda<0$, we have from (33) and (48) that

$$t_2^L(\lambda A)=(\lambda A)_-^L(1)=\lambda A_+^L(1)=\lambda t_3^L(A).\tag{113}$$

Similarly, we can prove that

$$\begin{gathered}
t_3^L(\lambda A)=\lambda t_2^L(A),\qquad t_2^U(\lambda A)=\lambda t_3^U(A),\\
t_3^U(\lambda A)=\lambda t_2^U(A).
\end{gathered}\tag{114}$$

Furthermore, it follows from (33) and (34) that

$$\begin{aligned}
&\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot(\lambda A)_-^L(1)-(\lambda A)_-^L(\alpha)\right]d\alpha\\
&\quad=\lambda\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_+^L(1)-A_+^L(\alpha)\right]d\alpha,\\
&\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot(\lambda A)_-^U(1)-(\lambda A)_-^U(\alpha)\right]d\alpha\\
&\quad=\lambda\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_+^U(1)-A_+^U(\alpha)\right]d\alpha,\\
&\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot(\lambda A)_+^L(1)-(\lambda A)_+^L(\alpha)\right]d\alpha\\
&\quad=\lambda\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_-^L(1)-A_-^L(\alpha)\right]d\alpha,\\
&\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot(\lambda A)_+^U(1)-(\lambda A)_+^U(\alpha)\right]d\alpha\\
&\quad=\lambda\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_-^U(1)-A_-^U(\alpha)\right]d\alpha.
\end{aligned}\tag{115}$$

Thus, λA is in the case (i) of Theorem 16 if and only if A is in the case (iii) of Theorem 16. Then making use of (115) and Theorem 16, we get

$$\begin{aligned}
t_1^L(\lambda A)&=\lambda t_4^L(A),\qquad t_4^L(\lambda A)=\lambda t_1^L(A),\\
t_1^U(\lambda A)&=\lambda t_4^U(A),\qquad t_4^U(\lambda A)=\lambda t_1^U(A).
\end{aligned}\tag{116}$$

Therefore, combine the above results with (113) and (114) and according to (33) and (34) we have

$$\begin{aligned}
T(\lambda A)&=\left[\left(t_1^L(\lambda A),t_2^L(\lambda A),t_3^L(\lambda A),t_4^L(\lambda A)\right),\right.\\
&\qquad\left.\left(t_1^U(\lambda A),t_2^U(\lambda A),t_3^U(\lambda A),t_4^U(\lambda A)\right)\right]\\
&=\left[\left(\lambda t_4^L,\lambda t_3^L,\lambda t_2^L,\lambda t_1^L\right),\left(\lambda t_4^U,\lambda t_3^U,\lambda t_2^U,\lambda t_1^U\right)\right]\\
&=\lambda\left[\left(t_1^L,t_2^L,t_3^L,t_4^L\right),\left(t_1^U,t_2^U,t_3^U,t_4^U\right)\right]\\
&=\lambda T(A).
\end{aligned}\tag{117}$$

Analogously, λA is in the case (ii) of Theorem 16 if and only if A is in the case (ii) of Theorem 16. λA is in the case (iii) of Theorem 16 if and only if A is in the case (i) of Theorem 16. λA is in the case (iv) of Theorem 16 if and only if A is in the case (iv) of Theorem 16. In each case $t_k^L(\lambda A)=\lambda t_{5-k}^L(A)$, $t_k^U(\lambda A)=\lambda t_{5-k}^U(A)$, for every $k\in\{1,2,3,4\}$; therefore, $T(\lambda A)=\lambda T(A)$.

(iii) If $A\in\mathrm{IF}^T(R)$, then A is in the case (iv) of Theorem 16 and $T(A)=A$.

(iv) and (v) are the direct consequences of Theorem 16.

(vi) By (22) and Theorem 16, we can obtain the conclusion. □

The continuity is considered the essential property for an approximation operator. However, the approximation operator given by Theorem 16 is not continuous, as the following example proves.

Example 23 (see [25]). Let us consider $A \in F(R) \subset \mathrm{IF}(R)$, $A_\alpha = [A_-(\alpha), A_+(\alpha)]$, $\alpha \in [0,1]$, such that $A_-(1) < A_+(1)$ and the sequence of fuzzy numbers $(A_n)_{n\in N}$ is given by

$$\begin{aligned} (A_n)_-(\alpha) &= A_-(\alpha) + \alpha^n (A_+(1) - A_-(1)), \\ (A_n)_+(\alpha) &= A_+(\alpha), \\ &\alpha \in [0,1]. \end{aligned} \tag{118}$$

It is easy to check that the function $(A_n)_-(\alpha)$ is nondecreasing and $(A_n)_-(1) = A_+(1) = (A_n)_+(1)$; therefore, A_n is a fuzzy number, for any $n \in N$. Then, according to the weighted distance d_f defined by (20), we have

$$\begin{aligned} d_f^2(A_n, A) &= (A_+(1) - A_-(1))^2 \int_0^1 f(\alpha)\alpha^{2n} d\alpha \\ &\le f(1)(A_+(1) - A_-(1))^2 \int_0^1 \alpha^{2n} d\alpha \\ &= \frac{f(1)(A_+(1) - A_-(1))^2}{2n+1}. \end{aligned} \tag{119}$$

It is immediate that $\lim_{n\to\infty} A_n = A$. Now, denote

$$\begin{aligned} T(A) &= (t_1, t_2, t_3, t_4), \\ T(A_n) &= (t_1(n), t_2(n), t_3(n), t_4(n)), \\ &n \in N. \end{aligned} \tag{120}$$

Because operator T preserves the core of fuzzy number A, by (48) we have

$$\lim_{n\to\infty} \mathrm{t}_2(n) = \lim_{n\to\infty} (A_n)_-(1) = A_+(1) > A_-(1) = t_2. \tag{121}$$

By seeing Lemma 3 [25], we cannot have $\lim_{n\to\infty} T(A_n) = T(A)$ with respect to the weighted distance d_f. It follows from Heine's criterion that T is discontinuous.

To overcome the handicap of discontinuity of the approximation operator T we present the following distance property.

Lemma 24. *Let $T_n = [(t_1^L(n), t_2^L(n), t_3^L(n), t_4^L(n)), (t_1^U(n), t_2^U(n), t_3^U(n), t_4^U(n))]$ be a sequence of interval-valued trapezoidal fuzzy numbers. If $\lim_{n\to\infty} t_i^L(n) = t_i^L$, $\lim_{n\to\infty} t_i^U(n) = t_i^U$, $i \in \{1,2,3,4\}$, then $\lim_{n\to\infty} T_n = T$ with respect to the weighted distance D_I, where $T = [(t_1^L, t_2^L, t_3^L, t_4^L), (t_1^U, t_2^U, t_3^U, t_4^U)] \in IF^T(R)$.*

Proof. It is similar to the proof of Lemma 2 in the paper [25]. □

Theorem 25. *Let $A = [A^L, A^U] \in IF(R)$, $A_\alpha = \{(x,y) \in R^2 : x \in [A_-^L(\alpha), A_+^L(\alpha)], y \in [A_-^U(\alpha), A_+^U(\alpha)]\}$, $\alpha \in [0,1]$, and $A_n = [A_n^L, A_n^U] (n \in N)$ be a sequence of interval-valued fuzzy numbers, where $(A_n)_\alpha = \{(x,y) \in R^2 : x \in [(A_n^L)_-(\alpha), (A_n^L)_+(\alpha)], y \in [(A_n^U)_-(\alpha), (A_n^U)_+(\alpha)]\}$, $\alpha \in [0,1]$. If $(A_n^L)_-(\alpha), (A_n^L)_+(\alpha), (A_n^U)_-(\alpha)$ and $(A_n^U)_+(\alpha)$ are uniform convergent sequences to $A_-^L(\alpha), A_+^L(\alpha), A_-^U(\alpha)$ and $A_+^U(\alpha)$, respectively, then*

$$\lim_{n\to\infty} T(A_n) = T(A), \tag{122}$$

with respect to the weighted distance D_I.

Proof. We denote

$$\begin{aligned} T(A) &= [(t_1^L, t_2^L, t_3^L, t_4^L), (t_1^U, t_2^U, t_3^U, t_4^U)], \\ T(A_n) &= [(t_1^L(n), t_2^L(n), t_3^L(n), t_4^L(n)), \\ &\quad (t_1^U(n), t_2^U(n), t_3^U(n), t_4^U(n))], \quad n \in N. \end{aligned} \tag{123}$$

Because $(A_n^L)_-(\alpha)$, $(A_n^L)_+(\alpha)$, $(A_n^U)_-(\alpha)$ and $(A_n^U)_+(\alpha)$ are uniform convergent sequences to $A_-^L(\alpha)$, $A_+^L(\alpha)$, $A_-^U(\alpha)$ and $A_+^U(\alpha)$, respectively, we have

$$\begin{aligned} &\lim_{n\to\infty} \int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot(A_n^L)_-(1) - (A_n^L)_-(\alpha)\right] d\alpha \\ &\quad = \int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_-^L(1) - A_-^L(\alpha)\right] d\alpha, \\ &\lim_{n\to\infty} \int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot(A_n^L)_+(1) - (A_n^L)_+(\alpha)\right] d\alpha \\ &\quad = \int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_+^L(1) - A_+^L(\alpha)\right] d\alpha, \\ &\lim_{n\to\infty} \int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot(A_n^U)_-(1) - (A_n^U)_-(\alpha)\right] d\alpha \\ &\quad = \int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_-^U(1) - A_-^U(\alpha)\right] d\alpha, \\ &\lim_{n\to\infty} \int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot(A_n^U)_+(1) - (A_n^U)_+(\alpha)\right] d\alpha \\ &\quad = \int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_+^U(1) - A_+^U(\alpha)\right] d\alpha, \end{aligned} \tag{124}$$

and by (48) we obtain

$$\begin{aligned} \lim_{n\to\infty} t_2^L(n) &= \lim_{n\to\infty} (A_n^L)_-(1) = A_-^L(1) = t_2^L, \\ \lim_{n\to\infty} t_2^U(n) &= \lim_{n\to\infty} (A_n^U)_-(1) = A_-^U(1) = t_2^U, \\ \lim_{n\to\infty} t_3^L(n) &= \lim_{n\to\infty} (A_n^L)_+(1) = A_+^L(1) = t_3^L, \\ \lim_{n\to\infty} t_3^U(n) &= \lim_{n\to\infty} (A_n^U)_+(1) = A_+^U(1) = t_3^U. \end{aligned} \tag{125}$$

Considering the following cases.

(i) $A = [A^L, A^U]$ satisfies condition (i) of Theorem 16; the following situations are possible.

($\mathrm{i_a}$) If

$$\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_-^U(1) - A_-^U(\alpha)\right]d\alpha - \int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_-^L(1) - A_-^L(\alpha)\right]d\alpha < 0,$$
$$\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_+^U(1) - A_+^U(\alpha)\right]d\alpha - \int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_+^L(1) - A_+^L(\alpha)\right]d\alpha < 0, \quad (126)$$

then there exists N, when $n > N$, A_n satisfies condition (i) of Theorem 16. We have from (124) that

$$\begin{aligned}
&\lim_{n\to\infty} t_1^L(n) \\
&= -\lim_{n\to\infty}\left(\left(\int_0^1 f(\alpha)(1-\alpha)\times\left[\alpha\cdot\left(A_n^L\right)_-(1) - \left(A_n^L\right)_-(\alpha)\right]d\alpha + \int_0^1 f(\alpha)(1-\alpha)\times\left[\alpha\cdot\left(A_n^U\right)_-(1) - \left(A_n^U\right)_-(\alpha)\right]d\alpha\right)\times\left(2\int_0^1 f(\alpha)(1-\alpha)^2 d\alpha\right)^{-1}\right) \\
&= -\left(\left(\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_-^L(1) - A_-^L(\alpha)\right]d\alpha + \int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_-^U(1) - A_-^U(\alpha)\right]d\alpha\right)\times\left(2\int_0^1 f(\alpha)(1-\alpha)^2 d\alpha\right)^{-1}\right) = t_1^L, \\
&\lim_{n\to\infty} t_1^U(n) = \lim_{n\to\infty} t_1^L(n) = t_1^L = t_1^U, \\
&\lim_{n\to\infty} t_4^L(n) \\
&= -\lim_{n\to\infty}\frac{\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot\left(A_n^L\right)_+(1) - \left(A_n^L\right)_+(\alpha)\right]d\alpha}{\int_0^1 f(\alpha)(1-\alpha)^2 d\alpha} \\
&= -\frac{\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_+^L(1) - A_+^L(\alpha)\right]d\alpha}{\int_0^1 f(\alpha)(1-\alpha)^2 d\alpha} \\
&= t_4^L, \\
&\lim_{n\to\infty} t_4^U(n) \\
&= -\lim_{n\to\infty}\frac{\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot\left(A_n^U\right)_+(1) - \left(A_n^U\right)_+(\alpha)\right]d\alpha}{\int_0^1 f(\alpha)(1-\alpha)^2 d\alpha} \\
&= -\frac{\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_+^U(1) - A_+^U(\alpha)\right]d\alpha}{\int_0^1 f(\alpha)(1-\alpha)^2 d\alpha} \\
&= t_4^U.
\end{aligned} \quad (127)$$

According to (125), and Lemma 24, we have $\lim_{n\to\infty} T(A_n) = T(A)$ with respect to the weighted distance D_I.

($\mathrm{i_b}$) If

$$\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_-^U(1) - A_-^U(\alpha)\right]d\alpha - \int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_-^L(1) - A_-^L(\alpha)\right]d\alpha < 0,$$
$$\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_+^U(1) - A_+^U(\alpha)\right]d\alpha - \int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_+^L(1) - A_+^L(\alpha)\right]d\alpha = 0, \quad (128)$$

then there exists N, when $n > N$, A_n satisfies condition (i) or condition (ii) of Theorem 16 and

$$\lim_{n\to\infty}\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot\left(A_n^L\right)_+(1) - \left(A_n^L\right)_+(\alpha)\right]d\alpha = \lim_{n\to\infty}\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot\left(A_n^U\right)_+(1) - \left(A_n^U\right)_+(\alpha)\right]d\alpha. \quad (129)$$

In both two cases, we can prove

$$\begin{aligned}
&\lim_{n\to\infty} t_1^L(n) \\
&= -\lim_{n\to\infty}\left(\left(\int_0^1 f(\alpha)(1-\alpha)\times\left[\alpha\cdot\left(A_n^L\right)_-(1) - \left(A_n^L\right)_-(\alpha)\right]d\alpha + \int_0^1 f(\alpha)(1-\alpha)\times\left[\alpha\cdot\left(A_n^U\right)_-(1) - \left(A_n^U\right)_-(\alpha)\right]d\alpha\right)\times\left(2\int_0^1 f(\alpha)(1-\alpha)^2 d\alpha\right)^{-1}\right)
\end{aligned}$$

$$= -\left(\left(\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_-^L(1)-A_-^L(\alpha)\right]d\alpha + \int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_-^U(1)-A_-^U(\alpha)\right]d\alpha\right) \times\left(2\int_0^1 f(\alpha)(1-\alpha)^2 d\alpha\right)^{-1}\right) = t_1^L,$$

$$\lim_{n\to\infty} t_1^U(n) = \lim_{n\to\infty} t_1^L(n) = t_1^L = t_1^U,$$

$$\lim_{n\to\infty} t_4^L(n) = -\lim_{n\to\infty}\left(\left(\int_0^1 f(\alpha)(1-\alpha) \times\left[\alpha\cdot\left(A_n^L\right)_+(1)-\left(A_n^L\right)_+(\alpha)\right]d\alpha + \int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot\left(A_n^U\right)_+(1) - \left(A_n^U\right)_+(\alpha)\right]d\alpha\right) \times\left(2\int_0^1 f(\alpha)(1-\alpha)^2 d\alpha\right)^{-1}\right)$$

$$= -\lim_{n\to\infty}\frac{\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot\left(A_n^L\right)_+(1)-\left(A_n^L\right)_+(\alpha)\right]d\alpha}{\int_0^1 f(\alpha)(1-\alpha)^2 d\alpha}$$

$$= -\frac{\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_+^L(1)-A_+^L(\alpha)\right]d\alpha}{\int_0^1 f(\alpha)(1-\alpha)^2 d\alpha} = t_4^L,$$

$$\lim_{n\to\infty} t_4^U(n) = \lim_{n\to\infty} t_4^L(n) = t_4^L = t_4^U. \tag{130}$$

Then according to (125), and Lemma 24, we have $\lim_{n\to\infty}T(A_n) = T(A)$ with respect to the weighted distance D_I.

(ii) $A = [A^L, A^U]$ satisfies condition (ii) of Theorem 16. The proof is analogous with the proof of case (i_a).

(iii) $A = [A^L, A^U]$ satisfies condition (iii) of Theorem 16. The proof is analogous with the proof of case (i).

(iv) $A = [A^L, A^U]$ satisfies condition (iv) of Theorem 16; the following situations are possible.

(iv_a) If

$$\begin{aligned}&\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_-^U(1)-A_-^U(\alpha)\right]d\alpha - \int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_-^L(1)-A_-^L(\alpha)\right]d\alpha > 0,\\ &\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_+^U(1)-A_+^U(\alpha)\right]d\alpha - \int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_+^L(1)-A_+^L(\alpha)\right]d\alpha < 0,\end{aligned} \tag{131}$$

the proof is analogous with the proof of (i_a).

(iv_b) If

$$\begin{aligned}&\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_-^U(1)-A_-^U(\alpha)\right]d\alpha - \int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_-^L(1)-A_-^L(\alpha)\right]d\alpha = 0,\\ &\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_+^U(1)-A_+^U(\alpha)\right]d\alpha - \int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_+^L(1)-A_+^L(\alpha)\right]d\alpha = 0,\end{aligned} \tag{132}$$

then there exists N, when $n > N$, A_n satisfies condition (i), (ii), (iii), or (iv) of Theorem 16, and

$$\begin{aligned}&\lim_{n\to\infty}\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot\left(A_n^L\right)_-(1)-\left(A_n^L\right)_-(\alpha)\right]d\alpha = \lim_{n\to\infty}\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot\left(A_n^U\right)_-(1)-\left(A_n^U\right)_-(\alpha)\right]d\alpha,\\ &\lim_{n\to\infty}\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot\left(A_n^L\right)_+(1)-\left(A_n^L\right)_+(\alpha)\right]d\alpha = \lim_{n\to\infty}\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot\left(A_n^U\right)_+(1)-\left(A_n^U\right)_+(\alpha)\right]d\alpha.\end{aligned} \tag{133}$$

In either cases among (i), (ii), (iii), and (iv), it follows from (133) that

$$\lim_{n\to\infty} t_1^L(n) = -\lim_{n\to\infty}\left(\left(\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot\left(A_n^L\right)_-(1)-\left(A_n^L\right)_-(\alpha)\right]d\alpha + \int_0^1 f(\alpha)(1-\alpha) \times\left[\alpha\cdot\left(A_n^U\right)_-(1)-\left(A_n^U\right)_-(\alpha)\right]d\alpha\right) \times\left(2\int_0^1 f(\alpha)(1-\alpha)^2 d\alpha\right)^{-1}\right)$$

$$= -\lim_{n\to\infty}\frac{\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot\left(A_n^L\right)_-(1)-\left(A_n^L\right)_-(\alpha)\right]d\alpha}{\int_0^1 f(\alpha)(1-\alpha)^2 d\alpha}$$

$$= -\frac{\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_-^L(1)-A_-^L(\alpha)\right]d\alpha}{\int_0^1 f(\alpha)(1-\alpha)^2 d\alpha}$$

$$= t_1^L,$$

$$\lim_{n\to\infty} t_1^U(n) = \lim_{n\to\infty} t_1^L(n) = t_1^L = t_1^U,$$

$$\lim_{n\to\infty} t_4^L(n)$$

$$= -\lim_{n\to\infty}\Bigg(\Bigg(\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot\left(A_n^L\right)_+(1)-\left(A_n^L\right)_+(\alpha)\right]d\alpha + \int_0^1 f(\alpha)(1-\alpha) \times\left[\alpha\cdot\left(A_n^U\right)_+(1)-\left(A_n^U\right)_+(\alpha)\right]d\alpha\Bigg) \times\left(2\int_0^1 f(\alpha)(1-\alpha)^2 d\alpha\right)^{-1}\Bigg)$$

$$= -\lim_{n\to\infty}\frac{\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot\left(A_n^L\right)_+(1)-\left(A_n^L\right)_+(\alpha)\right]d\alpha}{\int_0^1 f(\alpha)(1-\alpha)^2 d\alpha}$$

$$= -\frac{\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_+^L(1)-A_+^L(\alpha)\right]d\alpha}{\int_0^1 f(\alpha)(1-\alpha)^2 d\alpha} = t_4^L,$$

$$\lim_{n\to\infty} t_4^U(n) = \lim_{n\to\infty} t_4^L(n) = t_4^L = t_4^U. \tag{134}$$

Then according to (125) and Lemma 24, we have $\lim_{n\to\infty}T(A_n) = T(A)$ with respect to the weighted distance D_I.

(iv_c) If

$$\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_-^U(1)-A_-^U(\alpha)\right]d\alpha - \int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_-^L(1)-A_-^L(\alpha)\right]d\alpha = 0,$$
$$\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_+^U(1)-A_+^U(\alpha)\right]d\alpha - \int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_+^L(1)-A_+^L(\alpha)\right]d\alpha < 0, \tag{135}$$

the proof is analogous with the proof of (i_b).

(iv_d) If

$$\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_-^U(1)-A_-^U(\alpha)\right]d\alpha - \int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_-^L(1)-A_-^L(\alpha)\right]d\alpha > 0,$$
$$\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_+^U(1)-A_+^U(\alpha)\right]d\alpha - \int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_+^L(1)-A_+^L(\alpha)\right]d\alpha = 0, \tag{136}$$

the proof is analogous with the proof of (i_b). □

After we analyze all the cases, the theorem is proven. Next we will give an example to illustrate Theorem 25.

Example 26. Let us consider interval-valued fuzzy number $A = [A^L, A^U]$, $A_\alpha = \{(x,y) \in R^2 : x \in [e^{\alpha^2}, 4-\alpha], y \in [(1/2)e^{\alpha^2}, 5-\alpha]\}$, $\alpha \in [0,1]$. We will determine $T(A)$ with an error less than 10^{-2} with respect to the weighted distance D_I.

Let $A_n = [A_n^L, A_n^U]$ $(n \in N)$ be a sequence of interval-valued fuzzy numbers and

$$\left(A_n^L\right)_-(\alpha) = 1+\alpha^2+\frac{\alpha^4}{2!}+\cdots+\frac{\alpha^{2n}}{n!},$$
$$\left(A_n^U\right)_-(\alpha) = \frac{1}{2}\left(1+\alpha^2+\frac{\alpha^4}{2!}+\cdots+\frac{\alpha^{2n}}{n!}\right),$$
$$\left(A_n^L\right)_+(\alpha) = 4-\alpha, \qquad \left(A_n^U\right)_+(\alpha) = 5-\alpha,$$
$$\alpha \in [0,1]. \tag{137}$$

From the Taylor formula we have

$$e^{\alpha^2} = 1+\alpha^2+\frac{\alpha^4}{2!}+\cdots+\frac{\alpha^{2n}}{n!} + \int_0^{\alpha^2}\frac{\left(\alpha^2-t\right)^n}{n!}e^t dt, \quad \alpha \in [0,1]. \tag{138}$$

Therefore, for any $\alpha \in [0,1]$, we can prove that

$$0 \le A_-^L(\alpha) - \left(A_n^L\right)_-(\alpha) = \int_0^{\alpha^2}\frac{\left(\alpha^2-t\right)^n}{n!}e^t dt \le e\int_0^{\alpha^2}\frac{\left(\alpha^2-t\right)^n}{n!}dt = e\cdot\frac{\alpha^{2n+2}}{(n+1)!} \le \frac{e}{(n+1)!}, \tag{139}$$

$$0 \le A_-^U(\alpha) - \left(A_n^U\right)_-(\alpha) = \frac{1}{2}\int_0^{\alpha^2}\frac{\left(\alpha^2-t\right)^n}{n!}e^t dt \le \frac{e}{2}\int_0^{\alpha^2}\frac{\left(\alpha^2-t\right)^n}{n!}dt = \frac{e}{2}\cdot\frac{\alpha^{2n+2}}{(n+1)!} \le \frac{e}{2(n+1)!}. \tag{140}$$

That is, A and A_n satisfy the hypothesis in Theorem 25.

If $f(\alpha) = \alpha$, then

$$\int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_-^U(1)-A_-^U(\alpha)\right]d\alpha - \int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_-^L(1)-A_-^L(\alpha)\right]d\alpha$$
$$= \int_0^1 f(\alpha)(1-\alpha) \times\left[\alpha\left(A_-^U(1)-A_-^L(1)\right)-\left(A_-^U(\alpha)-A_-^L(\alpha)\right)\right]d\alpha$$

$$= \int_0^1 f(\alpha)(1-\alpha)\left[\alpha\left(\frac{e}{2}-e\right)-\left(\frac{1}{2}e^{\alpha^2}-e^{\alpha^2}\right)\right]d\alpha$$
$$= \frac{e}{2}\int_0^1 f(\alpha)(1-\alpha)\left(e^{\alpha^2-1}-\alpha\right)d\alpha$$
$$> 0, \tag{141}$$

$$\begin{aligned}\int_0^1 & f(\alpha)(1-\alpha)\left[\alpha\cdot A_+^U(1)-A_+^U(\alpha)\right]d\alpha \\ &- \int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot A_+^L(1)-A_+^L(\alpha)\right]d\alpha \\ &= \int_0^1 f(\alpha)(1-\alpha)\left[\alpha\left(A_+^U(1)-A_+^L(1)\right)\right. \\ &\qquad \left.-\left(A_+^U(\alpha)-A_+^L(\alpha)\right)\right]d\alpha \\ &= \int_0^1 f(\alpha)(1-\alpha)(\alpha-1)\,d\alpha \\ &< 0,\end{aligned} \tag{142}$$

such that A satisfies condition (iv) of Theorem 16. Furthermore, let

$$\begin{aligned}G(n) &= \int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot\left(A_n^U\right)_-(1)-\left(A_n^U\right)_-(\alpha)\right]d\alpha \\ &\quad - \int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot\left(A_n^L\right)_-(1)-\left(A_n^L\right)_-(\alpha)\right]d\alpha \\ &= \int_0^1 f(\alpha)(1-\alpha)\left[\alpha\left(\left(A_n^U\right)_-(1)-\left(A_n^L\right)_-(1)\right)\right. \\ &\qquad \left.-\left(\left(A_n^U\right)_-(\alpha)-\left(A_n^L\right)_-(\alpha)\right)\right]d\alpha \\ &= \frac{1}{2}\int_0^1 f(\alpha)(1-\alpha)\left[(1-\alpha)+\left(\alpha^2-\alpha\right)+\frac{1}{2!}\left(\alpha^4-\alpha\right)\right. \\ &\qquad \left.+\cdots+\frac{1}{n!}\left(\alpha^{2n}-\alpha\right)\right]d\alpha.\end{aligned} \tag{143}$$

It is obvious that $G(n)$ is decreasing, and by (141) we have

$$\begin{aligned}\lim_{n\to\infty} G(n) &= \lim_{n\to\infty}\int_0^1 f(\alpha)(1-\alpha) \\ &\qquad \times\left[\alpha\left(\left(A_n^U\right)_-(1)-\left(A_n^L\right)_-(1)\right)\right. \\ &\qquad \left.-\left(\left(A_n^U\right)_-(\alpha)-\left(A_n^L\right)_-(\alpha)\right)\right]d\alpha \\ &= \int_0^1 f(\alpha)(1-\alpha) \\ &\qquad \times\left[\alpha\left(A_-^U(1)-A_-^L(1)\right)-\left(A_-^U(\alpha)-A_-^L(\alpha)\right)\right]d\alpha \\ &> 0.\end{aligned} \tag{144}$$

Therefore, we can conclude that

$$\begin{aligned}\int_0^1 & f(\alpha)(1-\alpha)\left[\alpha\cdot\left(A_n^U\right)_-(1)-\left(A_n^U\right)_-(\alpha)\right]d\alpha \\ &- \int_0^1 f(\alpha)(1-\alpha)\left[\alpha\cdot\left(A_n^L\right)_-(1)-\left(A_n^L\right)_-(\alpha)\right]d\alpha > 0, \\ &\qquad n\in N.\end{aligned} \tag{145}$$

It follows that A_n satisfies condition (iv) of Theorem 16. We denote

$$\begin{aligned}T(A) &= \left[\left(t_1^L,t_2^L,t_3^L,t_4^L\right),\left(t_1^U,t_2^U,t_3^U,t_4^U\right)\right], \\ T(A_n) &= \left[\left(t_1^L(n),t_2^L(n),t_3^L(n),t_4^L(n)\right),\right. \\ &\qquad \left.\left(t_1^U(n),t_2^U(n),t_3^U(n),t_4^U(n)\right)\right] \\ &\qquad (n\in N).\end{aligned} \tag{146}$$

Using Theorem 16 (iv) together with (139), we have

$$\begin{aligned}&\left|t_1^L-t_1^L(n)\right| \\ &= 12\left|\int_0^1 \alpha(1-\alpha)\left[\alpha\left(\left(A_n^L\right)_-(1)-A_-^L(1)\right)\right.\right. \\ &\qquad \left.\left.-\left(\left(A_n^L\right)_-(\alpha)-A_-^L(\alpha)\right)\right]d\alpha\right| \\ &= \left|\left(\left(A_n^L\right)_-(1)-A_-^L(1)\right)\right. \\ &\qquad \left.-12\int_0^1\alpha(1-\alpha)\left(\left(A_n^L\right)_-(\alpha)-A_-^L(\alpha)\right)d\alpha\right| \\ &\le \left|\left(A_n^L\right)_-(1)-A_-^L(1)\right| \\ &\quad + 12\int_0^1\alpha(1-\alpha)\left|\left(A_n^L\right)_-(\alpha)-A_-^L(\alpha)\right|d\alpha \\ &\le \frac{e}{(n+1)!}+\frac{2e}{(n+1)!}=\frac{3e}{(n+1)!}.\end{aligned} \tag{147}$$

Similarly, we can prove that

$$\left|t_1^U-t_1^U(n)\right| \le \frac{e}{2(n+1)!}+\frac{2e}{2(n+1)!}=\frac{3e}{2(n+1)!}. \tag{148}$$

Combing (48), (139), and (140) we obtain

$$\begin{aligned}\left|t_2^L-t_2^L(n)\right| &= \left|A_-^L(1)-\left(A_n^L\right)_-(1)\right| \le \frac{e}{(n+1)!}, \\ \left|t_2^U-t_2^U(n)\right| &= \left|A_-^U(1)-\left(A_n^U\right)_-(1)\right| \le \frac{e}{2(n+1)!}.\end{aligned} \tag{149}$$

Therefore, by making use of (147), (148), and (149), we get

$$\begin{aligned}&D_I\left(T(A),T(A_n)\right) \\ &= \frac{1}{2}\left[\left(\int_0^1\left[\alpha\left(t_1^L+\left(t_2^L-t_1^L\right)\alpha\right)\right.\right.\right. \\ &\qquad \left.\left.-\left(t_1^L(n)+\left(t_2^L(n)-t_1^L(n)\right)\alpha\right)\right]^2 d\alpha\right)^{1/2}\end{aligned}$$

$$+\left(\int_0^1 \alpha\left[\left(t_1^U+\left(t_2^U-t_1^U\right)\alpha\right)\right.\right.$$
$$\left.\left.-\left(t_1^U(n)+\left(t_2^U(n)-t_1^U(n)\right)\alpha\right)\right]^2 d\alpha\right)^{1/2}\Bigg]$$
$$=\frac{1}{2}\left[\sqrt{\int_0^1 \alpha\left(\left(t_1^L-t_1^L(n)\right)(1-\alpha)+\left(t_2^L-t_2^L(n)\right)\alpha\right)^2 d\alpha}\right.$$
$$\left.+\sqrt{\int_0^1 \alpha\left(\left(t_1^U-t_1^U(n)\right)(1-\alpha)+\left(t_2^U-t_2^U(n)\right)\alpha\right)^2 d\alpha}\right]$$
$$=\frac{1}{2}\left[\left(\frac{1}{12}\left(t_1^L-t_1^L(n)\right)^2+\frac{1}{6}\left(t_1^L-t_1^L(n)\right)\left(t_2^L-t_2^L(n)\right)\right.\right.$$
$$\left.+\frac{1}{4}\left(t_2^L-t_2^L(n)\right)^2\right)^{1/2}$$
$$+\left(\frac{1}{12}\left(t_1^U-t_1^U(n)\right)^2+\frac{1}{6}\left(t_1^U-t_1^U(n)\right)\left(t_2^U-t_2^U(n)\right)\right.$$
$$\left.\left.+\frac{1}{4}\left(t_2^U-t_2^U(n)\right)^2\right)^{1/2}\right]$$
$$\leq\frac{1}{2}\left[\sqrt{\frac{1}{6}\left(t_1^L-t_1^L(n)\right)^2+\frac{1}{3}\left(t_2^L-t_2^L(n)\right)^2}\right.$$
$$\left.+\sqrt{\frac{1}{6}\left(t_1^U-t_1^U(n)\right)^2+\frac{1}{3}\left(t_2^U-t_2^U(n)\right)^2}\right]$$
$$<\frac{2e}{(n+1)!}. \tag{150}$$

It is obvious that for $n \geq 5$, we have $D_I(T(A), T(A_n)) < 10^{-2}$. For $n = 5$, case (iv) in Theorem 16 is applicable to compute the nearest interval-valued trapezoidal fuzzy number of interval-valued fuzzy number A_5, and we obtain

$$T(A_5)=\left[\left(\frac{21317}{360360},\frac{163}{60},3,4\right),\left(\frac{21317}{720720},\frac{163}{120},4,5\right)\right]. \tag{151}$$

Then we obtain an interval-valued trapezoidal approximation $T(A_5)$ with an error less than 10^{-2}.

5. Fuzzy Risk Analysis Based on Interval-Valued Fuzzy Numbers

Recently, a lot of methods have been presented for handling fuzzy risk analysis problems. However, these researches did not consider the risk analysis problems based on interval-valued fuzzy numbers. Following, we will use the approximation operator presented in Section 3.2 to deal with fuzzy risk analysis problems.

Assume that there is a component A consisting of n subcomponents $A_1, A_2, \ldots, A_n$ and each subcomponent is evaluated by two evaluating items "probability of failure" and "severity of loss." We want to evaluate the probability of failure and severity of loss of component A. Assume that R_i denotes the probability of failure and ω_i denotes the severity of loss of the subcomponent A_i, respectively, where R_i and ω_i are interval-valued fuzzy numbers and $1 \leq i \leq n$. The algorithm for dealing with fuzzy risk analysis is now presented as follows.

Table 1: A 9-member linguistic term set (Schmucker, 1984) [10].

Linguistic terms	Trapezoidal fuzzy numbers
Absolutely low	(0, 0, 0, 0)
Very low	(0, 0, 0.02, 0.07)
Low	(0.04, 0.1, 0.18, 0.23)
Fairly low	(0.17, 0.22, 0.36, 0.42)
Medium	(0.32, 0.41, 0.58, 0.65)
Fairly high	(0.58, 0.63, 0.80, 0.86)
High	(0.72, 0.78, 0.92, 0.97)
Very high	(0.93, 0.98, 1.0, 1.0)
Absolutely high	(1.0, 1.0, 1.0, 1.0)

Step 1. Use the fuzzy weighted mean method to integrate the evaluating values R_i and ω_i of each subcomponent A_i, where $1 \leq i \leq n$.

Step 2. Transform interval-valued fuzzy numbers R_i and ω_i into interval-valued trapezoidal fuzzy numbers $T(R_i)$ and $T(\omega_i)$ by means of the approximation operator T.

Step 3. Use the interval-valued fuzzy number arithmetic operations defined as [8] to calculate the probability of failure R of component A:

$$R=\left[\sum_{i=1}^{n}\left(T(R_i)\otimes T(\omega_i)\right)\right]\oslash\sum_{i=1}^{n}T(\omega_i)$$
$$=\left[\left(r_1^L,r_2^L,r_3^L,r_4^L\right),\left(r_1^U,r_2^U,r_3^U,r_4^U\right)\right]. \tag{152}$$

Without a doubt R is an interval-valued trapezoidal fuzzy number.

Step 4. Transform the interval-valued trapezoidal fuzzy number R into a standardized interval-valued trapezoidal fuzzy number R^*:

$$R^*=\left[\left(\frac{r_1^L}{k},\frac{r_2^L}{k},\frac{r_3^L}{k},\frac{r_4^L}{k}\right),\left(\frac{r_1^U}{k},\frac{r_2^U}{k},\frac{r_3^U}{k},\frac{r_4^U}{k}\right)\right], \tag{153}$$

where $k = \max\{\lceil |r_j^L| \rceil, \lceil |r_j^U| \rceil, 1\}$, $||$ denotes the absolute value and $\lceil\rceil$ denotes the upper bound and $1 \leq j \leq 4$.

Step 5. Use the similarity measure of interval-valued fuzzy numbers introduced in [26] to calculate the similarity measure of R^* and each linguistic term shown in Table 1. The larger the similarity measure, the higher the probability of failure of component A.

TABLE 2: Evaluating values of the subcomponents A_1, A_2, and A_3.

Subcomponents A_i	Probability of failure R_i	Severity of loss ω_i
A_1	$[(0.3, 0.5, 0.8, 1.0)_2, (0.1, 0.4, 0.9, 1.0)_2]$	$[(0.3, 0.5, 0.6, 1.0)_2, (0.1, 0.4, 0.9, 1.0)_2]$
A_2	$[(0.4, 0.8, 0.8, 1.0)_2, (0.4, 0.4, 1.0, 1.1)_2]$	$[(0.4, 0.5, 0.8, 1.0)_2, (0.4, 0.4, 1.0, 1.1)_2]$
A_3	$[(0.3, 0.7, 0.8, 1.0)_2, (0.1, 0.4, 0.8, 1.0)_2]$	$[(0.3, 0.7, 0.7, 1.0)_2, (0.1, 0.4, 0.8, 1.0)_2]$

TABLE 3: Interval-valued trapezoidal approximation of R_i and ω_i.

Subcomponents A_i	$T(R_i)$	$T(\omega_i)$
A_1	$\left[\left(\frac{13.1}{35}, 0.5, 0.8, \frac{32.4}{35}\right), \left(\frac{7.4}{35}, 0.4, 0.9, \frac{33.7}{35}\right)\right]$	$\left[\left(\frac{13.1}{35}, 0.5, 0.6, \frac{29.8}{35}\right), \left(\frac{7.4}{35}, 0.4, 0.9, \frac{33.7}{35}\right)\right]$
A_2	$\left[\left(\frac{19.2}{35}, 0.8, 0.8, \frac{32.4}{35}\right), \left(\frac{14}{35}, 0.4, 1.0, \frac{37.2}{35}\right)\right]$	$\left[\left(\frac{15.3}{35}, 0.5, 0.8, \frac{32.4}{35}\right), \left(\frac{14}{35}, 0.4, 1.0, \frac{37.2}{35}\right)\right]$
A_3	$\left[\left(\frac{15.7}{35}, 0.7, 0.8, \frac{32.4}{35}\right), \left(\frac{7.4}{35}, 0.4, 0.8, \frac{32.4}{35}\right)\right]$	$\left[\left(\frac{15.7}{35}, 0.7, 0.7, \frac{31.1}{35}\right), \left(\frac{7.4}{35}, 0.4, 0.8, \frac{32.4}{35}\right)\right]$

Example 27. Assume that the component A consists of three subcomponents A_1, A_2, and A_3; we evaluate the probability of failure of the component A. There are some evaluating values represented by interval-valued fuzzy numbers shown in Table 2, where R_i denotes the probability of failure and ω_i denotes the severity of loss of subcomponent A_i, and $1 \leq i \leq 3$.

Step 1. Let $f(\alpha) = \alpha$. According to Corollary 18, we obtain interval-valued trapezoidal fuzzy numbers $T(R_i)$ and $T(\omega_i)$ as shown in Table 3.

Step 2. Calculate the probability of failure R of component A. By the interval-valued fuzzy number arithmetic operations defined as [8], we have

$$\begin{aligned} R &= \left[\sum_{i=1}^{3} \left(T(R_i) \otimes T(\omega_i)\right)\right] \oslash \sum_{i=1}^{3} T(\omega_i) \\ &= \left[T(R_1) \otimes T(\omega_1) \oplus T(R_2) \otimes T(\omega_2) \oplus T(R_3) \otimes T(\omega_3)\right] \\ &\quad \oslash \left[T(\omega_1) \oplus T(\omega_2) \oplus T(\omega_3)\right] \\ &\approx \left[(0.218, 0.54, 0.99, 1.959), (0.085, 0.18, 2.04, 3.541)\right]. \end{aligned} \tag{154}$$

Step 3. Transform the interval-valued trapezoidal fuzzy number R into a standardized interval-valued trapezoidal fuzzy number R^*;

$$\begin{aligned} R^* = [&(0.0545, 0.1350, 0.2475, 0.4898), \\ &(0.0213, 0.045, 0.5100, 0.8853)]. \end{aligned} \tag{155}$$

Step 4. Calculate the similarity measure between the interval-valued trapezoidal fuzzy number R^* and the linguistic terms shown in Table 1, we have

$$\begin{gathered} S_F(R^*, \text{absolutely} - \text{low}) \approx 0.2797, \\ S_F(R^*, \text{very} - \text{low}) \approx 0.3131, \\ S_F(R^*, \text{low}) \approx 0.4174, \\ S_F(R^*, \text{fairly} - \text{low}) \approx 0.4747, \\ S_F(R^*, \text{medium}) \approx 0.4748, \\ S_F(R^*, \text{fairly} - \text{high}) \approx 0.3445, \\ S_F(R^*, \text{high}) \approx 0.2545, \\ S_F(R^*, \text{very} - \text{high}) \approx 0.1364, \\ S_F(R^*, \text{absolutely} - \text{high}) \approx 0.1166. \end{gathered} \tag{156}$$

It is obvious that $S_F(R^*, \text{medium}) \approx 0.4748$ is the largest value; therefore, the interval-valued trapezoidal fuzzy number R^* is translated into the linguistic term "medium." That is, the probability of failure of the component A is medium.

6. Conclusion

In this paper, we use the α-level set of interval-valued fuzzy numbers to investigate interval-valued trapezoidal approximation of interval-valued fuzzy numbers and discuss some properties of the approximation operator including translation invariance, scale invariance, identity, nearness criterion, and ranking invariance. However, Example 23 proves that the approximation operator suggested in Section 3.2 is not continuous. Nevertheless, Theorem 25 shows that the interval-valued trapezoidal approximation has a relative good behavior. As an application, we use interval-valued trapezoidal approximation to handle fuzzy risk analysis problems, which provides us with a useful way to deal with fuzzy risk analysis problems.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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