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Research Article

Generalized *s*-Convex Functions on Fractal Sets

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We introduce two kinds of generalized s-convex functions on real linear fractal sets $\mathbb{R}^{\alpha}$ $(0 < \alpha < 1)$. And similar to the class situation, we also study the properties of these two kinds of generalized s-convex functions and discuss the relationship between them. Furthermore, some applications are given.

1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$. For any $u, v \in I$ and $t \in [0,1]$, if the following inequality,

$$f(tu + (1-t)v) \leq tf(u) + (1-t)f(v), \qquad (1)$$

holds, then f is called a convex function on I.

The convexity of functions plays a significant role in many fields, such as in biological system, economy, and optimization [1, 2]. In [3], Hudzik and Maligranda generalized the definition of convex function and considered, among others, two kinds of functions which are s-convex.

Let $0 < s \leq 1$ and $\mathbb{R}_+ = [0, \infty)$, and then the two kinds of s-convex functions are defined, respectively, in the following way.

Definition 1. A function, $f : \mathbb{R}_+ \to \mathbb{R}$, is said to be s-convex in the first sense if

$$f(\alpha u + \beta v) \leq \alpha^s f(u) + \beta^s f(v), \qquad (2)$$

for all $u, v \in \mathbb{R}_+$ and all $\alpha, \beta \geq 0$ with $\alpha^s + \beta^s = 1$. One denotes this by $f \in K_s^1$.

Definition 2. A function, $f : \mathbb{R}_+ \to \mathbb{R}$, is said to be s-convex in the second sense if

$$f(\alpha u + \beta v) \leq \alpha^s f(u) + \beta^s f(v), \qquad (3)$$

for all $u, v \in \mathbb{R}_+$ and all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. One denotes this by $f \in K_s^2$.

It is obvious that the s-convexity means just the convexity when $s = 1$, no matter whether it is in the first sense or in the second sense. In [3], some properties of s-convex functions in both senses are considered and various examples and counterexamples are given. There are many research results related to the s-convex functions; see [4–6] and so on.

In recent years, the fractal has received significantly remarkable attention from scientists and engineers. In the sense of Mandelbrot, a fractal set is the one whose Hausdorff dimension strictly exceeds the topological dimension [7–12].

The calculus on fractal set can lead to better comprehension for the various real world models from science and engineering [8]. Researchers have constructed many kinds of fractional calculus on fractal sets by using different approaches. Particularly, in [13], Yang stated the analysis of local fractional functions on fractal space systematically, which includes local fractional calculus. In [14], the authors introduced the generalized convex function on fractal sets and established the generalized Jensen inequality and generalized Hermite-Hadamard inequality related to generalized convex function. And, in [15], Wei et al. established a local fractional integral inequality on fractal space analogous to Anderson's inequality for generalized convex functions. The generalized convex function on fractal sets $\mathbb{R}^{\alpha}$ $(0 < \alpha < 1)$ can be stated as follows.

Let $f : I \subset \mathbb{R} \to \mathbb{R}^{\alpha}$. For any $u, v \in I$ and $t \in [0,1]$, if the following inequality,

$$f(tu + (1-t)v) \leq t^{\alpha} f(u) + (1-t)^{\alpha} f(v), \qquad (4)$$

holds, then f is called a generalized convex on I.

Inspired by these investigations, we will introduce the generalized s-convex function in the first or second sense on fractal sets and study the properties of generalized s-convex functions.

The paper is organized as follows. In Section 2, we state the operations with real line number fractal sets and give the definitions of the local fractional calculus. In Section 3, we introduce the definitions of two kinds of generalized s-convex functions and study the properties of these functions. In Section 4, we give some applications for the two kinds of generalized s-convex functions on fractal sets.

2. Preliminaries

Let us recall the operations with real line number on fractal space and use Gao-Yang-Kang's idea to describe the definitions of the local fractional derivative and local fractional integral [13, 16–19].

If a^α, b^α, and c^α belong to the set $\mathbb{R}^\alpha$ $(0 < \alpha \leq 1)$ of real line numbers, then one has the following:

(1) $a^\alpha + b^\alpha$ and $a^\alpha b^\alpha$ belong to the set $\mathbb{R}^\alpha$;

(2) $a^\alpha + b^\alpha = b^\alpha + a^\alpha = (a+b)^\alpha = (b+a)^\alpha$;

(3) $a^\alpha + (b^\alpha + c^\alpha) = (a^\alpha + b^\alpha) + c^\alpha$;

(4) $a^\alpha b^\alpha = b^\alpha a^\alpha = (ab)^\alpha = (ba)^\alpha$;

(5) $a^\alpha (b^\alpha c^\alpha) = (a^\alpha b^\alpha) c^\alpha$;

(6) $a^\alpha (b^\alpha + c^\alpha) = a^\alpha b^\alpha + a^\alpha c^\alpha$;

(7) $a^\alpha + 0^\alpha = 0^\alpha + a^\alpha = a^\alpha$ and $a^\alpha \cdot 1^\alpha = 1^\alpha \cdot a^\alpha = a^\alpha$.

Let us now state some definitions about the local fractional calculus on $\mathbb{R}^\alpha$.

Definition 3 (see [13]). A nondifferentiable function $f : \mathbb{R} \to \mathbb{R}^\alpha$, $x \to f(x)$ is called to be local fractional continuous at x_0, if, for any $\varepsilon > 0$, there exists $\delta > 0$, such that

$$\left| f(x) - f(x_0) \right| < \varepsilon^\alpha \tag{5}$$

holds for $|x - x_0| < \delta$, where $\varepsilon, \delta \in \mathbb{R}$. If f is local fractional continuous on the interval (a, b), one denotes $f \in C_\alpha(a, b)$.

Definition 4 (see [13]). The local fractional derivative of function f of order α at $x = x_0$ is defined by

$$f^{(\alpha)}(x_0) = \left.\frac{d^\alpha f(x)}{dx^\alpha}\right|_{x=x_0} = \lim_{x \to x_0} \frac{\Delta^\alpha (f(x) - f(x_0))}{(x - x_0)^\alpha}, \tag{6}$$

where $\Delta^\alpha (f(x) - f(x_0)) = \Gamma(1 + a)(f(x) - f(x_0))$ and the Gamma function is defined by $\Gamma(t) = \int_0^{+\infty} x^{t-1} e^{-x} dx$.

If there exists $f^{((k+1)\alpha)}(x) = \overbrace{D_x^\alpha \cdots D_x^\alpha}^{k+1 \text{ times}} f(x)$ for any $x \in I \subseteq \mathbb{R}$, then one denoted $f \in D_{(k+1)\alpha}(I)$, where $k = 0, 1, 2, \ldots$.

Definition 5 (see [13]). Let $f \in C_\alpha[a, b]$. Then the local fractional integral of the function f of order α is defined by

$$\begin{aligned} {}_aI_b^{(\alpha)} f &= \frac{1}{\Gamma(1+a)} \int_a^b f(t)(dt)^\alpha \\ &= \frac{1}{\Gamma(1+a)} \lim_{\Delta t \to 0} \sum_{j=0}^{N} f(t_j)(\Delta t_j)^\alpha, \end{aligned} \tag{7}$$

with $\Delta t_j = t_{j+1} - t_j$, $\Delta t = \max\{\Delta t_1, \Delta t_2, \Delta t_j, \ldots, \Delta t_N - 1\}$, and $[t_j, t_j + 1]$, $j = 0, \ldots, N-1$, where $t_0 = a < t_1 < \cdots < t_i < \cdots < t_N = b$ is a partition of the interval $[a, b]$.

Lemma 6 (see [13]). *Suppose that $f, g \in C_\alpha[a, b]$ and $f, g \in D_\alpha(a, b)$. If $\lim_{x \to x_0} f(x) = 0^\alpha$, $\lim_{x \to x_0} g(x) = 0^\alpha$ and $g^{(\alpha)}(x) \neq 0^\alpha$. Suppose that $\lim_{x \to x_0} (f^{(\alpha)}(x) / g^{(\alpha)}(x)) = A^\alpha$, and then*

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = A^\alpha. \tag{8}$$

Lemma 7 (see [13]). *Suppose that $f(x) \in C_\alpha[a, b]$; then*

$$\frac{d^\alpha \left({}_aI_x^{(\alpha)} f \right)}{dx^\alpha} = f(x), \quad a < x < b. \tag{9}$$

3. Generalized s-Convexity Functions

The convexity of functions plays a significant role in many fields. In this section, let us introduce two kinds of generalized s-convex functions on fractal sets. And then, we study the properties of the two kinds of generalized s-convex functions.

Definition 8. Let $\mathbb{R}_+ = [0, +\infty)$. A function $f : \mathbb{R}_+ \to \mathbb{R}^\alpha$ is said to be generalized s-convex $(0 < s < 1)$ in the first sense, if

$$f(\lambda_1 u + \lambda_2 v) \leq \lambda_1^{s\alpha} f(u) + \lambda_2^{s\alpha} f(v), \tag{10}$$

for all $u, v \in \mathbb{R}_+$ and all $\lambda_1, \lambda_2 \geq 0$ with $\lambda_1^s + \lambda_2^s = 1$. One denotes this by $f \in GK_s^1$.

Definition 9. A function $f : \mathbb{R}_+ \to \mathbb{R}^\alpha$ is said to be generalized s-convex $(0 < s < 1)$ in the second sense, if

$$f(\lambda_1 u + \lambda_2 v) \leq \lambda_1^{s\alpha} f(u) + \lambda_2^{s\alpha} f(v), \tag{11}$$

for all $u, v \in \mathbb{R}_+$ and all $\lambda_1, \lambda_2 \geq 0$ with $\lambda_1 + \lambda_2 = 1$. One denotes this by $f \in GK_s^2$.

Note that, when $s = 1$, the generalized s-convex functions in both senses are the generalized convex functions; see [14].

Theorem 10. *Let $0 < s < 1$.*

(a) *If $f \in GK_s^1$, then f is nondecreasing on $(0, +\infty)$ and*

$$f(0^+) = \lim_{u \to 0^+} f(u) \leq f(0). \tag{12}$$

(b) *If $f \in GK_s^2$, then f is nonnegative on $[0, +\infty)$.*

Proof. (a) Since $f \in GK_s^1$, we have, for $u > 0$ and $\lambda \in [0,1]$,

$$f\left[\left(\lambda^{1/s} + (1-\lambda)^{1/s}\right)u\right] \leq \lambda^{\alpha} f(u) + (1-\lambda)^{\alpha} f(u) = f(u). \tag{13}$$

The function

$$h(\lambda) = \lambda^{1/s} + (1-\lambda)^{1/s} \tag{14}$$

is continuous on $[0,1]$, decreasing on $[0,1/2]$, and increasing on $[1/2,1]$ and $h([0,1]) = [h(1/2), h(1)] = [2^{1-1/s}, 1]$. This yields that

$$f(tu) \leq f(u), \tag{15}$$

for $u > 0$ and $t \in [2^{1-1/s}, 1]$. If $t \in [2^{2(1-1/s)}, 1]$, then $t^{1/2} \in [2^{1-1/s}, 1]$. Therefore, by the fact that (15) holds, we get

$$f(tu) = f\left(t^{1/2}\left(t^{1/2}u\right)\right) \leq f\left(t^{1/2}u\right) \leq f(u), \tag{16}$$

for all $u > 0$. So we can obtain that

$$f(tu) \leq f(u), \quad \forall u > 0,\ t \in (0,1]. \tag{17}$$

So, taking $0 < u < v$, we get

$$f(u) = f\left(\left(\frac{u}{v}\right)v\right) \leq f(v), \tag{18}$$

which means that f is nondecreasing on $(0, +\infty)$.

As for the second part, for $u > 0$ and $\lambda_1, \lambda_2 \geq 0$ with $\lambda_1^s + \lambda_2^s = 1$, we have

$$f(\lambda_1 u) = f(\lambda_1 u + \lambda_2 0) \leq \lambda_1^{s\alpha} f(u) + \lambda_2^{s\alpha} f(0). \tag{19}$$

And taking $u \to 0^+$, we get

$$\lim_{u\to 0^+} f(u) = \lim_{u\to 0^+} f(\lambda_1 u) \leq \lambda_1^{s\alpha} \lim_{u\to 0^+} f(u) + \lambda_2^{s\alpha} f(0). \tag{20}$$

So,

$$\lim_{u\to 0^+} f(u) \leq f(0). \tag{21}$$

(b) For $f \in GK_s^2$, we can get that, for $u \in \mathbb{R}_+$,

$$f(u) = f\left(\frac{u}{2} + \frac{u}{2}\right) \leq \frac{f(u)}{2^{s\alpha}} + \frac{f(u)}{2^{s\alpha}} = 2^{(1-s)\alpha} f(u). \tag{22}$$

So, $(2^{1-s} - 1)^{\alpha} f(u) \geq 0^{\alpha}$. This means that $f(u) \geq 0^{\alpha}$, since $0 < s < 1$. □

Remark 11. The above results do not hold, in general, in the case of generalized convex functions, that is, when $s = 1$, because a generalized convex function, $f : \mathbb{R}_+ \to \mathbb{R}^{\alpha}$, need not be either nondecreasing or nonnegative.

Remark 12. If $0 < s < 1$, then the function $f \in GK_s^1$ is nondecreasing on $(0, +\infty)$ but not necessarily on $[0, +\infty)$.

Function $F : \mathbb{R}^2 \to \mathbb{R}^{\alpha}$ is called to be generalized convex in each variable, if

$$F(\lambda_1 u + \lambda_2 v, \lambda_1 r + \lambda_2 t) \leq \lambda_1^{\alpha} F(u, r) + \lambda_2^{\alpha} F(v, t). \tag{23}$$

For all (u, r), $(v, t) \in \mathbb{R}^2$ and $\lambda_1, \lambda_2 \in [0,1]$ with $\lambda_1 + \lambda_2 = 1$.

Theorem 13. *Let $0 < s < 1$. If $f, g : \mathbb{R} \to \mathbb{R}$ and $f, g \in K_s^1$ and if $F : \mathbb{R}^2 \to \mathbb{R}^{\alpha}$ is a generalized convex and nondecreasing function in each variable, then the function $h : \mathbb{R}_+ \to \mathbb{R}^{\alpha}$ defined by*

$$h(u) = F(f(u), g(u)) \tag{24}$$

is in GK_s^1. In particular, if $f, g \in K_s^1$, then $f^{\alpha} + g^{\alpha}$, $\max\{f^{\alpha}, g^{\alpha}\} \in GK_s^1$.

Proof. If $u, v \in \mathbb{R}_+$, then for all $\lambda_1, \lambda_2 \geq 0$ with $\lambda_1^s + \lambda_2^s = 1$,

$$\begin{aligned} h(\lambda_1 u + \lambda_2 v) &= F(f(\lambda_1 u + \lambda_2 v), g(\lambda_1 u + \lambda_2 v)) \\ &\leq F(\lambda_1^s f(u) + \lambda_2^s f(v), \lambda_1^s g(u) + \lambda_2^s g(v)) \\ &\leq \lambda_1^{s\alpha} F(f(u), g(u)) + \lambda_2^{s\alpha} F(f(v), g(v)) \\ &= \lambda_1^{s\alpha} h(u) + \lambda_2^{s\alpha} h(v). \end{aligned} \tag{25}$$

Thus, $h \in GK_s^1$.

Moreover, since $F(u, v) = u^{\alpha} + v^{\alpha}$, $F(u, v) = \max\{u^{\alpha}, v^{\alpha}\}$ are nondecreasing generalized convex functions on R^2, so they yield particular cases of our theorem. □

Let us pay attention to the situation when the condition $\lambda_1^s + \lambda_2^s = 1$ $(\lambda_1 + \lambda_2 = 1)$ in the definition of $GK_s^1 (GK_s^2)$ can be equivalently replaced by the condition $\lambda_1^s + \lambda_2^s \leq 1$ $(\lambda_1 + \lambda_2 \leq 1)$.

Theorem 14. *(a) Let $f \in GK_s^1$. Then inequality* (10) *holds for all $u, v \in R_+$ and all $\lambda_1, \lambda_2 \geq 0$ with $\lambda_1^s + \lambda_2^s < 1$ if and only if $f(0) \leq 0^{\alpha}$.*

(b) Let $f \in GK_s^2$. Then inequality (11) *holds for all $u, v \in R_+$ and all $\lambda_1, \lambda_2 \geq 0$ with $\lambda_1 + \lambda_2 < 1$ if and only if $f(0) = 0^{\alpha}$.*

Proof. (a) Necessity is obvious by taking $u = v = 0$ and $\lambda_1 = \lambda_2 = 0$. Let us show the sufficiency.

Assume that $u, v \in \mathbb{R}_+$ and $\lambda_1, \lambda_2 \geq 0$ with $0 < \lambda_3 = \lambda_1^s + \lambda_2^s < 1$. Put $a = \lambda_1 \lambda_3^{-1/s}$ and $b = \lambda_2 \lambda_3^{-1/s}$. Then $a^s + b^s = \lambda_1^s/\lambda_3 + \lambda_2^s/\lambda_3 = 1$ and

$$\begin{aligned} f(\lambda_1 u + \lambda_2 v) &= f\left(a\lambda_3^{1/s} u + b\lambda_3^{1/s} v\right) \\ &\leq a^{s\alpha} f\left(\lambda_3^{1/s} u\right) + b^{s\alpha} f\left(\lambda_3^{1/s} v\right) \end{aligned}$$

$$
\begin{aligned}
&= a^{s\alpha} f\left[\lambda_3^{1/s} u + (1-\lambda_3)^{1/s} 0\right] \\
&\quad + b^{s\alpha} f\left[\lambda_3^{1/s} v + (1-\lambda_3)^{1/s} 0\right] \\
&\le a^{s\alpha}\left[\lambda_3^{\alpha} f(u) + (1-\lambda_3)^{\alpha} f(0)\right] \\
&\quad + b^{s\alpha}\left[\lambda_3^{\alpha} f(v) + (1-\lambda_3)^{\alpha} f(0)\right] \\
&= a^{s\alpha}\lambda_3^{\alpha} f(u) + b^{s\alpha}\lambda_3^{\alpha} f(v) + (1-\lambda_3)^{\alpha} f(0) \\
&\le \lambda_1^{s\alpha} f(u) + \lambda_2^{s\alpha} f(v).
\end{aligned} \tag{26}
$$

(b) Necessity. Taking $u = v = \lambda_1 = \lambda_2 = 0$, we obtain $f(0) \le 0^{\alpha}$. And using Theorem 10(b), we get $f(0) \ge 0^{\alpha}$. Therefore $f(0) = 0^{\alpha}$.

Sufficiency. Let $u, v \in \mathbb{R}_+$ and $\lambda_1, \lambda_2 \ge 0$ with $0 < \lambda_3 = \lambda_1 + \lambda_2 < 1$. Put $a = \lambda_1/\lambda_3$ and $b = \lambda_2/\lambda_3$, and then $a + b = 1$. So,

$$
\begin{aligned}
& f(\lambda_1 u + \lambda_2 v) \\
&\quad = f(a\lambda_3 u + b\lambda_3 v) \\
&\quad \le a^{s\alpha} f(\lambda_3 u) + b^{s\alpha} f(\lambda_3 v) \\
&\quad = a^{s\alpha} f[\lambda_3 u + (1-\lambda_3) 0] \\
&\qquad + b^{s\alpha} f[\lambda_3 v + (1-\lambda_3) 0] \\
&\quad \le a^{s\alpha}\left[\lambda_3^{s\alpha} f(u) + (1-\lambda_3)^{s\alpha} f(0)\right] \\
&\qquad + b^{s\alpha}\left[\lambda_3^{s\alpha} f(v) + (1-\lambda_3)^{s\alpha} f(0)\right] \\
&\quad = a^{s\alpha}\lambda_3^{s\alpha} f(u) + b^{s\alpha}\lambda_3^{s\alpha} f(v) \\
&\qquad + (1-\lambda_3)^{s\alpha} f(0) \\
&\quad = \lambda_1^{s\alpha} f(u) + \lambda_2^{s\alpha} f(v).
\end{aligned} \tag{27}
$$

□

Theorem 15. *(a) Let $0 < s \le 1$. If $f \in GK_s^2$ and $f(0) = 0^{\alpha}$, then $f \in GK_s^1$.*

(b) Let $0 < s_1 \le s_2 \le 1$. If $f \in GK_{s_2}^2$ and $f(0) = 0^{\alpha}$, then $f \in GK_{s_1}^2$.

(c) Let $0 < s_1 \le s_2 \le 1$. If $f \in GK_{s_2}^1$ and $f(0) \le 0^{\alpha}$, then $f \in GK_{s_1}^1$.

Proof. (a) Assume that $f \in GK_s^2$ and $f(0) = 0^{\alpha}$. Let $\lambda_1, \lambda_2 \ge 0$ with $\lambda_1^s + \lambda_2^s = 1$, and we have $\lambda_1 + \lambda_2 \le \lambda_1^s + \lambda_2^s = 1$. From Theorem 14(b), we can get

$$
f(\lambda_1 u + \lambda_2 v) \le \lambda_1^{s\alpha} f(u) + \lambda_2^{s\alpha} f(v), \tag{28}
$$

for $u, v \in \mathbb{R}_+$, and then $f \in GK_s^1$.

(b) Assume that $f \in GK_{s_2}^2$, $u, v \in \mathbb{R}_+$, and $\lambda_1, \lambda_2 \ge 0$ with $\lambda_1 + \lambda_2 = 1$. Then we have

$$
\begin{aligned}
f(\lambda_1 u + \lambda_2 v) &\le \lambda_1^{s_2\alpha} f(u) + \lambda_2^{s_2\alpha} f(v) \\
&\le \lambda_1^{s_1\alpha} f(u) + \lambda_2^{s_1\alpha} f(v).
\end{aligned} \tag{29}
$$

So $f \in GK_{s_1}^2$.

(c) Assume that $f \in GK_{s_2}^1$, $u, v \in R_+$, and $\lambda_1, \lambda_2 \ge 0$ with $\lambda_1^{s_1} + \lambda_2^{s_1} = 1$. Then $\lambda_1^{s_2} + \lambda_2^{s_2} \le \lambda_1^{s_1} + \lambda_2^{s_1} = 1$. Thus, according to Theorem 14(a), we have

$$
\begin{aligned}
f(\lambda_1 u + \lambda_2 v) &\le \lambda_1^{s_2\alpha} f(u) + \lambda_2^{s_2\alpha} f(v) \\
&\le \lambda_1^{s_1\alpha} f(u) + \lambda_2^{s_1\alpha} f(v).
\end{aligned} \tag{30}
$$

So, $f \in GK_{s_1}^1$. □

Theorem 16. *Let $0 < s < 1$ and $p : \mathbb{R}_+ \to \mathbb{R}_+^{\alpha}$ be a nondecreasing function. Then the function f defined for $u \in \mathbb{R}_+$ by*

$$
f(u) = u^{(s/(1-s))\alpha} p(u) \tag{31}
$$

belongs to GK_s^1.

Proof. Let $v \ge u \ge 0$ and $\lambda_1, \lambda_2 \ge 0$ with $\lambda_1^s + \lambda_2^s = 1$. We consider two cases.

Case I. It is easy to see that f is a nondecreasing function. Let $\lambda_1 u + \lambda_2 v \le u$, and then

$$
\begin{aligned}
f(\lambda_1 u + \lambda_2 v) &\le f(u) = \left(\lambda_1^{s\alpha} + \lambda_2^{s\alpha}\right) f(u) \\
&\le \lambda_1^{s\alpha} f(u) + \lambda_2^{s\alpha} f(v).
\end{aligned} \tag{32}
$$

Case II. Let $\lambda_1 u + \lambda_2 v > u$, and then $\lambda_2 v > (1-\lambda_1)u$. So, $\lambda_2 > 0$ and $\lambda_1 \le \lambda_1^s$. Thus,

$$
\lambda_1 - \lambda_1^{s+1} \le \lambda_1^s - \lambda_1^{s+1}; \tag{33}
$$

that is,

$$
\begin{aligned}
\frac{\lambda_1}{(1-\lambda_1)} &\le \frac{\lambda_1^s}{(1-\lambda_1^s)} = \frac{(1-\lambda_2^s)}{\lambda_2^s}, \\
\frac{\lambda_1\lambda_2}{(1-\lambda_1)} &\le \lambda_2^{1-s} - \lambda_2.
\end{aligned} \tag{34}
$$

Thus, we can get that

$$
\begin{aligned}
\lambda_1 u + \lambda_2 v &\le (\lambda_1 + \lambda_2) v \le \left(\lambda_1^s + \lambda_2^s\right) v = v, \\
\lambda_1 u + \lambda_2 v &\le \frac{\lambda_1\lambda_2 v}{(1-\lambda_1)} + \lambda_2 v \\
&\le \left(\lambda_2^{1-s} - \lambda_2\right) v + \lambda_2 v = \lambda_2^{1-s} v.
\end{aligned} \tag{35}
$$

Then,

$$
(\lambda_1 u + \lambda_2 v)^{s/(1-s)} \le \lambda_2^s v^{s/(1-s)}. \tag{36}
$$

We obtain

$$\begin{aligned} f(\lambda_1 u + \lambda_2 v) &= (\lambda_1 u + \lambda_2 v)^{(s/(1-s))\alpha} p(\lambda_1 u + \lambda_2 v) \\ &\le \lambda_2^{s\alpha} v^{(s/(1-s))\alpha} p(v) \\ &= \lambda_2^{s\alpha} f(v) \le \lambda_1^{s\alpha} f(u) + \lambda_2^{s\alpha} f(v). \end{aligned} \tag{37}$$

□

Theorem 17. *(a) Let $f \in GK_{s_1}^1$ and $g \in K_{s_2}^1$, where $0 < s_1, s_2 \le 1$. If f is a nondecreasing function and g is a nonnegative function such that $f(0) \le 0^\alpha$ and $g(0) = 0$, then the composition $f \circ g$ of f with g belongs to GK_s^1, where $s = s_1 s_2$.*

(b) Let $f \in GK_{s_1}^1$ and $g \in GK_{s_2}^1$, where $0 < s_1, s_2 \le 1$. Assume that $0 < s_1, s_2 < 1$. If f and g are nonnegative functions such that either $f(0) = 0^\alpha$ and $g(0^+) = g(0)$, or $g(0) = 0^\alpha$ and $f(0^+) = f(0)$, then the product fg of f and g belongs to GK_s^1, where $s = \min\{s_1, s_2\}$.

Proof. (a) Let $u, v \in \mathbb{R}_+$, $\lambda_1, \lambda_2 \ge 0$ with $\lambda_1^s + \lambda_2^s = 1$, where $s = s_1 s_2$. Since $\lambda_1^{s_i} + \lambda_2^{s_i} \le \lambda_1^{s_1 s_2} + \lambda_2^{s_1 s_2} = 1$ for $i = 1, 2$, then according to Theorem 3(a) in [3] and Theorem 14(a) in the paper, we have

$$\begin{aligned} f \circ g&(\lambda_1 u + \lambda_2 v) \\ &= f(g(\lambda_1 u + \lambda_2 v)) \\ &\le f(\lambda_1^{s_2} g(u) + \lambda_2^{s_2} g(v)) \\ &\le \lambda_1^{s_1 s_2 \alpha} f(g(u)) + \lambda_2^{s_1 s_2 \alpha} f(g(v)) \\ &= \lambda_1^{s\alpha} f \circ g(u) + \lambda_2^{s\alpha} f \circ g(v) \end{aligned} \tag{38}$$

which means that $f \circ g \in GK_s^1$.

(b) According to Theorem 10(a), f, g are nondecreasing on $(0, +\infty)$.

So,

$$(f(u) - f(v))(g(v) - g(u)) \le 0^\alpha, \tag{39}$$

or, equivalently,

$$f(u) g(v) + f(v) g(u) \le f(u) g(u) + f(v) g(v), \tag{40}$$

for all $v > u > 0$.

If $v > u = 0$, then the inequality is still true because f, g are nonnegative and either $f(0) = 0^\alpha$ and $g(0^+) = g(0)$ or $g(0) = 0^\alpha$ and $f(0^+) = f(0)$.

Now let $u, v \in \mathbb{R}_+$ and $\lambda_1, \lambda_2 \ge 0$ with $\lambda_1^s + \lambda_2^s = 1$, where $s = \min\{s_1, s_2\}$. Then $\lambda_1^{s_i} + \lambda_2^{s_i} \le \lambda_1^s + \lambda_2^s = 1$ for $i = 1, 2$. And by Theorem 14(a), we have

$$\begin{aligned} f&(\lambda_1 u + \lambda_2 v) g(\lambda_1 u + \lambda_2 v) \\ &\le (\lambda_1^{s_1\alpha} f(u) + \lambda_2^{s_1\alpha} f(v)) \\ &\quad \times (\lambda_1^{s_2\alpha} g(u) + \lambda_2^{s_2\alpha} g(v)) \\ &= \lambda_1^{(s_1+s_2)\alpha} f(u) g(u) + \lambda_1^{s_1\alpha} \lambda_2^{s_2\alpha} f(u) g(v) \\ &\quad + \lambda_1^{s_2\alpha} \lambda_2^{s_1\alpha} f(v) g(u) + \lambda_2^{(s_1+s_2)\alpha} f(v) g(v) \\ &\le \lambda_1^{2s\alpha} f(u) g(u) \\ &\quad + \lambda_1^{s\alpha} \lambda_2^{s\alpha} (f(u) g(v) + f(v) g(u)) \\ &\quad + \lambda_2^{2s\alpha} f(v) g(v) \\ &\le \lambda_1^{2s\alpha} f(u) g(u) \\ &\quad + \lambda_1^{s\alpha} \lambda_2^{s\alpha} (f(u) g(u) + f(v) g(v)) \\ &\quad + \lambda_2^{2s\alpha} f(v) g(v) \\ &= \lambda_1^{s\alpha} f(u) g(u) + \lambda_2^{s\alpha} f(v) g(v), \end{aligned} \tag{41}$$

which means that $fg \in GK_s^1$. □

Remark 18. From the above proof, we can get that if f is a nondecreasing function in GK_s^2 and g is a nonnegative convex function on $[0, +\infty)$, then the composition $f \circ g$ of f with g belongs to GK_s^2.

Remark 19. Generalized convex functions on $[0, +\infty)$ need not be monotonic. However, if f and g are nonnegative, generalized convex and either both are nondecreasing or both are nonincreasing on $[0, +\infty)$, then the product fg is also a generalized convex function.

Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function. Then f is said to be a φ-function if $f(0) = 0$ and f is nondecreasing on $\mathbb{R}_+$. Similarly, we can define the φ-type function on fractal sets as follows. A function $f : \mathbb{R}_+ \to \mathbb{R}_+^\alpha$ is said to be a φ-type function if $f(0) = 0^\alpha$ and $f \in C_\alpha(\mathbb{R}_+)$ is nondecreasing.

Corollary 20. *If Φ is a generalized convex φ-type function and $g \in K_s^1$ is a φ-function, then the composition $\Phi \circ g$ belongs to GK_s^1. In particular, the φ-type function $h(u) = \Phi(u^s)$ belongs to GK_s^1.*

Corollary 21. *If Φ is a convex φ-function and $f \in GK_s^2$ is a φ-type function, then the composition $f \circ \Phi$ belongs to GK_s^2. In particular, the φ-type function $h(u) = [\Phi(u)]^{s\alpha}$ belongs to GK_s^2.*

Theorem 22. *If $0 < s < 1$ and $f \in GK_s^1$ is a φ-type function, then there exists a generalized convex φ-type function Φ such that*

$$f(2^{-1/s} u) \le \Phi(u^s) \le f(u), \tag{42}$$

for all $u \ge 0$.

Proof. By the generalized s-convexity of the function f and by $f(0) = 0^\alpha$, we obtain $f(\lambda_1 u) \le \lambda_1^{s\alpha} f(u)$ for all $u \ge 0$ and all $\lambda_1 \in [0, 1]$.

Assume now that $v > u > 0$. Then

$$f\left(u^{1/s}\right) \leq f\left(\left(\frac{u}{v}\right)^{1/s} v^{1/s}\right) \leq \left(\frac{u^{\alpha}}{v^{\alpha}}\right) f\left(v^{1/s}\right); \tag{43}$$

that is,

$$\frac{f\left(u^{1/s}\right)}{u^{\alpha}} \leq \frac{f\left(v^{1/s}\right)}{v^{\alpha}}. \tag{44}$$

Inequality (44) means that the function $f(u^{1/s})/u^{\alpha}$ is a nondecreasing function on $(0, +\infty)$. And, since f is a φ-type function, thus f is local fractional continuous $[0, +\infty)$.

Define

$$\Phi(u) = \begin{cases} 0^{\alpha}, & u = 0, \\ \Gamma(1+\alpha)\,{}_0I_u^{(\alpha)}\left(\dfrac{f\left(t^{1/s}\right)}{t^{\alpha}}\right), & u > 0. \end{cases} \tag{45}$$

From Lemmas 6 and 7, it is easy to see that Φ is a generalized convex φ-type function and

$$\begin{aligned} \Phi\left(u^{s}\right) &= \Gamma(1+\alpha)\,{}_0I_{u^s}^{(\alpha)}\left(\frac{f\left(t^{1/s}\right)}{t^{\alpha}}\right) \\ &\leq \left(\frac{f\left(\left(u^{s}\right)^{1/s}\right)}{u^{s\alpha}}\right) u^{s\alpha} = f(u). \end{aligned} \tag{46}$$

Moreover,

$$\begin{aligned} \Phi\left(u^{s}\right) &\geq \Gamma(1+\alpha)_{(u^s/2)}I_{u^s}^{(\alpha)}\left(\frac{f\left(t^{1/s}\right)}{t^{\alpha}}\right) \\ &\geq \frac{\left(f\left(\left(u^{s}/2\right)^{1/s}\right)2^{\alpha}u^{-s\alpha}\right)u^{s\alpha}}{2^{\alpha}} = f\left(2^{-1/s}u\right). \end{aligned} \tag{47}$$

Therefore,

$$f\left(2^{-1/s}u\right) \leq \Phi\left(u^{s}\right) \leq f(u), \tag{48}$$

for all $u \geq 0$. □

4. Applications

Based on the properties of the two kinds of generalized s-convex functions in the above section, some applications are given.

Example 1. Let $0 < s < 1$, and $a^{\alpha}, b^{\alpha}, c^{\alpha} \in \mathbb{R}^{\alpha}$. For $u \in \mathbb{R}_{+}$, define

$$f(u) = \begin{cases} a^{\alpha}, & u = 0, \\ b^{\alpha}u^{s\alpha} + c^{\alpha}, & u > 0. \end{cases} \tag{49}$$

We have the following conclusions.

(i) If $b^{\alpha} \geq 0^{\alpha}$ and $c^{\alpha} \leq a^{\alpha}$, then $f \in GK_s^1$.

(ii) If $b^{\alpha} \geq 0^{\alpha}$ and $c^{\alpha} < a^{\alpha}$, then f is nondecreasing on $(0, +\infty)$ but not on $[0, +\infty)$.

(iii) If $b^{\alpha} \geq 0^{\alpha}$ and $0^{\alpha} \leq c^{\alpha} \leq a^{\alpha}$, then $f \in GK_s^2$.

(iv) If $b^{\alpha} > 0^{\alpha}$ and $c^{\alpha} < 0^{\alpha}$, then $f \notin GK_s^2$.

Proof. (i) Let $\lambda_1, \lambda_2 \geq 0$ with $\lambda_1^s + \lambda_2^s = 1$. Then, there are two nontrivial cases.

Case I. Let $u, v > 0$. Then $\lambda_1 u + \lambda_2 v > 0$.

Thus,

$$\begin{aligned} f(\lambda_1 u + \lambda_2 v) &= b^{\alpha}(\lambda_1 u + \lambda_2 v)^{s\alpha} + c^{\alpha} \\ &\leq b^{\alpha}\left(\lambda_1^{s\alpha}u^{s\alpha} + \lambda_2^{s\alpha}v^{s\alpha}\right) + c^{\alpha} \\ &= b^{\alpha}\left(\lambda_1^{s\alpha}u^{s\alpha} + \lambda_2^{s\alpha}v^{s\alpha}\right) + c^{\alpha}\left(\lambda_1^{s\alpha} + \lambda_2^{s\alpha}\right) \\ &= \lambda_1^{s\alpha}\left(b^{\alpha}u^{s\alpha} + c^{\alpha}\right) + \lambda_2^{s\alpha}\left(b^{\alpha}v^{s\alpha} + c^{\alpha}\right) \\ &= \lambda_1^{s\alpha} f(u) + \lambda_2^{s\alpha} f(v). \end{aligned} \tag{50}$$

Case II. Let $v > u = 0$. We need only to consider $\lambda_2 > 0$.

Thus, we have

$$\begin{aligned} f(\lambda_1 0 + \lambda_2 v) &= f(\lambda_2 v) \\ &= b^{\alpha}\lambda_2^{s\alpha}v^{s\alpha} + c^{\alpha} \\ &= b^{\alpha}\lambda_2^{s\alpha}v^{s\alpha} + c^{\alpha}\left(\lambda_1^{s\alpha} + \lambda_2^{s\alpha}\right) \\ &= \lambda_1^{s\alpha}c^{\alpha} + \lambda_2^{s\alpha}\left(b^{\alpha}v^{s\alpha} + c^{\alpha}\right) \\ &= \lambda_1^{s\alpha}c^{\alpha} + \lambda_2^{s\alpha} f(v) \\ &\leq \lambda_1^{s\alpha}a^{\alpha} + \lambda_2^{s\alpha} f(v) \\ &= \lambda_1^{s\alpha} f(0) + \lambda_2^{s\alpha} f(v). \end{aligned} \tag{51}$$

So, $f \in GK_s^1$.

(ii) From Theorem 10, we can see that property (ii) is true.

(iii) Let $\lambda_1, \lambda_2 \geq 0$ with $\lambda_1 + \lambda_2 = 1$. Similar to the estimate of (i), there are also two cases.

Let $v, v > 0$. Then $\lambda_1 u + \lambda_2 v > 0$,

Thus,

$$\begin{aligned} & f(\lambda_1 u + \lambda_2 v) \\ &\quad = b^{\alpha}(\lambda_1 u + \lambda_2 v)^{s\alpha} + c^{\alpha} \\ &\quad < b^{\alpha}\left(\lambda_1^{s\alpha}u^{s\alpha} + \lambda_2^{s\alpha}v^{s\alpha}\right) + c^{\alpha}\left(\lambda_1^{\alpha} + \lambda_2^{\alpha}\right) \\ &\quad \leq b^{\alpha}\left(\lambda_1^{s\alpha}u^{s\alpha} + \lambda_2^{s\alpha}v^{s\alpha}\right) + c^{\alpha}\left(\lambda_1^{s\alpha} + \lambda_2^{s\alpha}\right) \\ &\quad = \lambda_1^{s\alpha}\left(b^{\alpha}u^{s\alpha} + c^{\alpha}\right) + \lambda_2^{s\alpha}\left(b^{\alpha}v^{s\alpha} + c^{\alpha}\right) \\ &\quad \leq \lambda_1^{s\alpha} f(u) + \lambda_2^{s\alpha} f(v). \end{aligned} \tag{52}$$

Let $v > u = 0$. We need only to consider $\lambda_2 > 0$.

Thus, we have

$$\begin{aligned} f\left(\lambda_1 0+\lambda_2 v\right) &= f\left(\lambda_2 v\right) \\ &= b^\alpha \lambda_2^{s\alpha} v^{s\alpha} + c^\alpha\left(\lambda_1^\alpha+\lambda_2^\alpha\right) \\ &< b^\alpha \lambda_2^{s\alpha} v^{s\alpha} + c^\alpha\left(\lambda_1^{s\alpha}+\lambda_2^{s\alpha}\right) \\ &= \lambda_2^{s\alpha}\left(b^\alpha v^{s\alpha}+c^\alpha\right)+c^\alpha\lambda_1^{s\alpha} \\ &\le \lambda_2^{s\alpha}\left(b^\alpha v^{s\alpha}+c^\alpha\right)+a^\alpha\lambda_1^{s\alpha} \\ &= \lambda_2^{s\alpha} f(v)+\lambda_1^{s\alpha} f(0). \end{aligned} \tag{53}$$

So, $f \in GK_s^2$.

(iv) Assume that $f \in GK_s^2$, and then f is nonnegative on $(0, \infty)$. On the other hand, we can take $u_1 > 0$, $c_1 < 0$ such that $f(u_1) = b^\alpha u_1^{s\alpha} + c_1^\alpha < 0^\alpha$, which contradict the assumption. □

Example 2. Let $0 < s < 1$ and $k > 1$. For $u \in R_+$, define

$$f(u) = \begin{cases} u^{(s/(1-s))\alpha}, & 0 \le u \le 1, \\ k^\alpha u^{(s/(1-s))\alpha}, & u > 1. \end{cases} \tag{54}$$

The function f is nonnegative, not local fractional continuous at $u = 1$ and belongs to GK_s^1 but not to GK_s^2.

Proof. From Theorem 16, we have that $f \in GK_s^1$. In the following, let us show that f is not in GK_s^2.

Take an arbitrary $a > 1$ and put $u = 1$. Consider all $v > 1$ such that $\lambda_1 u + \lambda_2 v = \lambda_1 + \lambda_2 v = a$, where $\lambda_1, \lambda_2 \ge 0$ and $\lambda_1 + \lambda_2 = 1$.

If $f \in GK_s^2$, it must be

$$\begin{aligned} & k^\alpha a^{(s/(1-s))\alpha} \\ & \quad \le \lambda_1^{s\alpha} + k^\alpha (1-\lambda_1)^{s\alpha}\left[\frac{(a-\lambda_1)}{(1-\lambda_1)}\right]^{(s/(1-s))\alpha}, \end{aligned} \tag{55}$$

for all $a > 1$ and all $0 \le \lambda_1 \le 1$.

Define the function

$$\begin{aligned} f_{\lambda_1}(a) = {} & \lambda_1^{s\alpha} + k^\alpha (1-\lambda_1)^{s\alpha}\left[\frac{(a-\lambda_1)}{(1-\lambda_1)}\right]^{(s/(1-s))\alpha} \\ & - k^\alpha a^{(s/(1-s))\alpha}. \end{aligned} \tag{56}$$

Then the function is local fractional continuous on the (λ_1, ∞) and

$$g(\lambda_1) = f_{\lambda_1}(1) = \lambda_1^{s\alpha} + k^\alpha (1-\lambda_1)^{s\alpha} - k^\alpha. \tag{57}$$

It is easy to see that g is local fractional continuous on $[0, 1]$ and $g(1) = 1^\alpha - k^\alpha < 0^\alpha$. So there is a number λ_{1_0}, $0 < \lambda_{1_0} < 1$, such that $g(\lambda_{1_0}) = f_{\lambda_{1_0}}(1) < 0^\alpha$. The local fractional continuity of $f_{\lambda_{1_0}}$ yields that $f_{\lambda_{1_0}}(a) < 0^\alpha$ for a certain $a > 1$, that is, inequality (55) does not hold, which means that $f \notin GK_s^2$. □

5. Conclusion

In the paper, we introduce the definitions of two kinds of generalized s-convex function on fractal sets and study the properties of these generalized s-convex functions. When $\alpha = 1$, these results are the classical situation.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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