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Research Article

Probabilistic Decomposition Method on the Server Indices of an $M^{\xi}/G/1$ Vacation Queue

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This paper develops a probabilistic decomposition method for an $M^{\xi}/G/1$ repairable queueing system with multiple vacations, in which the customers who arrive during server vacations enter the system with probability p. Such a novel method is used to analyze the main performance indices of the server, such as the unavailability and the mean failure number during $(0, t]$. It is derived that the structures of server indices are two convolution equations. Further, comparisons with existing methods indicate that our method is effective and applicable for studying server performances in single-server $M^{\xi}/G/1$ vacation queues and their complex variants. Finally, a stochastic order and production system with a multipurpose production facility is numerically presented for illustrative purpose.

1. Introduction

There are some effective and convenient analytic methods for single-server queues with a repairable server or service station. For example, the Markov renewal process method is used to study an M/G/1 queueing system with repairable service station in [1], the geometric process method introduced by Lam is applied to analyze the lifetime behaviors and repair times of deteriorating service station in [2, 3], and the matrix-geometric method is available for GI/M/1 and $M/E_k/1$ repairable queues in [4, 5]. It is well known that the supplementary variable method posed by Cox [6] is very important in dealing with some Poisson input queues with a repairable server. Many researchers, such as Wang [7], Ke et al. [8], Liu et al. [9], and Cao [10], have utilized this method for lots of repairable single-server queueing systems. The above approaches were applied to analyze some queueing indices, such as queue size, waiting time, and their stochastic decompositions, and the performance measures of the server, such as the mean times to the first failure, unavailability and failure frequency. However, the common methods mentioned above usually become too complicated to be solved especially when dealing with some Poisson input bulk arrival queues with a repairable server and their complex vacation variants.

In this paper, based on the renewal process theory and Laplace and Laplace-Stieltjes transforms we develop a probabilistic decomposition method to analyze the performance measures of the repairable server for a single-server $M^{\xi}/G/1$ queue with variable input rate and multiple vacations. Our method is completely different from the methods used in [1–10] and reveals that the structures of the server indices in Poisson input single-server bulk arrival vacation queues are two convolution equations. Our analytic idea is presented as follows: (1) with the definition of "generalized server busy period", we get the conditional probability that the time t is during the generalized server busy period; (2) according to this probability and our probabilistic decomposition method, we obtain the unavailability and average failure number of the server, which derive two convolution equations; (3) finally, by means of a special case, comparisons are made between our new method and supplementary variable method. Comparisons indicate that our method is more effective and applicable for Poisson input single-server bulk arrival queues with a repairable server and their complex vacation variants.

The rest of the paper is organized as follows. Sections 2 and 3 give the queue assumptions and preliminaries, respectively. In Section 4 a probabilistic decomposition method is developed to analyze main server indices. A special case is presented to validate our results and make comparisons between our new method and supplementary variable method. In Section 5 as a real world example we numerically analyze the influences of system parameters on main facility indices for a stochastic order and production system with a multi-purpose production facility. Conclusions are finally drawn in Section 6.

2. Assumptions

we consider an $M^{\xi}/G/1$ vacation queueing system with variable input rate as follows.

(1) The interarrival times between batch customers, $\{\tau_i,\ i \geq 1\}$, are independent identically distributed (i.i.d) random variables with distribution function $F(t) = 1 - e^{-\lambda t}, t \geq 0$. Each batch size ξ is a random variable following distribution $P(\xi = k) = e_k, k \geq 1$ with finite mean e and probability-generating function (PGF) $A(z) = \sum_{k=1}^{\infty} e_k z^k,\ |z| < 1$.

(2) The service order for customers in different batch arrivals is under the rule of FCFS, and the order in one batch arrival is arbitrary. The server can serve only one customer at a time. The service times $\{\chi_n,\ n \geq 1\}$ are i.i.d random variables each with arbitrary distribution $G(t)$, $t \geq 0$ with finite mean μ.

(3) The server takes multiple vacations when the system becomes empty. Let V_n be the server's the nth vacation time. Assume that $V_n, n \geq 1$ are i.i.d random variables with distribution function $V(t)$, $t \geq 0$ and finite mean $E(V)$. The customers who arrive during server vacations enter the system with probability $p\,(0 < p < 1)$ or lose with probability $1 - p$. Upon returning from a vacation, the server immediately serves one by one when there is a waiting queue or leaves for another vacation when there is an empty queue.

(4) The server consists of r unreliable units; these units may possibly fail if and only if the server is serving a customer. Once a unit fails, the server breaks down and cannot continue to serve. The failed unit will be repaired immediately. After the repair is completed, the server resumes operating and continues to serve the customer whose service has not been finished yet. The service time for a customer is cumulative.

(5) During the repair of a unit, the server cannot operate and the other units cannot fail. After repair, the unit is as good as new. The lifetime X_i of unit i of the server has an exponential distribution $X_i(t) = 1-e^{-\alpha_i t}$, $t \geq 0$, and its repair time Y_i obeys an arbitrary distribution $Y_i(t)$, $t \geq 0$ with a mean repair time β_i, $i = 1, 2, \ldots, r$.

(6) All random variables are mutually independent. At the initial time $t = 0$, the server begins to serve when the number of customers presented in the system $N(0) > 0$, or the server is idle and waits for the first batch arrival when $N(0) = 0$.

Remark 1. Assumption (6) is practical and reasonable. But it is later proved that the steady-state performance indices of server are independent of initial state $N(0) = i$, $i \geq 0$.

Remark 2. Throughout this paper, we take some *notations* as follows: $\tilde{\rho}$ denotes the traffic intensity of the considered queue; $N(t)$ is the customer number of system at time t; $G^{(k)}(t)$ denotes the k-fold convolution of corresponding function $G(t)$, $G^{(0)}(t) = 1$; $g^*(s) = \int_0^{\infty} e^{-st} G(t)dt$ and $g(s) = \int_0^{\infty} e^{-st} dG(t)$ denote Laplace and Laplace-Stieltjes transforms of corresponding $G(t)$, respectively; $E(X)$ is the mean of random variable X; $P(Q)$ is the probability of event Q; $\Re(s)$ denotes the real part of complex number s.

3. Preliminaries

Let X and Y denote the lifetime and repair time of server, respectively; then for $t \geq 0$, the distribution functions of X and Y are given, respectively, by

$$\begin{aligned} X(t) &= P(X \leq t) \\ &= P(\min(X_1, X_2, \ldots, X_r) \leq t) \\ &= 1 - e^{-\alpha t}, \quad \left(\alpha = \sum_{i=1}^{r} \alpha_i\right), \end{aligned} \tag{1}$$

$$\begin{aligned} Y(t) &= P(Y \leq t) \\ &= \sum_{i=1}^{r} P(\min(X_1, X_2, \ldots, X_r) = X_i, Y = Y_i \leq t) \\ &= \sum_{i=1}^{r} P(X_1 > X_i, \ldots, X_{i-1} > X_i, \\ &\qquad X_{i+1} > X_i, \ldots, X_r > X_i, Y_i \leq t) \\ &= \sum_{i=1}^{r} Y_i(t) \int_0^{\infty} P(X_1 > x, \ldots, X_{i-1} > x, \\ &\qquad X_{i+1} > x, \ldots, X_r > x)\, dX_i(x) \\ &= \frac{1}{\alpha} \sum_{i=1}^{r} \alpha_i Y_i(t). \end{aligned} \tag{2}$$

Thus, the mean repair time of server is given by

$$\beta = \int_0^{\infty} t dY(t) = \frac{1}{\alpha} \sum_{i=1}^{r} \alpha_i \beta_i. \tag{3}$$

Definition 3. The "service completion time of a customer" represents the time interval from the epoch when the service for a customer begins to the epoch when the service of this customer ends, which includes possible repair times of server due to its unit failures in the process of serving this customer. Denote by $\tilde{\chi}_k$ the service completion time of customer k; it is obvious that $\tilde{\chi}_k$, $k \geq 1$, are i.i.d. random variables.

Lemma 4 (see [1]). *Let $\widetilde{G}(t) = P(\widetilde{\chi}_k \le t)$, $k \ge 1$, then*

$$\widetilde{G}(t) = \sum_{k=0}^{\infty} \int_0^t Y^{(k)}(t-x) \frac{(\alpha x)^k}{k!} e^{-\alpha x} dG(x), \quad t \ge 0,$$
$$\widetilde{g}(s) = \int_0^{\infty} e^{-st} d\widetilde{G}(t) = g(s+\alpha-\alpha y(s)), \quad \mathfrak{R}(s) > 0, \tag{4}$$
$$E(\widetilde{\chi}) = -\left.\frac{d\widetilde{g}(s)}{ds}\right|_{s=0} = \mu(1+\alpha\beta),$$

where $g(s) = \int_0^{\infty} e^{-st} dG(t)$ and $y(s) = \int_0^{\infty} e^{-st} dY(t)$.

Definition 5. The "generalized server busy period" represents the time interval from the epoch when the service begins to the epoch when the system becomes empty, which also contains possible repair times of server due to its unit failures in the process of service.

Let $\widetilde{b}$ denote the generalized server busy period initiated with one customer and its distribution function is $\widetilde{B}(t)$ with Laplace-Stieltjes transform $\widetilde{b}(s)$. Similar to the discussions in an M/G/1 queue with generalization vacations [11], the following lemma holds.

Lemma 6. *If $\mathfrak{R}(s) > 0$, then $\widetilde{b}(s)$ is the solution with smallest absolute value in z of the equation $z = \widetilde{g}(s+\lambda-\lambda A(z))$, and*

$$E(\widetilde{b}) = \begin{cases} \dfrac{\mu(1+\alpha\beta)}{1-\lambda e\mu(1+\alpha\beta)}, & \widetilde{\rho} < 1, \\ \infty, & \widetilde{\rho} \ge 1, \end{cases} \tag{5}$$

where $\widetilde{\rho} = \lambda e\mu(1+\alpha\beta)$ denotes the traffic intensity of the considered queue.

Denote by $\widetilde{b}^{\langle i\rangle}$ the generalized server busy period initiated with i customers; then $\widetilde{b}^{\langle i\rangle}$ can be expressed as $\widetilde{b}^{\langle i\rangle} = \sum_{k=1}^{i} \widetilde{b}_k$, where $\widetilde{b}_k$, $1 \le k \le i$, are mutually independent with the same distribution function as $\widetilde{b}$. Let $\widetilde{B}^{\langle i\rangle}(t) = P(\widetilde{b}^{\langle i\rangle} \le t)$; then we can get $\widetilde{B}^{\langle i\rangle}(t) = \widetilde{B}^{(i)}(t)$; that is, $\widetilde{B}^{\langle i\rangle}(t)$ is the i-fold convolution of $\widetilde{B}(t)$.

Definition 7. The "system idle period" represents the time interval from the epoch when the system becomes empty to the epoch when batch customers enter the system.

Denote by I_k the kth system idle period, then by the queue assumptions, $\{I_k,\ k \ge 1\}$ are independent of each other, and their distribution functions are as follows:

(1) if $N(0) = 0$, then

$$I_k(t) = P(I_k \le t) = \begin{cases} 1 - e^{-\lambda t}, & k = 1, \\ 1 - e^{-\lambda p t}, & k = 2, 3, \ldots, \end{cases} \quad t \ge 0, \tag{6}$$

(2) if $N(0) > 0$, then

$$I_k(t) = P(I_k \le t) = 1 - e^{-\lambda p t}, \quad k = 1, 2, 3, \ldots, \quad t \ge 0. \tag{7}$$

4. Performance Indices of the Server

In this section, we develop a probabilistic decomposition method to analyze main performance indices of the server in the considered queue, including the conditional probability that the time t is during the generalized server busy period, the unavailability and the average failure number during $(0, t]$. Further, it is derived that the structures of server indices are two convolution equations. Finally, a special case is presented to validate our results and make comparisons between our method and supplementary variable method.

4.1. The Conditional Probability That the Time t is during the Generalized Server Busy Period

Theorem 8. *For $i \ge 0$, let $A_i(t) = P$ (the time t is during the generalized server busy period $|N(0) = i$); then for $\mathfrak{R}(s) > 0$, Laplace transforms of $A_i(t)$, $i \ge 0$ are*

$$a_0^*(s) = \frac{\lambda}{s(s+\lambda)} \left\{ 1 - \frac{A(\widetilde{b}(s))[1-\nu(s)]}{1-\nu(s+\lambda p - \lambda p A(\widetilde{b}(s)))} \right\}, \tag{8}$$

$$a_i^*(s) = \frac{1}{s} \left\{ 1 - \frac{\widetilde{b}^i(s)[1-\nu(s)]}{1-\nu(s+\lambda p - \lambda p A(\widetilde{b}(s)))} \right\}, \quad i \ge 1, \tag{9}$$

and in steady state, for system traffic intensity $\widetilde{\rho} = \lambda e\mu(1+\alpha\beta)$ and $i \ge 0$, one has

$$\lim_{t\to\infty} A_i(t) = \lim_{s\to 0} s a_i^*(s) = \begin{cases} \dfrac{\lambda p e\mu(1+\alpha\beta)}{1-\lambda(1-p)e\mu(1+\alpha\beta)}, & \widetilde{\rho} < 1, \\ 1, & \widetilde{\rho} \ge 1, \end{cases} \tag{10}$$

where $\widetilde{b}(s)$ is determined by Lemma 6.

Proof. Let $s_k = \sum_{i=1}^{k} V_i$, $l_k = \sum_{i=1}^{k} \tau_i$, $k \ge 1$, $s_0 = l_0 = 0$, $\sum_{n[m]=m}^{\infty} = \sum_{n_1=1}^{\infty} \cdots \sum_{n_m=1}^{\infty}$, and $n[m] = n_1 + \cdots + n_m$. Denote by $\widetilde{b}^{\langle k\rangle}$ the generalized server busy period initiated with k customers with distribution function $\widetilde{B}^{\langle k\rangle}(t)$ (see Definition 5), and I_k is the kth system idle period with distribution function $I_k(t)$ (see Definition 7). Noting that the ending points of server vacation and generalized server busy period are renewal points, and the server takes no vacations when $N(0) = 0$, we have

$$A_0(t) = \sum_{k=1}^{\infty} e_k P(I_1 < t \le I_1 + \widetilde{b}^{\langle k\rangle}) + \sum_{k=1}^{\infty} e_k \sum_{j=1}^{\infty} \sum_{m=1}^{\infty} P(s_{j-1} < I_2 \le s_j,$$

$$I_2 + l_{m-1} < s_j \le I_2 + l_m,$$
$$t > I_1 + \widetilde{b}^{\langle k\rangle} + s_j,$$
$$\text{the time } t \text{ is during the generalized server busy period})$$

$$= \sum_{k=1}^{\infty} e_k \int_0^t \left[1 - \widetilde{B}^{(k)}(t-x)\right] dF(x)$$
$$+ \sum_{k=1}^{\infty} e_k \sum_{j=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n[m]=m}^{\infty} e_{n_1} \cdots e_{n_m}$$
$$\times \int_0^t \int_0^{t-x} \int_0^{t-x-y} A_{n[m]}(t-x-y-u) \frac{(\lambda p u)^m}{m!}$$
$$\times e^{-\lambda p(u+y)} dV(u)\, dV^{(j-1)}(y)\, d\left[F(x) * \widetilde{B}^{(k)}(x)\right], \tag{11}$$

where $F(x) * \widetilde{B}^{(k)}(x) = \int_0^x F(x-t) d\widetilde{B}^{(k)}(t), \quad x \ge 0$.

By means of the same decomposition way, for $i \ge 1$, we get

$$A_i(t) = P\left(t < \widetilde{b}^{\langle i\rangle}\right)$$
$$+ \sum_{j=1}^{\infty} \sum_{m=1}^{\infty} P\left(s_{j-1} < I_1 \le s_j,\right.$$
$$I_1 + l_{m-1} < s_j \le I_1 + l_m,$$
$$t > \widetilde{b}^{\langle i\rangle} + s_j,$$
$$\text{the time } t \text{ is during the generalized server busy period})$$
$$= 1 - \widetilde{B}^{(i)}(t) + \sum_{j=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n[m]=m}^{\infty} e_{n_1} \cdots e_{n_m}$$
$$\times \int_0^t \int_0^{t-x} \int_0^{t-x-y} A_{n[m]}(t-x-y-u)$$
$$\times \frac{(\lambda p u)^m}{m!} e^{-\lambda p(u+y)} dV(u)\, dV^{(j-1)}(y)\, d\widetilde{B}^{(i)}(x). \tag{12}$$

Taking Laplace transforms of (11) and (12), respectively, gives rise to

$$a_0^*(s) = \frac{\lambda}{s(s+\lambda)}\left[1 - A\left(\widetilde{b}(s)\right)\right] + \frac{f(s)A\left(\widetilde{b}(s)\right)}{1 - v(s+\lambda p)}$$
$$\times \sum_{m=1}^{\infty} \sum_{n[m]=m}^{\infty} e_{n_1} \cdots e_{n_m} a_{n[m]}^*(s) \int_0^{\infty} e^{-(s+\lambda p)t} \tag{13}$$
$$\times \frac{(\lambda p t)^m}{m!} dV(t),$$

$$a_i^*(s) = \frac{1 - \widetilde{b}^i(s)}{s}\left[1 - A\left(\widetilde{b}(s)\right)\right] + \frac{\widetilde{b}^i(s)}{1 - v(s+\lambda p)}$$
$$\times \sum_{m=1}^{\infty} \sum_{n[m]=m}^{\infty} e_{n_1} \cdots e_{n_m} a_{n[m]}^*(s) \int_0^{\infty} e^{-(s+\lambda p)t} \tag{14}$$
$$\times \frac{(\lambda p t)^m}{m!} dV(t), \quad i \ge 1.$$

By checking (13) and (14), we obtain the relation

$$a_i^*(s) = \frac{1}{s} - \frac{\widetilde{b}^i(s)\left[\lambda - s(s+\lambda)a_0^*(s)\right]}{s\lambda A\left(\widetilde{b}(s)\right)}, \quad i \ge 1. \tag{15}$$

Substituting (15) into (13) leads to (8). Equation (9) is obtained by (8) and (15). Applying Tauberian theorem [12] and L' Hospital's rule gives (10). □

In order to investigate the unavailability and the failure number during $(0, t]$ of server, we introduce a classical r-unit series repairable system [12]. For $t \ge 0$, let

$$\Phi(t) = P(\text{the system is repaired at time } t),$$
$$\varphi^*(s) = \int_0^{\infty} e^{-st}\Phi(t)\, dt,$$
$$M(t) = E(\text{the failure number of the system during } (0, t]),$$
$$m(s) = \int_0^{\infty} e^{-st} dM(t). \tag{16}$$

Lemma 9 (see [12]). *If* $\Re(s) > 0$, *then*

$$\varphi^*(s) = \frac{\alpha - \sum_{i=1}^r \alpha_i y_i(s)}{s\left[s + \alpha - \sum_{i=1}^r \alpha_i y_i(s)\right]},$$
$$m(s) = \frac{\alpha}{s + \alpha - \sum_{i=1}^r \alpha_i y_i(s)},$$
$$\lim_{t\to\infty} \Phi(t) = \lim_{s\to 0} s\varphi^*(s) \tag{17}$$
$$= \frac{\alpha\beta}{1 + \alpha\beta},$$
$$\lim_{t\to\infty} \frac{M(t)}{t} = \lim_{s\to 0} s m(s) = \frac{\alpha}{1 + \alpha\beta},$$

where $y_i(s) = \int_0^\infty e^{-st}\,dY_i(t)$, $\alpha = \sum_{i=1}^r \alpha_i$ *and* $\beta = (1/\alpha)\sum_{i=1}^r \alpha_i\beta_i$.

4.2. The Unavailability of the Server. The unavailability of the server at time t; that is, the probability that the server is repaired at time t.

Theorem 10. *Let* $\Phi_i(t) = P$ *(the server is repaired at time* $t|N(0) = i)$, $i \geq 0$*; then for* $\mathfrak{R}(s) > 0$*, Laplace transform of* $\Phi_i(t)$ *is*

$$\varphi_i^*(s) = \varphi^*(s)\left[sa_i^*(s)\right], \quad i \geq 0, \tag{18}$$

and for system traffic intensity $\tilde{\rho} = \lambda e\mu(1+\alpha\beta)$ *and* $i \geq 0$*, the steady-state unavailability of the server is given by*

$$\lim_{t\to\infty}\Phi_i(t) = \begin{cases} \dfrac{\lambda p e\mu\alpha\beta}{1-\lambda(1-p)e\mu(1+\alpha\beta)}, & \tilde{\rho} < 1 \\ \dfrac{\alpha\beta}{1+\alpha\beta}, & \tilde{\rho} \geq 1, \end{cases} \tag{19}$$

where $\varphi^*(s)$*, and* $a_i^*(s)$, $i \geq 0$ *are given by Lemma 9 and Theorem 8, respectively.*

Proof. (i) According to the queue assumptions, the server is repaired at time t if and only if the time t is during one generalized server busy period, and the server is repaired at time t. Consequently, using the law of total probability and renewal process theory, we have the decomposition of $\Phi_0(t)$ as follows:

$$\begin{aligned}\Phi_0(t) = {} & \sum_{k=1}^{\infty} e_k P\Big(I_1 < t \leq I_1 + \tilde{b}^{\langle k\rangle}, \\ & \qquad \text{the server is repaired at time } t\Big) \\ & + \sum_{k=1}^{\infty} e_k \sum_{j=1}^{\infty}\sum_{m=1}^{\infty} P\Big(s_{j-1} < I_2 \leq s_j, \\ & \qquad I_2 + l_{m-1} < s_j \leq I_2 + l_m, t > I_1 \\ & \qquad + \tilde{b}^{\langle k\rangle} + s_j, \text{the server is repaired} \\ & \qquad \text{at time } t\Big) \\ = {} & \sum_{k=1}^{\infty} e_k \int_0^t S_k(t-x)\,dF(x) \\ & + \sum_{k=1}^{\infty} e_k \sum_{j=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n[m]=m}^{\infty} e_{n_1}\cdots e_{n_m} \\ & \times \int_0^t\int_0^{t-x}\int_0^{t-x-y} \underset{n[m]}{\Phi}(t-x-y-u) \\ & \times \frac{(\lambda p u)^m}{m!} e^{-\lambda p(u+y)}\,dV(u)\,dV^{(j-1)} \\ & \times (y)\,d\left[F(x) * \tilde{B}^{(k)}(x)\right],\end{aligned} \tag{20}$$

where $S_k(t) = P(0 \leq t < b^{\langle k\rangle}$, the server is repaired at time $t)$, $k \geq 1$.

Similarly, for $i \geq 1$, $\Phi_i(t)$ is decomposed as

$$\begin{aligned}\Phi_i(t) = {} & S_i(t) + \sum_{j=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n[m]=m}^{\infty} e_{n_1}\cdots e_{n_m} \\ & \times \int_0^t\int_0^{t-x}\int_0^{t-x-y} \underset{n[m]}{\Phi}(t-x-y-u) \\ & \times \frac{(\lambda p u)^m}{m!} e^{-\lambda p(u+y)}\,dV(u)\,dV^{(j-1)}(y)\,d\tilde{B}^{(i)}(x).\end{aligned} \tag{21}$$

(ii) For $i \geq 1$,

$$S_i(t) = \Phi(t) - \int_0^t \Phi(t-x)\,d\tilde{B}^{(i)}(x), \tag{22}$$

$$\int_0^\infty e^{-st} S_i(t)\,dt = \varphi^*(s)\left[1-\tilde{b}^i(s)\right], \tag{23}$$

where $\Phi(t)$ and $\varphi^*(s)$ are determined by Lemma 9.

In reality, $\Phi(t)$ can be decomposed as

$$\begin{aligned}\Phi(t) = {} & P\left(\text{the system is repaired at time } t, \tilde{b}^{\langle i\rangle} \leq t\right) \\ & + P\left(\text{the system is repaired at time } t, \tilde{b}^{\langle i\rangle} > t\right) \\ = {} & \int_0^t \Phi(t-x)\,d\tilde{B}^{(i)}(x) + S_i(t),\end{aligned} \tag{24}$$

which leads to (22) and (23).

(iii) Taking Laplace transforms of (20) and (21), respectively, and utilizing (22) and (23), we get

$$\begin{aligned}\varphi_0^*(s) = {} & \frac{\lambda}{s+\lambda}\varphi^*(s)\left[1 - A\left(\tilde{b}(s)\right)\right] \\ & + \frac{\lambda A\left(\tilde{b}(s)\right)}{(s+\lambda)\left[1-v(s+\lambda p)\right]} \\ & \times \sum_{m=1}^{\infty}\sum_{n[m]=m}^{\infty} e_{n_1}\cdots e_{n_m}\varphi_{n[m]}^*(s) \\ & \times \int_0^\infty e^{-(s+\lambda p)t}\frac{(\lambda p t)^m}{m!}\,dV(t),\end{aligned} \tag{25}$$

$$\begin{aligned}\varphi_i^*(s) = {} & \varphi^*(s)\left[1-\tilde{b}^i(s)\right] \\ & + \frac{\tilde{b}^i(s)}{1-v(s+\lambda p)} \\ & \times \sum_{m=1}^{\infty}\sum_{n[m]=m}^{\infty} e_{n_1}\cdots e_{n_m}\varphi_{n[m]}^*(s) \\ & \times \int_0^\infty e^{-(s+\lambda p)t}\frac{(\lambda p t)^m}{m!}\,dV(t), \quad i \geq 1.\end{aligned} \tag{26}$$

Performing similar operations in the proof of Theorem 8 on (25) and (26), we can complete the proof by Theorem 8 and Lemma 9. □

4.3. The Mean Failure Number of Server During (0, t]

Theorem 11. *Let $M_i(t) = E$ (the failure number of server during $(0,t] | N(0) = i$); $i \geq 0$, then for $\Re(s) > 0$, Laplace-Stieltjes transform of $M_i(t)$ is*

$$m_i(s) = m(s)\left[s a_i^*(s)\right], \quad i \geq 0, \tag{27}$$

and for system traffic intensity $\tilde{\rho} = \lambda e\mu(1+\alpha\beta)$ and $i \geq 0$, the steady-state failure frequency of server, that is, in steady state, the rate of occurrence of server failures, is

$$\lim_{t\to\infty}\frac{M_i(t)}{t} = \lim_{s\to 0} s m_i(s) = \begin{cases} \dfrac{\lambda p e\mu\alpha}{1-\lambda(1-p)e\mu(1+\alpha\beta)}, & \tilde{\rho} < 1, \\ \dfrac{\alpha}{1+\alpha\beta}, & \tilde{\rho} \geq 1, \end{cases} \tag{28}$$

where $m(s)$ and $a_i^*(s)$, $i \geq 0$, are given by Lemma 9 and Theorem 8, respectively.

Proof. (1) For $i \geq 1$, let

$$\begin{aligned} H_i(t) &= E\left(0 \leq t < \tilde{b}^{\langle i\rangle},\right. \\ &\qquad \left.\text{the failure number of server during } (0,t]\right), \\ L_i(t) &= E\left(\tilde{b}^{\langle i\rangle} \leq t,\right. \\ &\qquad \left.\text{the failure number of server during } (0,\tilde{b}^{\langle i\rangle}]\right); \end{aligned} \tag{29}$$

then similar to (22), we have

$$H_i(t) + L_i(t) = M(t) - \int_0^t M(t-x)\, d\tilde{B}^{(i)}(x), \quad i \geq 1, \tag{30}$$

where $M(t)$ is determined by Lemma 9.

(2) By the law of total probability and renewal process theory, $M_0(t)$ is decomposed as

$$\begin{aligned} M_0(t) &= \sum_{k=1}^{\infty} e_k \left\{E\left(I_1 < t \leq I_1 + \tilde{b}^{\langle k\rangle},\right.\right. \\ &\qquad \text{the failure number of server} \\ &\qquad \left.\text{during } (0,t]\right) \\ &\quad + E\left(t > I_1 + \tilde{b}^{\langle k\rangle},\right. \\ &\qquad \text{the failure number of server} \\ &\qquad \left.\left.\text{during } \left(I_1, I_1 + \tilde{b}^{\langle k\rangle}\right]\right)\right\} \\ &\quad + \sum_{k=1}^{\infty} e_k \sum_{j=1}^{\infty}\sum_{m=1}^{\infty} E\left(t > I_1 + \tilde{b}^{\langle k\rangle} + s_j,\right. \\ &\qquad s_{j-1} < I_2 \leq s_j, \\ &\qquad I_2 + l_{m-1} < s_j \leq I_2 + l_m, \\ &\qquad \text{the failure number of server} \\ &\qquad \left.\text{during } \left(I_1 + \tilde{b}^{\langle k\rangle} + s_j, t\right]\right) \\ &= \sum_{k=1}^{\infty} e_k \int_0^t \left[H_k(t-x) + L_k(t-x)\right] dF(x) \\ &\quad + \sum_{k=1}^{\infty} e_k \sum_{j=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n[m]=m}^{\infty} \\ &\qquad \times e_{n_1}\cdots e_{n_m} \int_0^t \int_0^{t-x} \int_0^{t-x-y} \\ &\qquad \times M_{n[m]}(t-x-y-u) \\ &\qquad \times \frac{(\lambda p u)^m}{m!} e^{-\lambda p(u+y)} dV(u)\, dV^{(j-1)} \\ &\qquad \times (y)\, d\left[F(x) * \tilde{B}^{(k)}(x)\right]. \end{aligned} \tag{31}$$

Similarly, $M_i(t)$, $i \geq 1$, are decomposed as

$$\begin{aligned} M_i(t) &= H_i(t) + L_i(t) \\ &\quad + \sum_{j=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n[m]=m}^{\infty} \\ &\quad \times e_{n_1}\cdots e_{n_m} \int_0^t \int_0^{t-x} \int_0^{t-x-y} M_{n[m]}(t-x-y-u) \\ &\qquad \times \frac{(\lambda p u)^m}{m!} e^{-\lambda p(u+y)} \\ &\qquad \times dV(u)\, dV^{(j-1)}(y)\, d\tilde{B}^{(i)}(x), \\ &\qquad i \geq 1. \end{aligned} \tag{32}$$

Taking Laplace-Stieltjes transforms of (31) and (32) and using (30), Theorem 8, and Lemma 9, we get (27). Equation (28) is obtained by Tauberian theorem [12], Lemma 9, and (10). □

Remark 12 (a special example). If $p = 1$, $P(\xi = 1) = 1$, and $P(V_n = 0) = 1$, $n \geq 1$, then our model becomes an M/G/1 repairable queue with an unreliable server [10], in which the server consists of r repairable units and operates if and only if r units operate. In this case, for $\tilde{\rho} = \lambda\mu(1+\alpha\beta)$, we get

$$\lim_{t\to\infty} A_i(t) = \begin{cases} \lambda\mu(1+\alpha\beta), & \tilde{\rho} < 1 \\ 1, & \tilde{\rho} \geq 1, \end{cases} \quad (33)$$

$$\lim_{t\to\infty} \frac{M_i(t)}{t} = \begin{cases} \lambda\mu\alpha, & \tilde{\rho} < 1 \\ \dfrac{\alpha}{1+\alpha\beta}, & \tilde{\rho} \geq 1, \end{cases} \quad (34)$$

$$\begin{aligned} \widetilde{A} &= 1 - \lim_{t\to\infty} \Phi_i(t) \\ &= \begin{cases} 1 - \lambda\mu\alpha\beta, & \tilde{\rho} < 1, \\ \dfrac{1}{1+\alpha\beta}, & \tilde{\rho} \geq 1, \end{cases} \quad \left(\alpha = \sum_{i=1}^{r} \alpha_i, \beta = \frac{1}{\alpha}\sum_{i=1}^{r} \alpha_i\beta_i\right), \end{aligned} \quad (35)$$

where $\widetilde{A}$ denotes the steady-state availability of server.

In the above results, $\lim_{t\to\infty} M_i(t)/t$ and $\widetilde{A}$ agree with those in [10], which are derived with the help of the supplementary variable method. However, [10], didn't obtain $\lim_{t\to\infty} A_i(t)$ and $\lim_{t\to\infty} \Phi_i(t)$. Further, comparisons with our results indicate that using the supplementary variable method, [10] did not derive $a_i^*(s)$, $\varphi_i^*(s)$ and $m_i(s)$ for arbitrary initial state $N(0) = i$, $i > 0$, and arbitrary distributions $G(t)$ and $Y_k(t)$, $k = 1, 2, \ldots, r$.

Remark 13. Taking Laplace and Laplace-Stieltjes inverse transforms of (18) and (27), respectively, gives rise to the following two convolution equations:

$$\begin{aligned} \Phi_i(t) &= \Phi(t) * A_i(t) \\ &= \int_0^t \Phi(t-x)\, dA_i(x), \\ & \qquad t \geq 0, \quad i \geq 0, \end{aligned} \quad (36)$$

$$\begin{aligned} M_i(t) &= M(t) * A_i(t) \\ &= \int_0^t M(t-x)\, dA_i(x), \\ & \qquad t \geq 0, \quad i \geq 0. \end{aligned} \quad (37)$$

Since $\Phi(t)$ and $M(t)$ are known (see Lemma 9), it is indicated from (36) and (37) that discussing the unavailability $\Phi_i(t)$ and the mean failure number during $(0, t]$ of the server $M_i(t)$ can be simplified to discussing the conditional probability $A_i(t)$ presented in this paper. More importantly, (36) and (37) reveal the structures of the server indices, which are not derived by the supplementary variable method in [10].

Remark 14. From (10), (19), (28), and Lemma 9, we easily obtain two steady-state relation equations as follows:

$$\lim_{t\to\infty} \Phi_i(t) = \lim_{t\to\infty} \Phi(t) \lim_{t\to\infty} A_i(t), \quad i \geq 0, \quad (38)$$

$$\lim_{t\to\infty} \frac{M_i(t)}{t} = \lim_{t\to\infty} \frac{M(t)}{t} \lim_{t\to\infty} A_i(t), \quad i \geq 0. \quad (39)$$

What is more, we see that the two steady-state results are independent of arbitrary initial state $N(0) = i$, $i \geq 0$, and have nothing to do with server vacations. The above relations are also new, which are not obtained by the supplementary variable method in [10].

5. Numerical Examples

Our queueing model and its theoretical results obtained can be applied to model a stochastic order and production system with a multipurpose production facility (server). In such a system, customer orders for the product arrive in batch according to a compound Poisson process with mean arrival rate λ. The distribution for each batch order size ξ is geometric with mean $E(\xi) = 1/\theta$. The production time of each unit of the product is assumed to follow the 4-stage Erlang distribution with mean $\mu = 2$. Whenever all orders are completed and no new orders arrive, the production will be stopped and the facility may be available to perform some optional jobs (vacations). The optional jobs can make profits for the system. The orders which arrive during optional jobs will enter the queue for production with probability p $(0 < p < 1)$ or lose with probability $1-p$. Upon completion of each optional job, the system manager checks the orders and decides whether or not to resume the major production. If at this moment the orders are empty, a decision may be made for taking another optional job to be performed. If orders occur, production restarts. Moreover, the production may be interrupted due to some unpredictable events, which occur according to a Poisson process with rate α. The interrupted production is immediately recovered with a random time obeying the 2-stage Erlang distribution with mean $\beta = 0.8$. The production will continuously start when the interruption is recovered.

Tables 1–4 present several numerical results to illustrate the influences of varying system parameters on main performance measures of production facility. We consider three facility indices: the busy probability of production facility $\lim_{t\to\infty} A_i(t)$, the unavailability $\lim_{t\to\infty} \Phi_i(t)$, and the failure frequency $\lim_{t\to\infty} M_i(t)/t$. Moreover, in all the following cases, the system load value $\tilde{\rho}$ is also discussed.

By means of analysis results derived in Section 4, the effects of batch order arrival rate λ on production facility indices are presented in Table 1, where we set $(p, \theta, \mu, \alpha, \beta) = (0.8, 0.5, 2, 0.1, 0.8)$. As to be expected, the four performance indices all increase with increasing value of λ. But for $\lambda \geq 0.25$, three facility indices do not vary. This is because the order and production system becomes unstable. Table 2 shows that the effects of batch order entering probability p during optional jobs on production facility indices for the set of parameters $(\lambda, \theta, \mu, \alpha, \beta) = (0.2, 0.5, 2, 0.1, 0.8)$. It can be seen from Table 2 that batch order entering probability does not affect the system load and the system is always stable, whereas $\lim_{t\to\infty} A_i(t)$, $\lim_{t\to\infty} \Phi_i(t)$, and $\lim_{t\to\infty} M_i(t)/t$ all increase monotonously as the value of p increases, which coincides with the intuitive expectations. The effects of each batch order size parameter θ on production facility indices are shown in Table 3 with $(\lambda, p, \mu, \alpha, \beta) = (0.2, 0.8, 2, 0.1, 0.8)$.

TABLE 1: The effects of batch order rate λ on production facility indices ($p = 0.8$, $\theta = 0.5$, $\mu = 2$, $\alpha = 0.1$, and $\beta = 0.8$).

λ	$\widetilde{\rho}$	$\lim_{t\to\infty} A_i(t)$	$\lim_{t\to\infty} \Phi_i(t)$	$\lim_{t\to\infty} M_i(t)/t$
0.10	0.4320	0.3783	0.0280	0.0350
0.15	0.6480	0.5956	0.0441	0.0551
0.20	0.8640	0.8356	0.0619	0.0774
0.25	1.0800	1	0.0741	0.0926
0.30	1.2960	1	0.0741	0.0926
0.35	1.5120	1	0.0741	0.0926

TABLE 2: The effects of batch order entering probability p during optional jobs on production facility indices ($\lambda = 0.2$, $\theta = 0.5$, $\mu = 2$, $\alpha = 0.1$, and $\beta = 0.8$).

p	$\widetilde{\rho}$	$\lim_{t\to\infty} A_i(t)$	$\lim_{t\to\infty} \Phi_i(t)$	$\lim_{t\to\infty} M_i(t)/t$
0.5	0.8640	0.7606	0.0563	0.0704
0.6	0.8640	0.7922	0.0587	0.0733
0.7	0.8640	0.8164	0.0605	0.0756
0.8	0.8640	0.8356	0.0619	0.0774
0.9	0.8640	0.8511	0.0630	0.0788
1.0	0.8640	0.8640	0.0640	0.0800

TABLE 3: The effects of each batch order size parameter θ on production facility indices ($\lambda = 0.2$, $p = 0.8$, $\mu = 2$, $\alpha = 0.1$, and $\beta = 0.8$).

θ	$\widetilde{\rho}$	$\lim_{t\to\infty} A_i(t)$	$\lim_{t\to\infty} \Phi_i(t)$	$\lim_{t\to\infty} M_i(t)/t$
0.2	2.1600	1	0.0741	0.0926
0.3	1.4400	1	0.0741	0.0926
0.4	1.0800	1	0.0741	0.0926
0.5	0.8640	0.8356	0.0619	0.0774
0.6	0.7200	0.6729	0.0498	0.0623
0.7	0.6171	0.5632	0.0417	0.0522

From Table 3, we observe that the influence of θ on four indices is completely opposite to that of λ. Table 4 reports the effects of unpredictable events arrival rate α on production facility indices. Here we assume that $(\lambda, p, \theta, \mu, \beta) = (0.2, 0.8, 0.5, 2, 0.8)$. As shown in Table 4, when α increases, all production facility indices increase monotonously. Furthermore, for $\alpha \geq 0.32$, the system becomes unstable and the production facility is always busy. The trends shown by Tables 1–4 are as expected.

From the analysis presented in Tables 1–4, it can be concluded that under the stability condition, that is $\widetilde{\rho} < 1$, the performance indices of production facility are affected by batch order arrival, batch order entering probability, batch order size, and unpredictable events arrival. But as $\widetilde{\rho} \geq 1$, production facility indices are not affected by batch order arrival and batch order size, and production facility is always busy. In this case, the system is unstable.

6. Conclusions

In this paper, we develop a probabilistic decomposition method to analyze the performance measures of the repairable server in a single-server $M^{\xi}/G/1$ queue with p-entering discipline during server vacations. Our method is completely different from common methods used in [1–10] and reveals that the structures of server indices in Poisson input single-server bulk arrival vacation queues are two convolution equations. A special case and comparisons with supplementary variable method indicate that our method is effective and applicable for Poisson input bulk arrival vacation queues with a repairable server and their complex variants. Finally, a stochastic order and production system with a multipurpose production facility is numerically presented for illustrative purpose. In the future, the server performance indices of discrete time bulk arrival vacation queues will be our further work using similar probabilistic decomposition method.

TABLE 4: The effects of unpredictable events arrival rate α on production facility indices ($\lambda = 0.2$, $p = 0.8$, $\theta = 0.5$, $\mu = 2$, and $\beta = 0.8$).

α	$\widetilde{\rho}$	$\lim_{t\to\infty} A_i(t)$	$\lim_{t\to\infty} \Phi_i(t)$	$\lim_{t\to\infty} M_i(t)/t$
0.02	0.8128	0.7765	0.0122	0.0153
0.12	0.8768	0.8506	0.0745	0.0931
0.22	0.9408	0.9271	0.1387	0.1734
0.32	1.0048	1	0.2038	0.2548
0.42	1.0688	1	0.2515	0.3144
0.52	1.1328	1	0.2938	0.3672

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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