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*Research Article*

# The Determinants, Inverses, Norm, and Spread of Skew Circulant Type Matrices Involving Any Continuous Lucas Numbers

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We consider the skew circulant and skew left circulant matrices with any continuous Lucas numbers. Firstly, we discuss the invertibility of the skew circulant matrices and present the determinant and the inverse matrices by constructing the transformation matrices. Furthermore, the invertibility of the skew left circulant matrices is also discussed. We obtain the determinants and the inverse matrices of the skew left circulant matrices by utilizing the relationship between skew left circulant matrices and skew circulant matrix, respectively. Finally, the four kinds of norms and bounds for the spread of these matrices are given, respectively.

## 1. Introduction

Circulant and skew-circulant matrices are appearing increasingly often in scientific and engineering applications. Briefly, scanning the recent literature, one can see their utility is appreciated in the design of digital filters [1–3], image processing [4–6], communications [7], signal processing [8], and encoding [9]. They have been put on firm basis with the work of Davis [10] and Jiang and Zhou [11].

The skew circulant matrices as preconditioners for linear multistep formulae- (LMF-) based ordinary differential equations (ODEs) codes. Hermitian and skew-Hermitian Toeplitz systems are considered in [12–15]. Lyness and Sørevik employed a skew circulant matrix to construct $s$-dimensional lattice rules in [16]. Spectral decompositions of skew circulant and skew left circulant matrices were discussed in [17]. Compared with cyclic convolution algorithm, the skew cyclic convolution algorithm [8] is able to perform filtering procedure in approximate half of computational cost for real signals. In [2] two new normal-form realizations are presented which utilize circulant and skew circulant matrices as their state transition matrices. The well-known second-order coupled form is a special case of the skew circulant form. Li et al. [18] gave the style spectral decomposition of skew circulant matrix firstly and then dealt with the optimal backward perturbation analysis for the linear system with skew circulant coefficient matrix. In [3], a new fast algorithm for optimal design of block digital filters (BDFs) was proposed based on skew circulant matrix.

Besides, some scholars have given various algorithms for the determinants and inverses of nonsingular circulant matrices [10, 11]. Unfortunately, the computational complexity of these algorithms is very amazing with the order of matrix increasing. However, some authors gave the explicit determinants and inverse of circulant and skew circulant involving some famous numbers. For example, Jaiswal evaluated some determinants of circulant whose elements are the generalized Fibonacci numbers [19]. Lind presented the determinants of circulant and skew circulant involving the Fibonacci numbers [20]. Dazheng [21] gave the determinant of the Fibonacci-Lucas quasicyclic matrices. Shen et al. considered circulant matrices with the Fibonacci and Lucas numbers and presented their explicit determinants and inverses by constructing the transformation matrices [22]. Gao et al. [23] gave explicit determinants and inverses of skew circulant and skew left circulant matrices with the Fibonacci and Lucas numbers. Jiang et al. [24, 25] considered the skew circulant and skew left circulant matrices with the $k$-Fibonacci numbers and the $k$-Lucas numbers and discussed the invertibility of the these matrices and presented their determinant and the inverse matrix by constructing the transformation matrices, respectively.

Recently, there are several papers on the norms of some special matrices. Solak [26] established the lower and upper bounds for the spectral norms of circulant matrices with the classical Fibonacci and Lucas numbers entries. İpek [27] investigated an improved estimation for spectral norms of these matrices. Shen and Cen [28] gave upper and lower bounds for the spectral norms of $r$-circulant matrices in the forms of $A = C_r(F_0, F_1, \dots, F_{n-1})$, $B = C_r(L_0, L_1, \dots, L_{n-1})$, and they also obtained some bounds for the spectral norms of Kronecker and Hadamard products of matrix $A$ and matrix $B$. Akbulak and Bozkurt [29] found upper and lower bounds for the spectral norms of Toeplitz matrices such that $a_{ij} \equiv F_{i-j}$ and $b_{ij} \equiv L_{i-j}$. The convergence in probability and in distribution of the spectral norm of scaled Toeplitz, circulant, reverse circulant, symmetric circulant, and a class of $k$-circulant matrices is discussed in [30].

Beginning with Mirsky [31], several authors [32–38] have obtained bounds for the spread of a matrix.

The purpose of this paper is to obtain the explicit determinants, explicit inverses, norm, and spread of skew circulant type matrices involving any continuous Lucas numbers. And we generalize the result [23]. In passing, the norm and spread of skew circulant type matrices have not been researched. It is hoped that this paper will help in changing this. More work continuing the present paper is forthcoming.

In the following, let $r$ be a nonnegative integer. We adopt the following two conventions $0^0 = 1$, and, for any sequence $\{a_n\}$, $\sum_{k=i}^{n} a_k = 0$ in the case $i > n$.

The Lucas sequences are defined by the following recurrence relations [21–23, 27–29]:

$$L_{n+1} = L_n + L_{n-1}, \quad \text{where } L_0 = 2,\ L_1 = 1, \tag{1}$$

for $n \geq 0$. The first few values of the sequences are given by the following table:

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| $L_n$ | 2 | 1 | 3 | 4 | 7 | 11 | 18 | 29 | 47 | 76 |

(2)

The $\{L_n\}$ is given by the formula

$$L_n = \alpha^n + \beta^n, \tag{3}$$

where $\alpha$ and $\beta$ are the roots of the characteristic equation $x^2 - x - 1 = 0$.

*Definition 1* (see [17]). A skew circulant matrix over $C$ with the first row $(a_1, a_2, \dots, a_n)$ is meant a square matrix of the form

$$\begin{pmatrix} a_1 & a_2 & \dots & a_{n-1} & a_n \\ -a_n & a_1 & a_2 & \dots & a_{n-1} \\ \vdots & -a_n & a_1 & \ddots & \vdots \\ -a_3 & \vdots & \ddots & \ddots & a_2 \\ -a_2 & -a_3 & \dots & -a_n & a_1 \end{pmatrix}_{n\times n}, \tag{4}$$

denoted by $\text{SCirc}(a_1, a_2, \dots, a_n)$.

*Definition 2* (see [17]). A skew left circulant matrix over $C$ with the first row $(a_1, a_2, \dots, a_n)$ is meant a square matrix of the form

$$\begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ a_2 & a_3 & \cdots & a_n & -a_1 \\ a_3 & \iddots & \iddots & \iddots & \vdots \\ \vdots & a_n & -a_1 & \cdots & -a_{n-2} \\ a_n & -a_1 & \cdots & -a_{n-2} & -a_{n-1} \end{pmatrix}_{n\times n}, \tag{5}$$

denoted by $\text{SLCirc}(a_1, a_2, \dots, a_n)$.

**Lemma 3** (see [10, 17]). *Let $A = SCirc(a_1, a_2, \dots, a_n)$ be skew circulant matrix; then*

(i) *$A$ is invertible if and only if the eigenvalues of $A$*

$$\lambda_k = f\left(\omega^k \eta\right) \neq 0, \quad (k = 0, 1, 2, \dots, n-1), \tag{6}$$

*where $f(x) = \sum_{j=1}^{n} a_j x^{j-1}$, $\omega = \exp(2\pi i/n)$, and $\eta = \exp(\pi i/n)$;*

(ii) *if $A$ is invertible, then the inverse of $A$ is a skew circulant matrix.*

**Lemma 4** (see [17]). *Let $A = SLCirc(a_1, a_2, \dots, a_n)$ be skew left circulant matrix and let $n$ be odd; then*

$$\lambda_j = \pm\left|\sum_{k=1}^{n} a_k \omega^{(j-(1/2))(k-1)}\right|, \quad \left(j = 1, 2, \dots, \frac{n-1}{2}\right),$$
$$\lambda_{(n+1)/2} = \sum_{k=1}^{n}\left|a_k(-1)^{k-1}\right|, \tag{7}$$

*where $\lambda_j$, $j = 1, 2, \dots, (n-1)/2$, $(n+1)/2$ are the eigenvalues of $A$.*

**Lemma 5** (see [23]). *With the orthogonal skew left circulant matrix*

$$\Theta := \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & \cdots & -1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & -1 & \cdots & 0 & 0 \end{pmatrix}_{n\times n}, \tag{8}$$

*it holds that*

$$SCirc\left(a_1, a_2, \dots, a_n\right) = \Theta\, SLCirc\left(a_1, a_2, \dots, a_n\right). \tag{9}$$

**Lemma 6** (see [23]). *If*

$$\left[SCirc\left(a_1, a_2, \dots, a_n\right)\right]^{-1} = SCirc\left(b_1, b_2, \dots, b_n\right), \tag{10}$$

*then*

$$\left[SLCirc\left(a_1, a_2, \dots, a_n\right)\right]^{-1} = SLCirc\left(b_1, -b_n, \dots, -b_2\right). \tag{11}$$

**Lemma 7** (see [27, 28]). *Let $\{L_n\}$ be the Lucas numbers; then*

$$(i)\ \sum_{i=0}^{n-1} L_i = L_{n+1} - 1, \tag{12}$$

$$(ii)\ \sum_{i=0}^{n-1} L_i^2 = L_n L_{n-1} + 2, \tag{13}$$

$$(iii)\ \sum_{i=0}^{n-1} iL_i = (n-1)L_{n+1} - L_{n+2} + 4. \tag{14}$$

*Definition 8* (see [29]). Let $A = (a_{ij})$ be an $n \times n$ matrix. The maximum column sum matrix norm, the spectral norm, the Euclidean (or Frobenius) norm, and the maximum row sum matrix norm of matrix $A$ are, respectively,

$$\begin{aligned} \|A\|_1 &= \max_{1\le j\le n} \sum_{i=1}^{n} \left|a_{ij}\right|, \\ \|A\|_2 &= \left(\max_{1\le i\le n} \lambda_i\left(A^* A\right)\right)^{1/2}, \\ \|A\|_F &= \left(\sum_{i,j=1}^{n} \left|a_{ij}\right|^2\right)^{1/2}, \\ \|A\|_\infty &= \max_{1\le i\le n} \sum_{j=1}^{n} \left|a_{ij}\right|, \end{aligned} \tag{15}$$

where $A^*$ denotes the conjugate transpose of $A$.

**Lemma 9** (see [30]). *If $A$ is an $n\times n$ real symmetric or normal matrix, then one has*

$$\|A\|_2 = \max_{1\le i\le n} \left|\lambda_i\right|, \tag{16}$$

*where $\lambda_i$ $(i = 1, 2, \ldots, n)$ are the eigenvalues of $A$.*

*Definition 10* (see [31, 32]). Let $A = (a_{ij})$ be an $n \times n$ matrix with eigenvalues $\lambda_i$, $i = 1, 2, \ldots n$. The spread of $A$ is defined as

$$s(A) = \max_{i,j} \left|\lambda_i - \lambda_j\right|. \tag{17}$$

Beginning with Mirsky [31], several authors [32–38] have obtained bounds for the spread of a matrix.

**Lemma 11.** *Let $A = (a_{ij})$ be an $n \times n$ matrix. An upper bound for the spread due to Mirsky [31] states that*

$$s(A) \leqslant \sqrt{2\|A\|_F^2 - \frac{2}{n}|\mathrm{tr}\, A|^2}, \tag{18}$$

*where $\|A\|_F$ denotes the Frobenius norm of $A$ and* tr $A$ *is trace of $A$.*

**Lemma 12** (see [38]). *Let $A = (a_{ij})$ be an $n \times n$ matrix; then*

(i) *if $A$ is real and normal, then*

$$s(A) \ge \frac{1}{n-1} \left|\sum_{i\ne j} a_{ij}\right|, \tag{19}$$

(ii) *and if $A$ is Hermitian, then*

$$s(A) \ge 2 \max_{i\ne j} \left|a_{ij}\right|. \tag{20}$$

## 2. Determinant and Inverse of Skew Circulant Matrix with the Lucas Numbers

In this section, let $A_{r,n} = \mathrm{SCirc}(L_{r+1}, \ldots, L_{r+n})$ be skew circulant matrix. Firstly, we give a determinant explicit formula for the matrix $A_{r,n}$. Afterwards, we prove that $A_{r,n}$ is an invertible matrix for $n \ge 2$, and then we find the inverse of the matrix $A_{r,n}$.

In the following, let $x = -((L_r+L_{r+n})/(L_{r+1}+L_{r+n+1}))$, $t = L_{r+2}/L_{r+1}$, $c = L_{r+1}+L_{r+n+1}$, $d = L_r+L_{r+n}$, $l_n = L_{r+1}+tL_{r+n}+\sum_{k=1}^{n-2}(tL_{r+k+1}-L_{r+k+2})\cdot x^{n-(k+1)}$ and $l_n' = \sum_{k=1}^{n-1} L_{r+k+1}\cdot x^{n-(k+1)}$.

**Theorem 13.** *Let $A_{r,n} = SCirc(L_{r+1}, \ldots, L_{r+n})$ be skew circulant matrix; then*

$$\det A_{r,n} = L_{r+1}\left[L_{r+1} + tL_{r+n} + \sum_{i=1}^{n-2}\left(tL_{r+i+1} - L_{r+i+2}\right)x^{n-(i+1)}\right]\cdot c^{n-2}, \tag{21}$$

*where $L_{r+n}$ is the $(r+n)$th Lucas number. Specially, when $r = 0$, one gets the result of [23].*

*Proof.* Obviously, $\det A_{r,1} = L_{r+1}$ satisfies the equation. In the case $n > 1$, let

$$\Sigma = \begin{pmatrix} 1 &&&&&&& \\ t &&&&&&& 1 \\ 1 &&&&&& 1 & -1 \\ 0 &&& 0 && 1 & -1 & -1 \\ \vdots &&&& \iddots & \iddots & \iddots & \\ 0 &&& 1 & \iddots & \iddots && \\ 0 && 1 & -1 & \iddots && 0 & \\ 0 & 1 & -1 & -1 &&&& \end{pmatrix}, \tag{22}$$

$$\Omega_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & x^{n-2} & 0 & \cdots & 0 & 1 \\ 0 & x^{n-3} & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & x & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

be two $n \times n$ matrices; then we have

$$\Sigma A_{r,n}\Omega_1 = \begin{pmatrix} L_{r+1} & l_n' & c_{13} & \cdots & c_{1,n-1} & c_{1n} \\ 0 & l_n & c_{23} & \cdots & c_{2,n-1} & c_{2n} \\ 0 & 0 & c &&& \\ 0 & 0 & d & \ddots && \\ \vdots & \vdots && \ddots & c & \\ 0 & 0 &&& d & c \end{pmatrix}, \tag{23}$$

where

$$c_{1j} = L_{r+n+2-j}, \quad c_{2j} = tL_{r+n+2-j} - L_{r+n+3-j}, \quad (j = 3, 4, \dots, n). \tag{24}$$

So it holds that

$$\begin{aligned} &\det \Sigma \det A_{r,n} \det \Omega_1 \\ &= L_{r+1} \left[ L_{r+1} + tL_{r+n} + \sum_{k=1}^{n-2} \left(tL_{r+k+1} - L_{r+k+2}\right) x^{n-(i+1)} \right] \\ &\quad \cdot \left(L_{r+1} + L_{r+n+1}\right)^{n-2}. \end{aligned} \tag{25}$$

While taking $\det \Sigma = \det \Omega_1 = (-1)^{(n-1)(n-2)/2}$, we have

$$\begin{aligned} &\det A_{r,n} \\ &= L_{r+1} \left[ L_{r+1} + tL_{r+n} + \sum_{k=1}^{n-2} \left(tL_{r+k+1} - L_{r+k+2}\right) x^{n-(i+1)} \right] \\ &\quad \cdot \left(L_{r+1} + L_{r+n+1}\right)^{n-2}. \end{aligned} \tag{26}$$

This completes the proof. □

**Theorem 14.** *Let $A_{r,n} = SCirc(L_{r+1}, \dots, L_{r+n})$ be skew circulant matrix; then $A_{r,n}$ is an invertible matrix. Specially, when $r = 0$, one gets the result of [23].*

*Proof.* Taking $n = 2$ in, Theorem 13, we have $\det A_{r,2} = L_{r+1}^2 + L_{r+2}^2 \neq 0$. Hence $A_{r,2}$ is invertible. In the case $n > 2$, since $L_{r+n} = \alpha^{r+n} + \beta^{r+n}$, where $\alpha + \beta = 1$, $\alpha\beta = -1$, we have

$$\begin{aligned} f\left(\omega^k \eta\right) &= \sum_{j=1}^{n} L_{r+j} \left(\omega^k \eta\right)^{j-1} \\ &= \sum_{j=1}^{n} \left(\alpha^{r+j} + \beta^{r+j}\right) \left(\omega^k \eta\right)^{j-1} \\ &= \frac{\alpha^{r+1}\left(1 + \alpha^n\right)}{1 - \alpha\omega^k \eta} + \frac{\beta^{r+1}\left(1 + \beta^n\right)}{1 - \beta\omega^k \eta} \\ &= \frac{L_{r+1} + L_{r+n+1} + \left(L_r + L_{r+n}\right)\omega^k \eta}{1 - \omega^k \eta - \omega^{2k}\eta^2} \\ &\qquad (k = 1, 2, \dots, n-1), \end{aligned} \tag{27}$$

where $\omega = \exp(2\pi i/n)$, $\eta = \exp(\pi i/n)$. If there exists $\omega^l \eta$ ($l = 1, 2, \dots, n-1$) such that $f(\omega^l \eta) = 0$, we obtain $L_{r+1} + L_{r+n+1} + (L_r + L_{r+n})\omega^l \eta = 0$, for $1 - \omega^l \eta - \omega^{2l}\eta^2 \neq 0$, and hence it follows that $\omega^l \eta = -((L_{r+1} + L_{r+n+1})/(L_r + L_{r+n}))$ is a real number. Since

$$\begin{aligned} \omega^l \eta &= \exp \frac{(2l+1)\pi i}{n} \\ &= \cos \frac{(2l+1)\pi}{n} + i \sin \frac{(2l+1)\pi}{n}, \end{aligned} \tag{28}$$

it yields that $\sin((2l+1)\pi/n) = 0$, so we have $\omega^l \eta = -1$ for $0 < (2l+1)\pi/n < 2\pi$. Since $x = -1$ is not the root of the equation $L_{r+1} + L_{r+n+1} + (L_r + L_{r+n})x = 0$ $(n > 2)$. We obtain $f(\omega^k \eta) \neq 0$, for any $\omega^k \eta$ ($k = 1, 2, \dots, n-1$), while

$$\begin{aligned} f(\eta) &= \sum_{j=1}^{n} L_j \eta^{j-1} \\ &= \frac{L_{r+1} + L_{r+n+1} + \left(L_r + L_{r+n}\right)\eta}{1 - \eta - \eta^2} \neq 0. \end{aligned} \tag{29}$$

It follows from Lemma 3 that the conclusion holds. □

**Lemma 15.** *Let the matrix $\mathscr{H} = [h_{ij}]_{i,j=1}^{n-2}$ be of the form*

$$h_{ij} = \begin{cases} L_{r+1} + L_{r+n+1} = c, & i = j, \\ L_r + L_{r+n} = d, & i = j+1, \\ 0, & \text{otherwise}. \end{cases} \tag{30}$$

*Then the inverse $\mathscr{H}^{-1} = [h'_{i,j}]_{i,j=1}^{n-2}$ of the matrix $\mathscr{H}$ is equal to*

$$h'_{ij} = \begin{cases} \dfrac{(-d)^{i-j}}{c^{i-j+1}}, & i \geq j, \\ 0, & i < j. \end{cases} \tag{31}$$

*Specially, when $r = 0$, one gets the result of [23].*

*Proof.* Let $e_{ij} = \sum_{k=1}^{n-2} h_{ik} h'_{kj}$. Obviously, $e_{ij} = 0$ for $i < j$. In the case $i = j$, we obtain $e_{ii} = h_{ii} h'_{ii} = (L_{r+1} + L_{r+n+1}) \cdot (1/(L_{r+1} + L_{r+n+1})) = 1$. For $i \geq j+1$, we obtain

$$\begin{aligned} e_{ij} &= \sum_{k=1}^{n-2} h_{ik} h'_{kj} = h_{i,i-1} h'_{i-1,j} + h_{ii} h'_{ij} \\ &= d \cdot \frac{(-d)^{i-j-1}}{c^{i-j}} + c \cdot \frac{(-d)^{i-j}}{c^{i-j+1}} = 0. \end{aligned} \tag{32}$$

Hence, we get $\mathscr{H}\mathscr{H}^{-1} = I_{n-2}$, where $I_{n-2}$ is $(n-2) \times (n-2)$ identity matrix. Similarly, we can verify that $\mathscr{H}^{-1}\mathscr{H} = I_{n-2}$. Thus, the proof is completed. □

**Theorem 16.** *Let $A_{r,n} = SCirc(L_{r+1}, \dots, L_{r+n})$ be skew circulant matrix; then*

$$\left(A_{r,n}\right)^{-1} = \frac{1}{l_n} \cdot SCirc\left(y'_1, y'_2, \dots, y'_n\right), \tag{33}$$

*where*

$$y_1' = 1 - \left[ \left(L_{r+3} - tL_{r+2}\right) \cdot \frac{(-d)^{n-3}}{c^{n-2}} + \sum_{i=1}^{n-3} \left(L_{r+n+2-i} - tL_{r+n+1-i}\right) \cdot \frac{(-d)^{i-1}}{c^i} \right],$$

$$y_2' = -t - \sum_{i=1}^{n-2} \left(L_{r+n+1-i} - tL_{r+n-i}\right) \cdot \frac{(-d)^{i-1}}{c^i},$$

$$y_3' = -\left(L_{r+3} - tL_{r+2}\right) \cdot \frac{1}{c}, \quad (34)$$

$$y_4' = -\sum_{i=1}^{2} \left(L_{r+1+i} - tL_{r+i}\right) \cdot \frac{(-d)^{i-1}}{c^i},$$

$$y_k' = -\sum_{i=1}^{2} \left(L_{r+1+i} - tL_{r+i}\right) \cdot \frac{(-d)^{k-5+i}}{c^{k-4+i}} \quad (k = 5, 6, \ldots, n).$$

*Specially, when $r = 0$, one gets the result of [23].*

*Proof.* Let

$$\Omega_2 = \begin{pmatrix} 1 & -\dfrac{l_n'}{L_{r+1}} & \omega_{13} & \omega_{14} & \cdots & \omega_{1n} \\ 0 & 1 & \omega_{23} & \omega_{24} & \cdots & \omega_{2n} \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \quad (35)$$

where

$$\omega_{1j} = \frac{1}{L_{r+1}} \left[ \frac{l_n'}{l_n} \left(tL_{r+n+2-j} - L_{r+n+3-j}\right) - L_{r+n+2-j} \right]$$
$$\omega_{2j} = \frac{1}{l_n} \cdot \left(L_{r+n+3-j} - tL_{r+n+2-j}\right) \quad (j = 3, 4, \ldots, n). \quad (36)$$

Then, we have

$$\Sigma A_{r,n} \Omega_1 \Omega_2 = \begin{pmatrix} L_{r+1} & 0 & 0 & 0 & \cdots & 0 \\ 0 & l_n & 0 & 0 & \cdots & 0 \\ 0 & 0 & c & 0 & \cdots & 0 \\ 0 & 0 & d & c & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & c \end{pmatrix}, \quad (37)$$

so $\Sigma A_{r,n} \Omega_1 \Omega_2 = \mathscr{D} \oplus \mathscr{H}$, where $D = \operatorname{diag}(L_{r+1}, l_n)$ is a diagonal matrix and $\mathscr{D} \oplus \mathscr{H}$ is the direct sum of $\mathscr{D}$ and $\mathscr{H}$. If we denote $\Omega = \Omega_1 \Omega_2$, then we obtain $A_{r,n}^{-1} = \Omega(\mathscr{D}^{-1} \oplus \mathscr{H}^{-1})\Sigma$.

Since the last row elements of the matrix $\Omega$ are $(0, 1, \omega_{23}, \omega_{24}, \ldots, \omega_{2,n-1}, \omega_{2n})$, then the last row elements of the matrix $\Omega(\mathscr{D}^{-1} \oplus \mathscr{H}^{-1})$ are $(0, 1/l_n, T_{23}, T_{24}, \ldots, T_{2n})$, where

$$T_{23} = \sum_{i=1}^{n-2} \omega_{2,2+i} \cdot \frac{(-d)^{i-1}}{c^i},$$
$$T_{2k} = \sum_{i=1}^{n+1-k} \omega_{2,k-1+i} \cdot \frac{(-d)^{i-1}}{c^i} \quad (k = 3, 4, \ldots, n). \quad (38)$$

Hence, it follows from Lemma 15 that letting $A_{r,n}^{-1} = \operatorname{SCirc}(y_1, y_2, \ldots, y_n)$, then its last row elements are $(-y_2, -y_3, \ldots, -y_n, y_1)$ which are given by the following equations:

$$-y_2 = \frac{t}{l_n} + T_{23} = \frac{t}{l_n} + \frac{1}{l_n} \sum_{i=1}^{n-2} \left(L_{r+n+1-i} - tL_{r+n-i}\right) \cdot \frac{(-d)^{i-1}}{c^i},$$

$$-y_3 = T_{2,n} = \frac{1}{l_n} \left(L_{r+3} - tL_{r+2}\right) \cdot \frac{1}{c},$$

$$-y_4 = T_{2,n-1} - T_{2n} = \frac{1}{l_n} \sum_{i=1}^{2} \left(L_{r+1+i} - tL_{r+i}\right) \cdot \frac{(-d)^{i-1}}{c^i},$$

$$-y_5 = T_{2,n-2} - T_{2n-1} - T_{2n} = \frac{1}{l_n} \sum_{i=1}^{2} \left(L_{r+1+i} - tL_{r+i}\right) \cdot \frac{(-d)^{i}}{c^{i+1}},$$

$$-y_k = T_{2,n-k+3} - T_{2,n-k+4} - T_{2,n-k+5} = \frac{1}{l_n} \sum_{i=1}^{2} \left(L_{r+1+i} - tL_{r+i}\right) \cdot \frac{(-d)^{k-5+i}}{c^{k-4+i}},$$

$$\vdots$$

$$-y_n = T_{23} - T_{24} - T_{25} = \sum_{i=1}^{n-2} \omega_{2,2+i} \cdot \frac{(-d)^{i-1}}{c^i} - \sum_{i=1}^{n-3} \omega_{2,3+i} \cdot \frac{(-d)^{i-1}}{c^i} - \sum_{i=1}^{n-4} \omega_{2,4+i} \cdot \frac{(-d)^{i-1}}{c^i}$$
$$= \frac{1}{l_n} \sum_{i=1}^{2} \left(L_{r+1+i} - tL_{r+i}\right) \cdot \frac{(-d)^{n-5+i}}{c^{n-4+i}},$$

$$y_1 = \frac{1}{l_n} - T_{23} - T_{24}$$
$$= \frac{1}{l_n} - \frac{1}{l_n}\left[\left(L_{r+3} - tL_{r+2}\right) \cdot \frac{(-d)^{n-3}}{c^{n-2}} + \sum_{i=1}^{n-3}\left(L_{r+n+2-i} - tL_{r+n+1-i}\right) \cdot \frac{(-d)^{i-1}}{c^i}\right]. \quad (39)$$

Hence, we obtain

$$y_1 = \frac{1}{l_n} - \frac{1}{l_n}\left[\left(L_{r+3} - tL_{r+2}\right) \cdot \frac{(-d)^{n-3}}{c^{n-2}} + \sum_{i=1}^{n-3}\left(L_{r+n+2-i} - tL_{r+n+1-i}\right) \frac{(-d)^{i-1}}{c^i}\right],$$
$$y_2 = -\frac{t}{l_n} - \frac{1}{l_n}\sum_{i=1}^{n-2}\left(L_{r+n+1-i} - tL_{r+n-i}\right) \cdot \frac{(-d)^{i-1}}{c^i},$$
$$y_3 = -\frac{1}{l_n}\left(L_{r+3} - tL_{r+2}\right) \cdot \frac{1}{c},$$
$$y_4 = -\frac{1}{l_n}\sum_{i=1}^{2}\left(L_{r+1+i} - tL_{r+i}\right) \cdot \frac{(-d)^{i-1}}{c^i},$$
$$y_5 = -\frac{1}{l_n}\sum_{i=1}^{2}\left(L_{r+1+i} - tL_{r+i}\right) \cdot \frac{(-d)^{i}}{c^{i+1}}, \quad (40)$$
$$y_k = -\frac{1}{l_n}\sum_{i=1}^{2}\left(L_{r+1+i} - tL_{r+i}\right) \cdot \frac{(-d)^{k-5+i}}{c^{k-4+i}},$$
$$\vdots$$
$$y_n = \frac{1}{l_n}\sum_{i=1}^{2}\left(L_{r+1+i} - tL_{r+i}\right) \cdot \frac{(-d)^{n-5+i}}{c^{n-4+i}},$$
$$A_{r,n}^{-1} = \frac{1}{l_n} \cdot \mathrm{SCirc}\left(y_1', y_2', \ldots, y_n'\right),$$

where

$$y_1' = 1 - \left[\left(L_{r+3} - tL_{r+2}\right) \cdot \frac{(-d)^{n-3}}{c^{n-2}} + \sum_{i=1}^{n-3}\left(L_{r+n+2-i} - tL_{r+n+1-i}\right) \cdot \frac{(-d)^{i-1}}{c^i}\right],$$
$$y_2' = -t - \sum_{i=1}^{n-2}\left(L_{r+n+1-i} - tL_{r+n-i}\right) \cdot \frac{(-d)^{i-1}}{c^i},$$
$$y_3' = -\left(L_{r+3} - tL_{r+2}\right) \cdot \frac{1}{c},$$
$$y_4' = -\sum_{i=1}^{2}\left(L_{r+1+i} - tL_{r+i}\right) \cdot \frac{(-d)^{i-1}}{c^i},$$
$$y_k' = -\sum_{i=1}^{2}\left(L_{r+1+i} - tL_{r+i}\right) \cdot \frac{(-d)^{k-5+i}}{c^{k-4+i}}, \quad (k = 5, 6, \ldots, n). \quad (41)$$

This completes the proof. □

## 3. Norm and Spread of Skew Circulant Matrix with the Lucas Numbers

**Theorem 17.** *Let $A_{r,n} = SCirc(L_{r+1}, \ldots, L_{r+n})$ be skew circulant matrix; then three kinds of norms of $A_{r,n}$ are given by*

$$\left\|A_{r,n}\right\|_1 = \left\|A_{r,n}\right\|_\infty = L_{r+n+2} - L_{r+2}, \quad (42)$$
$$\left\|A_{r,n}\right\|_F = \sqrt{n\left(L_{r+n}L_{r+n+1} - L_rL_{r+1}\right)}. \quad (43)$$

*Proof.* By Definition 8 and (12), we have

$$\left\|A_{r,n}\right\|_1 = \left\|A_{r,n}\right\|_\infty = \sum_{i=1}^{n} L_{r+i} = L_{r+n+2} - L_{r+2}. \quad (44)$$

By Definition 8 and (13), we have

$$\left(\left\|A_{r,n}\right\|_F\right)^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}\left|a_{ij}\right|^2 = n\sum_{i=1}^{n} L_{r+i}^2 = n\left(\sum_{i=0}^{r+n} L_i^2 - \sum_{i=0}^{r} L_i^2\right) = n\left(L_{r+n}L_{r+n+1} - L_rL_{r+1}\right). \quad (45)$$

Thus

$$\left\|A_{r,n}\right\|_F = \sqrt{n\left(L_{r+n}L_{r+n+1} - L_rL_{r+1}\right)}. \quad (46)$$

□

**Theorem 18.** *Let*

$$A_{r,n}' = SCirc\left(L_{r+1}, -L_{r+2}, \ldots, -L_{r+n-1}, L_{r+n}\right) \quad (47)$$

*be an odd-order alternative skew circulant matrix and let n be odd. Then*

$$\left\|A_{r,n}'\right\|_2 = \sum_{i=1}^{n} L_{r+i} = L_{r+n+2} - L_{r+2}. \quad (48)$$

*Proof.* By Lemma 3, we have

$$\lambda_j\left(A_{r,n}'\right) = \sum_{i=1}^{n}(-1)^{i-1} L_{r+i}\left(\omega^j\eta\right)^{i-1}. \quad (49)$$

So

$$\left|\lambda_j\left(A'_{r,n}\right)\right| \leq \sum_{i=1}^{n}\left|(-1)^{i-1}L_{r+i}\right| \cdot \left|\left(\omega^j\eta\right)^{i-1}\right| = \sum_{i=1}^{n} L_{r+i}, \quad (50)$$

for all $j = 0, 1, \ldots, n-1$.

Since $n$ is odd, $\sum_{i=1}^{n} L_{r+i}$ is an eigenvalue of $A'_{r,n}$; that is,

$$\begin{pmatrix} L_{r+1} & -L_{r+2} & \vdots & L_{r+n} \\ -L_{r+n} & L_{r+1} & & -L_{r+n-1} \\ L_{r+n-1} & -L_{r+n} & & L_{r+n-2} \\ \vdots & \vdots & \vdots & \vdots \\ L_{r+2} & -L_{r+3} & & L_{r+1} \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ \vdots \\ 1 \end{pmatrix} = \sum_{i=1}^{n} L_{r+i} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ \vdots \\ 1 \end{pmatrix}. \quad (51)$$

To sum up, we have

$$\max_{0\leq j\leq n-1}\left|\lambda_j\left(A'_{r,n}\right)\right| = \sum_{i=1}^{n} L_{r+i}. \quad (52)$$

Since all skew circulant matrices are normal, by Lemma 9 and (12), and (52), we have

$$\left\|A'_{r,n}\right\|_2 = \sum_{i=1}^{n} L_{r+i} = L_{r+n+2} - L_{r+2}, \quad (53)$$

which completes the proof. □

**Theorem 19.** *Let $A_{r,n} = SCirc(L_{r+1}, \ldots, L_{r+n})$ be skew circulant matrix; then the bounds for the spread of $A_{r,n}$ are*

$$s\left(A_{r,n}\right) \leqslant \sqrt{2n\left(L_{r+n}L_{r+n+1} - L_{r+1}L_{r+2}\right)},$$
$$s\left(A_{r,n}\right) \geq \frac{1}{n-1}\left|2L_{r+n+3} - (n-2)L_{r+n+2} - nL_{r+3} - 2L_{r+4}\right|. \quad (54)$$

*Proof.* The trace of $A_{r,n}$, $\operatorname{tr} A_{r,n} = nL_{r+1}$. By (18) and (43), we have

$$s\left(A_{r,n}\right) \leqslant \sqrt{2n\left(L_{r+n}L_{r+n+1} - L_{r+1}L_{r+2}\right)}. \quad (55)$$

Since

$$\begin{aligned}\sum_{i\neq j} a_{ij} &= \sum_{k=2}^{n}(n-(k-1))L_{r+k} - \sum_{k=2}^{n}(k-1)L_{r+k} \\ &= (n+2)\sum_{k=2}^{n}L_{r+k} - 2\sum_{k=2}^{n}kL_{r+k} \\ &= (n+2)\left(L_{r+n+2} - L_{r+3}\right) \\ &\quad - 2\left[\sum_{k=2}^{n}(r+k)L_{r+k} - \sum_{k=2}^{n}rL_{r+k}\right],\end{aligned} \quad (56)$$

by (12) and (14),

$$\sum_{i\neq j} a_{ij} = 2L_{r+n+3} - (n-2)L_{r+n+2} - nL_{r+3} - 2L_{r+4}. \quad (57)$$

By (19), we have

$$s\left(A_{r,n}\right) \geq \frac{1}{n-1}\left|2L_{r+n+3} - (n-2)L_{r+n+2} - nL_{r+3} - 2L_{r+4}\right|. \quad (58)$$

□

## 4. Determinant and Inverse of Skew Left Circulant Matrix with the Lucas Numbers

In this section, let $A''_{r,n} = \mathrm{SLCirc}(L_{r+1}, \ldots, L_{r+n})$ be skew left circulant matrix. By using the obtained conclusions in Section 2, we give a determinant explicit formula for the matrix $A''_{r,n}$. Afterwards, we prove that $A''_{r,n}$ is an invertible matrix for any positive interger $n$. The inverse of the matrix $A''_{r,n}$ is also presented.

According to Lemmas 5 and 6 and Theorems 13, 14, and 16, we can obtain the following theorems.

**Theorem 20.** *Let $A''_{r,n} = SLCirc(L_{r+1}, \ldots, L_{r+n})$ be skew left circulant matrix; then*

$$\det A''_{r,n} = (-1)^{n(n-1)/2} L_{r+1} \times \left[L_{r+1} + tL_{r+n} + \sum_{k=1}^{n-2}\left(tL_{r+1+i} - L_{r+2+i}\right)x^{n-1-i}\right] \cdot c^{n-2}, \quad (59)$$

*where $L_{r+n}$ is the $(r+n)$th Lucas number.*

**Theorem 21.** *Let $A''_{r,n} = SLCirc(L_{r+1}, \ldots, L_{r+n})$ be skew left circulant matrix; then $A''_{r,n}$ is an invertible matrix.*

**Theorem 22.** *Let $A''_{r,n} = SLCirc(L_{r+1}, \ldots, L_{r+n})$ be skew left circulant matrix; then*

$$\left(A''_{r,n}\right)^{-1} = \frac{1}{l_n} SLCirc\left(y''_1, y''_2, \ldots, y''_n\right), \quad (60)$$

*where*

$$y''_1 = 1 - \left[\left(L_{r+3} - tL_{r+2}\right)\frac{(-d)^{n-3}}{c^{n-2}} + \sum_{i=1}^{n-3}\left(L_{r+n+2-i} - tL_{r+n+1-i}\right)\cdot\frac{(-d)^{i-1}}{c^i}\right],$$

$$\begin{aligned} y_k'' &= -y_{n-k+2}' \\ &= \sum_{i=1}^{2}\left(L_{r+1+i} - tL_{r+i}\right)\cdot\frac{(-d)^{n-k-3+i}}{c^{n-k-2+i}}, \\ &\qquad (k = 2, 3, \dots, n-2). \\ y_{n-1}'' &= -y_3' = \left(L_{r+3} - tL_{r+2}\right)\cdot\frac{1}{c}, \\ y_n'' &= -y_2' \\ &= t + \sum_{i=1}^{n-2}\left(L_{r+n+1-i} - tL_{r+n-i}\right)\cdot\frac{(-d)^{i-1}}{c^i}. \end{aligned} \tag{61}$$

## 5. Norm and Spread of Skew Left Circulant Matrix with the Lucas Numbers

**Theorem 23.** *Let $A''_{r,n} = SLCirc(L_{r+1}, \dots, L_{r+n})$ be skew left circulant matrix. Then three kinds of norms of $A''_{r,n}$ are given by*

$$\begin{aligned} \left\|A''_{r,n}\right\|_1 &= \left\|A_{r,n}\right\|_\infty = L_{r+n+2} - L_{r+2}, \\ \left\|A''_{r,n}\right\|_F &= \sqrt{n\left(L_{r+n}L_{r+n+1} - L_rL_{r+1}\right)}. \end{aligned} \tag{62}$$

*Proof.* Using the method in Theorem 17 similarly, the conclusion is obtained. □

**Theorem 24.** *Let*

$$A'''_{r,n} = SLCirc\left(L_{r+1}, -L_{r+2}, \dots, -L_{r+n-1}, L_{r+n}\right) \tag{63}$$

*be an odd-order alternative skew left circulant matrix; then*

$$\left\|A'''_{r,n}\right\|_2 = \sum_{i=1}^{n} L_{r+i} = L_{r+n+2} - L_{r+2}. \tag{64}$$

*Proof.* According to Lemma 4,

$$\lambda_j\left(A'''_{r,n}\right) = \pm\left|\sum_{i=1}^{n}(-1)^{i-1}L_{r+i}\omega^{(j-(1/2))(k-1)}\right|, \tag{65}$$

for $j = 1, 2, \dots, (n-1)/2$, and

$$\lambda_{(n+1)/2}\left(A'''_{r,n}\right) = \sum_{i=1}^{n} L_{r+i}. \tag{66}$$

So

$$\begin{aligned} \left|\lambda_j\left(A'''_{r,n}\right)\right| &\le \sum_{i=1}^{n}\left|(-1)^{i-1}L_{r+i}(-1)^{i-1}\right| \\ &= \sum_{i=1}^{n} L_{r+i}, \quad \left(j = 1, 2, \dots, \frac{n+1}{2}\right). \end{aligned} \tag{67}$$

By (66) and (67), we have

$$\max_{0\le i\le (n+1)/2}\left|\lambda_i\left(A'''_{r,n}\right)\right| = \sum_{i=1}^{n} L_{r+i}. \tag{68}$$

Since all skew left circulant matrices are symmetrical, by Lemma 9 and (12) and (68), we obtain

$$\left\|A'''_{r,n}\right\|_2 = L_{r+n+2} - L_{r+2}. \tag{69}$$

□

**Theorem 25.** *Let $A''_{r,n} = SLCirc(L_{r+1}, \dots, L_{r+n})$ be skew left circulant matrix; the bounds for the spread of $A''_{r,n}$ are*

$$2L_{r+n} \le s\left(A''_{r,n}\right) \le \begin{cases} \sqrt{M - \dfrac{2}{n}N^2}, & \text{if } n \text{ is odd}, \\ \sqrt{M}, & \text{if } n \text{ is even}, \end{cases} \tag{70}$$

*where*

$$\begin{aligned} M &= 2n\left(L_{r+n}L_{r+n+1} - L_{r+1}L_r\right), \\ N &= L_{r+n-1} + L_{r-1}. \end{aligned} \tag{71}$$

*Proof.* Since $A''_{r,n}$ is a symmetric matrix, by (20),

$$s\left(A''_{r,n}\right) \ge 2\max_{i\ne j}\left|a_{ij}\right| = 2L_{r+n}. \tag{72}$$

The trace of $A''_{r,n}$ is, if $n$ is odd,

$$\begin{aligned} \operatorname{tr}&\left(A''_{r,n}\right) \\ &= L_{r+1} - L_{r+2} + L_{r+3} - \cdots + L_{r+n} \\ &= L_{r+1} + L_{r+1} + L_{r+3} + \cdots + L_{r+n-2} \\ &= 2L_{r+1} + L_{r+1} + L_{r+2} + \cdots + L_{r+n-3} \\ &= 2L_{r+1} + \sum_{i=1}^{n-3} L_{r+i}. \end{aligned} \tag{73}$$

By (12), we have

$$\operatorname{tr}\left(A''_{r,n}\right) = L_{r+n-1} + L_{r-1} = N. \tag{74}$$

Let $M = 2n(L_{r+n}L_{r+n+1} - L_{r+1}L_r)$; then, by (18), (62), and (74), we obtain

$$s\left(A''_{r,n}\right) \leqslant \sqrt{M - \frac{2}{n}N^2}. \tag{75}$$

If $n$ is even, then

$$\begin{aligned} \operatorname{tr}\left(A''_{r,n}\right) &= L_{r+1} - L_{r+1} + L_{r+3} \\ &\quad - L_{r+3}\cdots - L_{r+n-1} = 0. \end{aligned} \tag{76}$$

By (18), (62), and (76), we have

$$s\left(A''_{r,n}\right) \leqslant \sqrt{M}. \tag{77}$$

So the result follows. □

## 6. Conclusion

We discuss the invertibility of the skew circulant type matrices with any continuous Lucas numbers and present the determinant and the inverse matrices by constructing the transformation matrices. The four kinds of norms and bounds for the spread of these matrices are given, respectively. In [3], a new fast algorithm for optimal design of block digital filters (BDFs) is proposed based on skew circulant matrix. The reason why we focus our attention on skew circulant is to explore the application of skew circulant in the related field in medicine image, image encryption, and real-time tracking. On the basis of existing application situation [4], we conjecture that SVD decomposition of skew circulant matrix will play an important role in CT-perfusion imaging of human brain. On the basis method of [8] and ideas of [5], we will exploit real-time tracking with kernel matrix of skew circulant structure. A novel chaotic image encryption scheme based on the time-delay Lorenz system is presented in [6] with the description of circulant matrix. We will exploit chaotic image encryption algorithm based on skew circulant operation.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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