

## Research Article

# On Positive Solutions of a Fourth Order Nonlinear Neutral Delay Difference Equation

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The existence results of uncountably many bounded positive solutions for a fourth order nonlinear neutral delay difference equation are proved by means of the Krasnoselskii's fixed point theorem and Schauder's fixed point theorem. A few examples are included.

## 1. Introduction

In the past few decades, the researchers [1–31] and others studied oscillation, asymptotic behavior, and solvability for a lot of second and third order nonlinear difference equations, some of which are as follows:

$$\begin{aligned}
 &\Delta(a_n \Delta(x_n + px_{n-\tau})) + F(n+1, x_{n+1-\sigma}) = 0, \quad n \geq 1, \\
 &\Delta(a_n \Delta(x_n + bx_{n-\tau})) \\
 &\quad + f(n, x_{n-d_{1n}}, \dots, x_{n-d_{kn}}) = c_n, \quad n \geq n_0, \\
 &\Delta^3 x_n + f(n, x_n, x_{n-\tau}) = 0, \quad n \geq n_0, \\
 &\Delta(a_n \Delta^2(x_n + P_n x_{n-\tau})) \\
 &\quad + f(n, x_{n-d_{1n}}, \dots, x_{n-d_{kn}}) = g_n, \quad n \geq n_0, \\
 &\Delta^2(a_n \Delta(x_n + \gamma_n x_{n-\tau})) + \Delta^2 f(n, x_{n-d_{1n}}, \dots, x_{n-d_{kn}}) \\
 &\quad + \Delta g(n, x_{n-d_{1n}}, \dots, x_{n-d_{kn}}) \\
 &= h(n, x_{n-d_{1n}}, \dots, x_{n-d_{kn}}), \quad n \geq n_0.
 \end{aligned} \tag{1}$$

By employing a few famous tools in nonlinear analysis including the nonlinear alternative of Leray-Schauder type, Banach's fixed point theorem, Schauder's fixed point theorem,

Krasnoselskii's fixed point theorem, coincidence degree theory and critical point theory, the authors [3, 4, 9, 13–16, 18, 25, 27, 28] and others proved the existence of nonoscillatory solutions, uncountably many bounded nonoscillatory solutions and periodic solutions for the difference equations above, where Lipschitz conditions were used in [14, 16]. Recently, the authors [32] used the Krasnoselskii's fixed point theorem to obtain  $h$ -asymptotic stability results about the zero solution for a very general first order nonlinear neutral differential equation with functional delay.

However, to our knowledge, no one studied the following fourth order nonlinear neutral delay difference equation:

$$\begin{aligned}
 &\Delta^3(a_n \Delta(x_n + \gamma_n x_{n-\tau})) + \Delta^3 f(n, x_{b_{1n}}, \dots, x_{b_{kn}}) \\
 &\quad + \Delta^2 g(n, x_{c_{1n}}, \dots, x_{c_{kn}}) + \Delta h(n, x_{d_{1n}}, \dots, x_{d_{kn}}) \\
 &\quad + p(n, x_{o_{1n}}, \dots, x_{o_{kn}}) = r_n, \quad n \geq n_0,
 \end{aligned} \tag{2}$$

where  $\tau, k \in \mathbb{N}$ ,  $n_0 \in \mathbb{N}_0$ ,  $f, g, h, p \in C(\mathbb{N}_{n_0} \times \mathbb{R}^k, \mathbb{R})$ ,  $\{a_n\}_{n \in \mathbb{N}_{n_0}}$ , and  $\{\gamma_n\}_{n \in \mathbb{N}_{n_0}}$  are real sequences with  $a_n \neq 0$  for  $n \in \mathbb{N}_{n_0}$  and  $\{b_{ln}, c_{ln}, d_{ln}, o_{ln} : n \in \mathbb{N}_{n_0}, l \in \{1, 2, \dots, k\}\} \subset \mathbb{Z}$  with

$$\lim_{n \rightarrow \infty} b_{ln} = \lim_{n \rightarrow \infty} c_{ln} = \lim_{n \rightarrow \infty} d_{ln} = \lim_{n \rightarrow \infty} o_{ln} = +\infty, \tag{3}$$

$l \in \{1, 2, \dots, k\}$ .

The purpose of this paper is to fill this gap in the literature and to study solvability of (2). Under certain conditions, we prove the existence of uncountably many bounded positive solutions of (2) by means of the Krasnoselskii's fixed point theorem and Schauder's fixed point theorem, respectively. Nine examples are included.

This paper is organized as follows. In Section 2 we present some notations, definitions, and lemmas. In Section 3 we establish nine sufficient conditions which guarantee the existence of uncountably many bounded positive solutions of (2) by using fixed point theorems and new techniques. In Section 4 we give nine examples to illustrate the effectiveness and applications of the results presented in Section 3.

## 2. Preliminaries

Throughout this paper, we assume that  $\mathbb{R} = (-\infty, +\infty)$ ,  $\mathbb{Z}$ ,  $\mathbb{N}$  and  $\mathbb{N}_0$  stand for the sets of all integers, positive integers, and nonnegative integers, respectively:

$$\begin{aligned} \mathbb{N}_{n_0} &= \{n : n \in \mathbb{N} \text{ with } n \geq n_0\}, \quad n_0 \in \mathbb{N}_0, \\ \alpha &= \inf \{b_{ln}, d_{ln}, c_{ln}, o_{ln} : 1 \leq l \leq k, n \in \mathbb{N}_{n_0}\}, \\ \beta &= \min \{n_0 - \tau, \alpha\}, \\ \mathbb{Z}_\beta &= \{n : n \in \mathbb{Z} \text{ with } n \geq \beta\}; \end{aligned} \quad (4)$$

$\Delta$  denotes the forward different operator defined by  $\Delta x_n = x_{n+1} - x_n$  and  $\Delta^i x_n = \Delta(\Delta^{i-1} x_n)$  for  $i \in \{2, 3, 4\}$ . Let  $l_\beta^\infty$  denote the Banach space of all bounded sequences on  $\mathbb{Z}_\beta$  with norm:

$$\|x\| = \sup_{n \in \mathbb{Z}_\beta} |x_n| \quad \text{for } x = \{x_n\}_{n \in \mathbb{Z}_\beta} \in l_\beta^\infty,$$

$$\Omega_1(N, M)$$

$$= \left\{ x = \{x_n\}_{n \in \mathbb{Z}_\beta} \in l_\beta^\infty : N \leq x_n \leq M, n \in \mathbb{Z}_\beta \right\};$$

$$\begin{aligned} \Omega_{2T}(N, M) &= \left\{ x = \{x_n\}_{n \in \mathbb{Z}_\beta} \in l_\beta^\infty : \frac{N}{\gamma_n} \leq x_n \leq \frac{M}{\gamma_n}, \right. \\ &\quad \left. n \geq T; \frac{N}{\gamma_T} \leq x_n \leq \frac{M}{\gamma_T}, \beta \leq n < T \right\}; \end{aligned}$$

$$\begin{aligned} \Omega_{3T}(N, M) &= \left\{ x = \{x_n\}_{n \in \mathbb{Z}_\beta} \in l_\beta^\infty : \frac{-N}{\gamma_n} \leq x_n \leq \frac{-M}{\gamma_n}, \right. \\ &\quad \left. n \geq T; \frac{-N}{\gamma_T} \leq x_n \leq \frac{-M}{\gamma_T}, \beta \leq n < T \right\}. \end{aligned} \quad (5)$$

Obviously,  $\Omega_1(N, M)$ ,  $\Omega_{2T}(N, M)$ , and  $\Omega_{3T}(N, M)$ , are closed bounded and convex subsets of  $l_\beta^\infty$  for any  $M > N > 0$ .

By a solution of (2), we mean a real sequence  $\{x_n\}_{n \in \mathbb{Z}_\beta}$  with a positive integer  $T \geq n_0 + \tau + |\beta|$  such that (2) is satisfied for all  $n \geq T$ .

*Definition 1* (see [5]). A subset  $D$  of  $l_\beta^\infty$  is said to be uniformly Cauchy (or equi-Cauchy) if for every  $\varepsilon > 0$  there exists a positive integer  $K > \beta$  such that

$$|x_i - x_j| < \varepsilon, \quad (6)$$

whenever  $i, j > K$  for any  $x = \{x_n\}_{n \in \mathbb{Z}_\beta} \in l_\beta^\infty$ .

**Lemma 2** (see [5]). *Each bounded and uniformly Cauchy subset of  $l_\beta^\infty$  is relatively compact.*

**Lemma 3** (Krasnoselskii's fixed point theorem). *Let  $X$  be a Banach space, let  $D$  be a bounded closed convex subset of  $X$ , and let  $S, G$  be mappings from  $D$  into  $X$  such that  $Sx + Gy \in D$  for every pair  $x, y \in D$ . If  $S$  is a contraction and  $G$  is completely continuous, then the equation*

$$Sx + Gx = x \quad (7)$$

has a solution in  $D$ .

**Lemma 4** (Schauder's fixed point theorem). *Let  $D$  be a nonempty closed convex subset of a Banach space  $X$ ,  $T : D \rightarrow D$  continuous, and  $f(D)$  relatively compact. Then  $f$  has at least one fixed point in  $D$ .*

**Lemma 5.** *Let  $\tau \in \mathbb{N}$ ,  $n_0 \in \mathbb{N}_0$ ,  $\{a_n\}_{n \in \mathbb{N}_{n_0}}$ ,  $\{b_n\}_{n \in \mathbb{N}_{n_0}}$ , and  $\{c_n\}_{n \in \mathbb{N}_{n_0}}$  be nonnegative sequences. If*

$$\sum_{j=n_0}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} j c_j a_s b_t < +\infty, \quad (8)$$

then

$$\begin{aligned} &\sum_{i=1}^{\infty} \sum_{j=n_0+i\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} c_j a_s b_t \\ &= \sum_{j=n_0+\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \left[ \frac{j-n_0}{\tau} \right] c_j a_s b_t \\ &\leq \frac{1}{\tau} \sum_{j=n_0+\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} j c_j a_s b_t, \end{aligned} \quad (9)$$

where  $[(j-n_0)/\tau]$  denotes the integer part of number  $(j-n_0)/\tau$ .

*Proof.* Notice that

$$\begin{aligned} &\sum_{i=1}^{\infty} \sum_{j=n_0+i\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} c_j a_s b_t \\ &= \sum_{j=n_0+\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} c_j a_s b_t + \sum_{j=n_0+2\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} c_j a_s b_t \\ &\quad + \sum_{j=n_0+3\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} c_j a_s b_t + \cdots \end{aligned}$$

$$\begin{aligned}
 &= \sum_{j=n_0+\tau}^{n_0+2\tau-1} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \left[ \frac{j-n_0}{\tau} \right] c_j a_s b_t \\
 &+ \sum_{j=n_0+2\tau}^{n_0+3\tau-1} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \left[ \frac{j-n_0}{\tau} \right] c_j a_s b_t \\
 &+ \sum_{j=n_0+3\tau}^{n_0+4\tau-1} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \left[ \frac{j-n_0}{\tau} \right] c_j a_s b_t + \dots \\
 &= \sum_{j=n_0+\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \left[ \frac{j-n_0}{\tau} \right] c_j a_s b_t \\
 &\leq \frac{1}{\tau} \sum_{j=n_0+\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} j c_j a_s b_t.
 \end{aligned} \tag{10}$$

That is, (9) holds. This completes the proof.  $\square$

### 3. Existence of Uncountably Many Bounded Positive Solutions

Now we study the existence of uncountably many bounded positive solutions for (2) by using the Krasnoselskii's fixed point theorem and Schauder's fixed point theorem, respectively.

**Theorem 6.** Assume that there exist constants  $n_1 \in \mathbb{N}_{n_0}$ ,  $M$ ,  $N$ , and  $c$  with  $M > N > 0$  and  $c \in [0, (M - N)/2M]$  and nonnegative sequences  $\{W_n\}_{n \in \mathbb{N}_{n_0}}$ ,  $\{P_n\}_{n \in \mathbb{N}_{n_0}}$ ,  $\{Q_n\}_{n \in \mathbb{N}_{n_0}}$ , and  $\{R_n\}_{n \in \mathbb{N}_{n_0}}$  satisfying

$$|\gamma_n| \leq c, \quad \forall n \geq n_1, \tag{11}$$

$$|f(n, u_1, u_2, \dots, u_k)| \leq W_n, \quad |g(n, u_1, u_2, \dots, u_k)| \leq P_n,$$

$$|h(n, u_1, u_2, \dots, u_k)| \leq Q_n, \quad |p(n, u_1, u_2, \dots, u_k)| \leq R_n,$$

$$\forall (n, u_1, u_2, \dots, u_k) \in \mathbb{N}_{n_0} \times [N, M]^k, \tag{12}$$

$$\begin{aligned}
 \max \left\{ \sum_{t=n_0}^{\infty} \frac{W_t}{|a_t|}, \sum_{s=n_0}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} [(t-s+1)Q_t + P_t], \right. \\
 \left. \sum_{j=n_0}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} (R_t + |r_t|) \right\} < +\infty.
 \end{aligned} \tag{13}$$

Then (2) possesses uncountably many bounded positive solutions in  $\Omega_1(N, M)$ .

*Proof.* Set  $L \in (N + cM, M(1 - c))$ . It follows from (13) that there exists  $T \geq n_0 + n_1 + \tau + |\beta|$  sufficiently large such that

$$\begin{aligned}
 &\sum_{t=T}^{\infty} \frac{W_t}{|a_t|} + \sum_{s=T}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} [(t-s+1)Q_t + P_t] \\
 &+ \sum_{j=T}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} (R_t + |r_t|) \\
 &< \min \{M(1-c) - L, L - cM - N\}.
 \end{aligned} \tag{14}$$

Define two mappings  $U_L$  and  $S_L: \Omega_1(N, M) \rightarrow I_{\beta}^{\infty}$  by

$$(U_L x)_n = \begin{cases} \frac{1}{2}L - \gamma_n x_{n-\tau}, & n \geq T, \\ (U_L x)_T, & \beta \leq n < T, \end{cases} \tag{15}$$

$$\begin{aligned}
 (S_L x)_n &= \begin{cases} \frac{1}{2}L + \sum_{t=n}^{\infty} \frac{1}{a_t} f(t, x_{b_{1t}}, \dots, x_{b_{kt}}) \\ + \sum_{s=n}^{\infty} \sum_{t=s}^{\infty} \frac{1}{a_s} [(t-s+1)h(t, x_{d_{1t}}, \dots, x_{d_{kt}}) \\ - g(t, x_{c_{1t}}, \dots, x_{c_{kt}})] \\ - \sum_{j=n}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{a_j} [p(t, x_{o_{1t}}, \dots, x_{o_{kt}}) - r_t], & n \geq T, \\ (S_L x)_T, & \beta \leq n < T \end{cases} \\
 & \tag{16}
 \end{aligned}$$

for each  $x = \{x_n\}_{n \in \mathbb{Z}_{\beta}} \in \Omega_1(N, M)$ .

Now we prove that

$$\begin{aligned}
 U_L x + S_L y &\in \Omega_1(N, M), \quad \forall x, y \in \Omega_1(N, M), \\
 \|U_L x - U_L y\| &\leq c \|x - y\|, \quad \forall x, y \in \Omega_1(N, M), \\
 \|S_L x\| &< M, \quad \forall x, y \in \Omega_1(N, M).
 \end{aligned} \tag{17}$$

In view of (11), (12), and (14)–(16), we conclude that for any  $x = \{x_n\}_{n \in \mathbb{Z}_{\beta}}$ ,  $y = \{y_n\}_{n \in \mathbb{Z}_{\beta}} \in \Omega_1(N, M)$ , and  $n \geq T$ ,

$$\begin{aligned}
 &|(U_L x)_n + (S_L y)_n - L| \\
 &= \left| -\gamma_n x_{n-\tau} + \sum_{t=n}^{\infty} \frac{1}{a_t} f(t, y_{b_{1t}}, \dots, y_{b_{kt}}) \right. \\
 &\quad + \sum_{s=n}^{\infty} \sum_{t=s}^{\infty} \frac{1}{a_s} [(t-s+1)h(t, y_{d_{1t}}, \dots, y_{d_{kt}}) \\
 &\quad \quad \quad \left. - g(t, y_{c_{1t}}, \dots, y_{c_{kt}})] \right. \\
 &\quad \left. - \sum_{j=n}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{a_j} \right. \\
 &\quad \quad \left. \times [p(t, y_{o_{1t}}, \dots, y_{o_{kt}}) - r_t] \right|
 \end{aligned}$$

$$\begin{aligned}
&\leq cM + \sum_{t=T}^{\infty} \frac{W_t}{|a_t|} + \sum_{s=T}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} \\
&\quad \times [(t-s+1)Q_t + P_t] \\
&\quad + \sum_{j=T}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} (R_t + |r_t|) \\
&< cM + \min \{M(1-c) - L, L - cM - N\}, \\
&\quad + \sum_{j=T}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} (R_t + |r_t|) \\
&< \frac{1}{2}L + \min \{M(1-c) - L, L - cM - N\} \\
&< M,
\end{aligned} \tag{18}$$

which yield that (17) hold.

In order to prove that  $S_L$  is completely continuous in  $\Omega_1(N, M)$ , we have to show that  $S_L$  is continuous in  $\Omega_1(N, M)$  and  $S_L(\Omega_1(N, M))$  is relatively compact. Suppose that  $\{x^m\}_{m \in \mathbb{N}}$  is an arbitrary sequence in  $\Omega_1(N, M)$  and  $x \in \Omega_1(N, M)$  with  $\lim_{m \rightarrow \infty} x^m = x$ , where  $x^m = \{x_n^m\}_{n \in \mathbb{Z}_\beta}$  for each  $m \in \mathbb{N}$  and  $x = \{x_n\}_{n \in \mathbb{Z}_\beta}$ . With the help of (12), (13),  $\lim_{m \rightarrow \infty} x^m = x$ , and the continuity of  $f, g, h$ , and  $p$ , we know that for given  $\varepsilon > 0$ , there exist  $T_1, T_2, T_3$ , and  $T_4 \in \mathbb{N}$  with  $T_4 > T_3 > T_2 > T_1 > T$  satisfying

$$\begin{aligned}
\|U_L x - U_L y\| &= \sup_{n \in \mathbb{Z}_\beta} |(U_L x)_n - (U_L y)_n| \\
&= \max \left\{ \sup_{T > n \geq \beta} |(U_L x)_n - (U_L y)_n|, \right. \\
&\quad \left. \sup_{n \geq T} |(U_L x)_n - (U_L y)_n| \right\} \\
&= \sup_{n \geq T} (|\gamma_n| |x_{n-\tau} - y_{n-\tau}|) \\
&\leq c \|x - y\|, \\
\|S_L x\| &= \sup_{n \in \mathbb{Z}_\beta} |(S_L x)_n| \\
&= \max \left\{ \sup_{T > n \geq \beta} |(S_L x)_n|, \sup_{n \geq T} |(S_L x)_n| \right\} \\
&= \sup_{n \geq T} \left| \frac{1}{2}L + \sum_{t=n}^{\infty} \frac{1}{a_t} f(t, x_{b_t}, \dots, x_{b_{kt}}) \right. \\
&\quad + \sum_{s=n}^{\infty} \sum_{t=s}^{\infty} \frac{1}{a_s} [(t-s+1) \\
&\quad \quad \times h(t, x_{d_t}, \dots, x_{d_{kt}}) \\
&\quad \quad \left. - g(t, x_{c_t}, \dots, x_{c_{kt}})] \right. \\
&\quad \left. - \sum_{j=n}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{a_j} [p(t, x_{o_t}, \dots, x_{o_{kt}}) \right. \\
&\quad \quad \left. - r_t] \right| \\
&\leq \sup_{n \geq T} \left( \frac{1}{2}L + \sum_{t=n}^{\infty} \frac{W_t}{|a_t|} + \sum_{s=n}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} [(t-s+1)Q_t \right. \\
&\quad \quad \left. + P_t] \right. \\
&\quad \left. + \sum_{j=n}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} (R_t + |r_t|) \right) \\
&\leq \frac{1}{2}L + \sum_{t=T}^{\infty} \frac{W_t}{|a_t|} + \sum_{s=T}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} [(t-s+1)Q_t + P_t]
\end{aligned}$$

$$\begin{aligned}
&\sum_{t=T_1}^{\infty} \frac{W_t}{|a_t|} + \sum_{s=T_1}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} [(t-s+1)Q_t + P_t] \\
&\quad + \sum_{j=T_1}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} R_t < \frac{\varepsilon}{18};
\end{aligned}$$

$$\begin{aligned}
&\max \left\{ \sum_{t=T_3}^{\infty} \frac{t-s+1}{|a_j|} R_t : T \leq j \leq T_1, j \leq s \leq T_2 \right\} \\
&< \frac{\varepsilon}{18T_1 T_2};
\end{aligned}$$

$$\max \left\{ \sum_{s=T_2}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} R_t : T \leq j \leq T_1 \right\} < \frac{\varepsilon}{18T_1};$$

$$\begin{aligned}
&\max \left\{ \sum_{t=T_2}^{\infty} \frac{1}{|a_s|} [(t-s+1)Q_t + P_t] : T \leq s \leq T_1 \right\} \\
&< \frac{\varepsilon}{18T_1};
\end{aligned}$$

$$\begin{aligned}
&\max \left\{ |q(t, x_{b_t}^m, \dots, x_{b_{kt}}^m) \right. \\
&\quad \left. - q(t, x_{b_t}, \dots, x_{b_{kt}}) \right\} : q \in \{f, g, h, p\} \\
&< \frac{\varepsilon}{18T_1 T_2 T_3 (A + B + E)}, \quad \forall m \geq T_4, T \leq t \leq T_3,
\end{aligned}$$

(19)

where

$$A = \max \left\{ \frac{1}{|a_s|} : T \leq s \leq T_1 \right\},$$

$$B = \max \left\{ \frac{t-s+1}{|a_s|} : T \leq s \leq T_1, s \leq t \leq T_2 \right\},$$

$$E = \max \left\{ \frac{t-s+1}{|a_j|} : T \leq j \leq T_1, j \leq s \leq T_2, s \leq t \leq T_3 \right\}. \tag{20}$$

By virtue of (16), (19), and (20), we get that

$$\begin{aligned} & \|S_L x^m - S_L x\| \\ &= \sup_{n \in \mathbb{Z}_\beta} |(S_L x^m)_n - (S_L x)_n| \\ &= \max \left\{ \sup_{T > n \geq \beta} |(S_L x^m)_n - (S_L x)_n|, \right. \\ & \quad \left. \sup_{n \geq T} |(S_L x^m)_n - (S_L x)_n| \right\} \\ &= \sup_{n \geq T} \left| \sum_{t=n}^{\infty} \frac{1}{a_t} f(t, x_{b_{1t}}^m, \dots, x_{b_{kt}}^m) \right. \\ & \quad + \sum_{s=n}^{\infty} \sum_{t=s}^{\infty} \frac{1}{a_s} [(t-s+1)h(t, x_{d_{1t}}^m, \dots, x_{d_{kt}}^m) \\ & \quad \quad \quad - g(t, x_{c_{1t}}^m, \dots, x_{c_{kt}}^m)] \\ & \quad - \sum_{j=n}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{a_j} [p(t, x_{o_{1t}}^m, \dots, x_{o_{kt}}^m) - r_t] \\ & \quad - \left( \sum_{t=n}^{\infty} \frac{1}{a_t} f(t, x_{b_{1t}}^m, \dots, x_{b_{kt}}^m) \right. \\ & \quad + \sum_{s=n}^{\infty} \sum_{t=s}^{\infty} \frac{1}{a_s} [(t-s+1)h(t, x_{d_{1t}}^m, \dots, x_{d_{kt}}^m) \\ & \quad \quad \quad - g(t, x_{c_{1t}}^m, \dots, x_{c_{kt}}^m)] \\ & \quad \left. - \sum_{j=n}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{a_j} [p(t, x_{o_{1t}}^m, \dots, x_{o_{kt}}^m) \right. \\ & \quad \quad \quad \left. - r_t] \right) \Big| \\ &\leq \sum_{t=T}^{\infty} \frac{1}{|a_t|} |f(t, x_{b_{1t}}^m, \dots, x_{b_{kt}}^m) - f(t, x_{b_{1t}}, \dots, x_{b_{kt}})| \\ & \quad + \sum_{s=T}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} [(t-s+1)|h(t, x_{d_{1t}}^m, \dots, x_{d_{kt}}^m) \\ & \quad \quad \quad - h(t, x_{d_{1t}}, \dots, x_{d_{kt}})| \\ & \quad \quad \quad + |g(t, x_{c_{1t}}^m, \dots, x_{c_{kt}}^m) \\ & \quad \quad \quad - g(t, x_{c_{1t}}, \dots, x_{c_{kt}})|] \end{aligned}$$

$$\begin{aligned} & + \sum_{j=T}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} |p(t, x_{o_{1t}}^m, \dots, x_{o_{kt}}^m) \\ & \quad \quad \quad - p(t, x_{o_{1t}}, \dots, x_{o_{kt}})| \\ &= \sum_{t=T}^{T_1} \frac{1}{|a_t|} |f(t, x_{b_{1t}}^m, \dots, x_{b_{kt}}^m) - f(t, x_{b_{1t}}, \dots, x_{b_{kt}})| \\ & \quad + \sum_{t=T_1+1}^{\infty} \frac{1}{|a_t|} |f(t, x_{b_{1t}}^m, \dots, x_{b_{kt}}^m) \\ & \quad \quad \quad - f(t, x_{b_{1t}}, \dots, x_{b_{kt}})| \\ & \quad + \sum_{s=T}^{T_1} \sum_{t=s}^{T_2} \frac{1}{|a_s|} [(t-s+1)|h(t, x_{d_{1t}}^m, \dots, x_{d_{kt}}^m) \\ & \quad \quad \quad - h(t, x_{d_{1t}}, \dots, x_{d_{kt}})| \\ & \quad \quad \quad + |g(t, x_{c_{1t}}^m, \dots, x_{c_{kt}}^m) \\ & \quad \quad \quad - g(t, x_{c_{1t}}, \dots, x_{c_{kt}})|] \\ & \quad + \sum_{s=T}^{T_1} \sum_{t=T_2+1}^{\infty} \frac{1}{|a_s|} [(t-s+1)|h(t, x_{d_{1t}}^m, \dots, x_{d_{kt}}^m) \\ & \quad \quad \quad - h(t, x_{d_{1t}}, \dots, x_{d_{kt}})| \\ & \quad \quad \quad + |g(t, x_{c_{1t}}^m, \dots, x_{c_{kt}}^m) \\ & \quad \quad \quad - g(t, x_{c_{1t}}, \dots, x_{c_{kt}})|] \\ & \quad + \sum_{s=T_1+1}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} [(t-s+1)|h(t, x_{d_{1t}}^m, \dots, x_{d_{kt}}^m) \\ & \quad \quad \quad - h(t, x_{d_{1t}}, \dots, x_{d_{kt}})| \\ & \quad \quad \quad + |g(t, x_{c_{1t}}^m, \dots, x_{c_{kt}}^m) \\ & \quad \quad \quad - g(t, x_{c_{1t}}, \dots, x_{c_{kt}})|] \\ & \quad + \sum_{j=T}^{T_1} \sum_{s=j}^{T_2} \sum_{t=s}^{T_3} \frac{t-s+1}{|a_j|} |p(t, x_{o_{1t}}^m, \dots, x_{o_{kt}}^m) \\ & \quad \quad \quad - p(t, x_{o_{1t}}, \dots, x_{o_{kt}})| \\ & \quad + \sum_{j=T}^{T_1} \sum_{s=j}^{T_2} \sum_{t=T_3+1}^{\infty} \frac{t-s+1}{|a_j|} |p(t, x_{o_{1t}}^m, \dots, x_{o_{kt}}^m) \\ & \quad \quad \quad - p(t, x_{o_{1t}}, \dots, x_{o_{kt}})| \\ & \quad + \sum_{j=T}^{T_1} \sum_{s=T_2+1}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} |p(t, x_{o_{1t}}^m, \dots, x_{o_{kt}}^m) \\ & \quad \quad \quad - p(t, x_{o_{1t}}, \dots, x_{o_{kt}})| \end{aligned}$$

$$\begin{aligned}
& + \sum_{j=T_1+1}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} \left| p(t, x_{o_{1t}}, \dots, x_{o_{kt}}^m) \right. \\
& \quad \left. - p(t, x_{o_{1t}}, \dots, x_{o_{kt}}) \right| \\
\leq & \sum_{t=T}^{T_1} A \left| f(t, x_{b_{1t}}^m, \dots, x_{b_{kt}}^m) - f(t, x_{b_{1t}}, \dots, x_{b_{kt}}) \right| \\
& + \sum_{s=T}^{T_1} \sum_{t=s}^{T_2} \left[ B \left| h(t, x_{d_{1t}}^m, \dots, x_{d_{kt}}^m) - h(t, x_{d_{1t}}, \dots, x_{d_{kt}}) \right| \right. \\
& \quad \left. + A \left| g(t, x_{c_{1t}}^m, \dots, x_{c_{kt}}^m) - g(t, x_{c_{1t}}, \dots, x_{c_{kt}}) \right| \right] \\
& + \sum_{j=T}^{T_1} \sum_{s=j}^{T_2} \sum_{t=s}^{T_3} E \left| p(t, x_{o_{1t}}^m, \dots, x_{o_{kt}}^m) - p(t, x_{o_{1t}}, \dots, x_{o_{kt}}) \right| \\
& + 2 \sum_{t=T_1+1}^{\infty} \frac{W_t}{|a_t|} + 2 \sum_{s=T}^{T_1} \sum_{t=T_2+1}^{\infty} \frac{1}{|a_s|} [(t-s+1)Q_t + P_t] \\
& + 2 \sum_{s=T_1+1}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} [(t-s+1)Q_t + P_t] \\
& + 2 \sum_{j=T}^{T_1} \sum_{s=j}^{T_2} \sum_{t=T_3+1}^{\infty} \frac{t-s+1}{|a_j|} R_t \\
& + 2 \sum_{j=T}^{T_1} \sum_{s=T_2+1}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} R_t \\
& + 2 \sum_{j=T_1+1}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} R_t \\
\leq & \frac{\varepsilon A (T_1 - T + 1)}{18T_1 T_2 T_3 (A + B + E)} \\
& + \frac{\varepsilon (B + A) (T_1 - T + 1) (T_2 - T + 1)}{18T_1 T_2 T_3 (A + B + E)} \\
& + \frac{\varepsilon E (T_1 - T + 1) (T_2 - T + 1) (T_3 - T + 1)}{18T_1 T_2 T_3 (A + B + E)} \\
& + \frac{2\varepsilon (T_1 - T + 1)}{18T_1} \\
& + \frac{2\varepsilon (T_1 - T + 1) (T_2 - T + 1)}{18T_1 T_2} \\
& + \frac{2\varepsilon (T_1 - T + 1)}{18T_1} + \frac{\varepsilon}{18} < \varepsilon, \\
& \forall m \geq T_4,
\end{aligned} \tag{21}$$

which means that  $S_L$  is continuous in  $\Omega_1(N, M)$ .

Next we prove that  $S_L(\Omega_1(N, M))$  is uniformly Cauchy. It follows from (13) that for given  $\varepsilon > 0$ , there exists  $V > T$  satisfying

$$\begin{aligned}
& \sum_{t=V}^{\infty} \frac{W_t}{|a_t|} + \sum_{s=V}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} [(t-s+1)Q_t + P_t] \\
& + \sum_{j=V}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} (R_t + |r_t|) < \frac{\varepsilon}{4}.
\end{aligned} \tag{22}$$

Using (16) and (22), we know that for any  $x = \{x_n\}_{n \in \mathbb{Z}_\beta} \in \Omega_1(N, M)$  and  $m, n \geq V$ ,

$$\begin{aligned}
& |(S_L x)_m - (S_L x)_n| \\
& = \left| \sum_{t=m}^{\infty} \frac{1}{a_t} f(t, x_{b_{1t}}, \dots, x_{b_{kt}}) - \sum_{t=n}^{\infty} \frac{1}{a_t} f(t, x_{b_{1t}}, \dots, x_{b_{kt}}) \right. \\
& \quad + \sum_{s=m}^{\infty} \sum_{t=s}^{\infty} \frac{1}{a_s} [(t-s+1)h(t, x_{d_{1t}}, \dots, x_{d_{kt}}) \\
& \quad \quad \quad \left. - g(t, x_{c_{1t}}, \dots, x_{c_{kt}})] \right. \\
& \quad \left. - \sum_{s=n}^{\infty} \sum_{t=s}^{\infty} \frac{1}{a_s} [(t-s+1)h(t, x_{d_{1t}}, \dots, x_{d_{kt}}) \right. \\
& \quad \quad \quad \left. - g(t, x_{c_{1t}}, \dots, x_{c_{kt}})] \right. \\
& \quad + \sum_{j=n}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{a_j} [p(t, x_{o_{1t}}, \dots, x_{o_{kt}}) - r_t] \\
& \quad \left. - \sum_{j=m}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{a_j} [p(t, x_{o_{1t}}, \dots, x_{o_{kt}}) - r_t] \right| \\
& \leq \sum_{t=m}^{\infty} \frac{W_t}{|a_t|} + \sum_{t=n}^{\infty} \frac{W_t}{|a_t|} \\
& \quad + \sum_{s=m}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} [(t-s+1)Q_t + P_t] \\
& \quad + \sum_{s=n}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} [(t-s+1)Q_t + P_t] \\
& \quad + \sum_{j=n}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} (R_t + |r_t|) \\
& \quad + \sum_{j=m}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} (R_t + |r_t|)
\end{aligned} \tag{23}$$

$$\begin{aligned} &\leq 2 \sum_{t=V}^{\infty} \frac{W_t}{|a_t|} + 2 \sum_{s=V}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} [(t-s+1)Q_t + P_t] \\ &+ 2 \sum_{j=V}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} (R_t + |r_t|) \end{aligned} \tag{24}$$

$< \varepsilon$ ,

which yields that  $S_L(\Omega_1(N, M))$  is uniformly Cauchy. It follows from Lemma 2 that  $S_L(\Omega_1(N, M))$  is relatively compact. Consequently Lemma 3 guarantees that there exists  $x = \{x_n\}_{n \in \mathbb{Z}_\beta} \in \Omega_1(N, M)$  satisfying  $U_L x + S_L x = x$ , which together with (15) and (16) gives that

$$\begin{aligned} x_n &= L - \gamma_n x_{n-\tau} + \sum_{t=n}^{\infty} \frac{1}{a_t} f(t, x_{b_{1t}}, \dots, x_{b_{kn}}) \\ &+ \sum_{s=n}^{\infty} \sum_{t=s}^{\infty} \frac{1}{a_s} [(t-s+1)h(t, x_{d_{1t}}, \dots, x_{d_{kt}}) \\ &\quad - g(t, x_{c_{1t}}, \dots, x_{c_{kt}})] \\ &- \sum_{j=n}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{a_j} [p(t, x_{o_{1t}}, \dots, x_{o_{kt}}) - r_t], \end{aligned} \tag{25}$$

$\forall n \geq T,$

which yields that

$$\begin{aligned} &\Delta(x_n + \gamma_n x_{n-\tau}) + \frac{1}{a_n} f(n, x_{b_{1n}}, \dots, x_{b_{kn}}) \\ &= - \sum_{t=n}^{\infty} \frac{1}{a_n} [(t-n+1)h(t, x_{d_{1t}}, \dots, x_{d_{kt}}) \\ &\quad - g(t, x_{c_{1t}}, \dots, x_{c_{kt}})] \\ &+ \sum_{s=n}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{a_n} [p(t, x_{o_{1t}}, \dots, x_{o_{kt}}) - r_t], \end{aligned}$$

$\forall n \geq T,$

$$\begin{aligned} &a_n \Delta(x_n + \gamma_n x_{n-\tau}) + f(n, x_{b_{1n}}, \dots, x_{b_{kn}}) \\ &= - \sum_{t=n}^{\infty} (t-n+1)h(t, x_{d_{1t}}, \dots, x_{d_{kt}}) \\ &+ \sum_{t=n}^{\infty} g(t, x_{c_{1t}}, \dots, x_{c_{kt}}) \\ &+ \sum_{s=n}^{\infty} \sum_{t=s}^{\infty} (t-s+1) [p(t, x_{o_{1t}}, \dots, x_{o_{kt}}) - r_t], \end{aligned}$$

$\forall n \geq T,$

(26)

which means that

$$\begin{aligned} &\Delta^3(a_n \Delta(x_n + \gamma_n x_{n-\tau})) + \Delta^3 f(n, x_{b_{1n}}, \dots, x_{b_{kn}}) \\ &+ \Delta^2 g(n, x_{c_{1n}}, \dots, x_{c_{kn}}) + \Delta h(n, x_{d_{1n}}, \dots, x_{d_{kn}}) \\ &+ p(n, x_{o_{1n}}, \dots, x_{o_{kn}}) = r_t, \quad \forall n \geq T; \end{aligned} \tag{27}$$

that is,  $x = \{x_n\}_{n \in \mathbb{Z}_\beta}$  is a bounded positive solution of (2) in  $\Omega_1(N, M)$ .

Finally we prove that (2) has uncountably many bounded positive solutions in  $\Omega_1(N, M)$ . Let  $L_1, L_2 \in (cM + N, M(1 - c))$  and  $L_1 \neq L_2$ . For each  $j \in \{1, 2\}$ , we conclude similarly that there exist a positive integer  $T_j \geq n_0 + n_1 + \tau + |\beta|$  and two mappings  $U_{L_j}$  and  $S_{L_j}$  satisfying (14)–(16), where  $L$  and  $T$  are replaced by  $L_j$  and  $T_j$ , respectively, and  $U_{L_j} + S_{L_j}$  has a fixed point  $z_j = \{z_{jn}\}_{n \in \mathbb{Z}_\beta}$ , which is a bounded positive solution (2) in  $\Omega_1(N, M)$ ; that is,

$$\begin{aligned} z_{jn} &= L_j - \gamma_n z_{jn-\tau} + \sum_{t=n}^{\infty} \frac{1}{a_t} f(t, z_{j b_{1t}}, \dots, z_{j b_{kt}}) \\ &+ \sum_{s=n}^{\infty} \sum_{t=s}^{\infty} \frac{1}{a_s} [(t-s+1)h(t, z_{j d_{1t}}, \dots, z_{j d_{kt}}) \\ &\quad - g(t, z_{j c_{1t}}, \dots, z_{j c_{kt}})] \\ &- \sum_{j=n}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{a_j} [p(t, z_{j o_{1t}}, \dots, z_{j o_{kt}}) - r_t], \end{aligned} \tag{28}$$

$\forall n \geq T_j, j \in \{1, 2\}.$

Equation (13) ensures that there exists some  $T_3 > \max\{T_1, T_2\}$  satisfying

$$\begin{aligned} &\sum_{t=T_3}^{\infty} \frac{W_t}{|a_t|} + \sum_{s=T_3}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} [(t-s+1)Q_t + P_t] \\ &+ \sum_{j=T_3}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} R_t < \frac{|L_1 - L_2|}{4}. \end{aligned} \tag{29}$$

Combining (11), (12), (28), and (29), we deduce that for any  $n \geq T_3$ ,

$$\begin{aligned} &|z_{1n} - z_{2n}| \\ &= \left| L_1 - L_2 - \gamma_n z_{1n-\tau} + \gamma_n z_{2n-\tau} \right. \\ &\quad + \sum_{t=n}^{\infty} \frac{1}{a_t} f(t, z_{1 b_{1t}}, \dots, z_{1 b_{kt}}) \\ &\quad \left. - \sum_{t=n}^{\infty} \frac{1}{a_t} f(t, z_{2 b_{1t}}, \dots, z_{2 b_{kt}}) \right| \end{aligned}$$

$$\begin{aligned}
& + \sum_{s=n}^{\infty} \sum_{t=s}^{\infty} \frac{1}{a_s} \left[ (t-s+1) h(t, z_{1d_{1t}}, \dots, z_{1d_{kt}}) \right. \\
& \quad \left. - g(t, z_{1c_{1t}}, \dots, z_{1c_{kt}}) \right] \\
& - \sum_{s=n}^{\infty} \sum_{t=s}^{\infty} \frac{1}{a_s} \left[ (t-s+1) h(t, z_{2d_{1t}}, \dots, z_{2d_{kt}}) \right. \\
& \quad \left. - g(t, z_{2c_{1t}}, \dots, z_{2c_{kt}}) \right] \\
& - \sum_{j=n}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{a_j} \left[ p(t, z_{1o_{1t}}, \dots, z_{1o_{kt}}) - r_t \right] \\
& + \sum_{j=n}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{a_j} \left[ p(t, z_{2o_{1t}}, \dots, z_{2o_{kt}}) - r_t \right] \Big| \\
\geq & |L_1 - L_2| - |\gamma_n| |z_{1n-\tau} - z_{2n-\tau}| \\
& - \sum_{t=n}^{\infty} \frac{1}{a_t} |f(t, z_{1b_{1t}}, \dots, z_{1b_{kt}}) - f(t, z_{2b_{1t}}, \dots, z_{2b_{kt}})| \\
& - \sum_{s=n}^{\infty} \sum_{t=s}^{\infty} \frac{1}{a_s} \left[ (t-s+1) |h(t, z_{1d_{1t}}, \dots, z_{1d_{kt}}) \right. \\
& \quad \left. - h(t, z_{2d_{1t}}, \dots, z_{2d_{kt}}) \right] \\
& + |g(t, z_{1c_{1t}}, \dots, z_{1c_{kt}}) \\
& \quad - g(t, z_{2c_{1t}}, \dots, z_{2c_{kt}})| \\
& - \sum_{j=n}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{a_j} |p(t, z_{1o_{1t}}, \dots, z_{1o_{kt}}) \\
& \quad - p(t, z_{2o_{1t}}, \dots, z_{2o_{kt}})| \\
\geq & |L_1 - L_2| - c \|z_1 - z_2\| \\
& - 2 \left( \sum_{t=n}^{\infty} \frac{W_t}{|a_t|} + \sum_{s=n}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} [(t-s+1) Q_t + P_t] \right. \\
& \quad \left. + \sum_{j=n}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} R_t \right) \\
\geq & |L_1 - L_2| - c \|z_1 - z_2\| \\
& - 2 \left( \sum_{t=T_3}^{\infty} \frac{W_t}{|a_t|} + \sum_{s=T_3}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} [(t-s+1) Q_t + P_t] \right. \\
& \quad \left. + \sum_{j=T_3}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} R_t \right) \\
> & \frac{|L_1 - L_2|}{2} - c \|z_1 - z_2\|, \tag{30}
\end{aligned}$$

which implies that

$$\|z_1 - z_2\| > \frac{|L_1 - L_2|}{2(1+c)} > 0. \tag{31}$$

That is,  $z_1 \neq z_2$ . Thus (2) has uncountably many bounded positive solutions in  $\Omega_1(N, M)$ . This completes the proof.  $\square$

**Theorem 7.** Assume that there exist constants  $n_1 \in \mathbb{N}_{n_0}$ ,  $M$ ,  $N$ , and  $c$  with  $M > N > 0$  and  $c \in [0, (M - N)/M)$  and nonnegative sequences  $\{W_n\}_{n \in \mathbb{N}_{n_0}}$ ,  $\{P_n\}_{n \in \mathbb{N}_{n_0}}$ ,  $\{Q_n\}_{n \in \mathbb{N}_{n_0}}$ , and  $\{R_n\}_{n \in \mathbb{N}_{n_0}}$  satisfying (12), (13), and

$$0 \leq \gamma_n \leq c, \quad \forall n \geq n_1. \tag{32}$$

Then (2) possesses uncountably many bounded positive solutions in  $\Omega_1(N, M)$ .

*Proof.* Let  $L \in (N + cM, M)$ . It follows from (13) that there exists  $T \geq n_0 + n_1 + \tau + |\beta|$  sufficiently large such that

$$\begin{aligned}
& \sum_{t=T}^{\infty} \frac{W_t}{|a_t|} + \sum_{s=T}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} [(t-s+1) Q_t + P_t] \\
& + \sum_{j=T}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} (R_t + |r_t|) \\
& < \min \{M - L, L - cM - N\}. \tag{33}
\end{aligned}$$

Let the mappings  $U_L$  and  $S_L : \Omega_1(N, M) \rightarrow l_{\beta}^{\infty}$  be defined by (15) and (16), respectively. By means of (12), (15), (16), (32), and (33), we infer that for any  $x = \{x_n\}_{n \in \mathbb{Z}_{\beta}}$ ,  $y = \{y_n\}_{n \in \mathbb{Z}_{\beta}} \in \Omega_1(N, M)$ , and  $n \geq T$ ,

$$\begin{aligned}
& (U_L x)_n + (S_L y)_n \\
& = L - \gamma_n x_{n-\tau} + \sum_{t=n}^{\infty} \frac{1}{a_t} f(t, y_{b_{1t}}, \dots, y_{b_{kt}}) \\
& + \sum_{s=n}^{\infty} \sum_{t=s}^{\infty} \frac{1}{a_s} \left[ (t-s+1) h(t, y_{d_{1t}}, \dots, y_{d_{kt}}) \right. \\
& \quad \left. - g(t, y_{c_{1t}}, \dots, y_{c_{kt}}) \right] \\
& - \sum_{j=n}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{a_j} \left[ p(t, y_{o_{1t}}, \dots, y_{o_{kt}}) - r_t \right]
\end{aligned}$$



$$\begin{aligned}
 &\leq L + \sum_{t=T}^{\infty} \frac{W_t}{|a_t|} + \sum_{s=T}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} [(t-s+1)Q_t + P_t] \\
 &\quad + \sum_{j=T}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} (R_t + |r_t|) \\
 &< L + \min \{M - L, L - cM - N\} \\
 &\leq M, \\
 (U_L x)_n + (S_L y)_n &= L - \gamma_n x_{n-\tau} + \sum_{t=n}^{\infty} \frac{1}{a_t} f(t, y_{b_{1t}}, \dots, y_{b_{kt}}) \\
 &\quad + \sum_{s=n}^{\infty} \sum_{t=s}^{\infty} \frac{1}{a_s} [(t-s+1)h(t, y_{d_{1t}}, \dots, y_{d_{kt}}) \\
 &\quad \quad - g(t, y_{c_{1t}}, \dots, y_{c_{kt}})] \\
 &\quad - \sum_{j=n}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{a_j} [p(t, y_{o_{1t}}, \dots, y_{o_{kt}}) - r_t] \\
 &\geq L - cM - \sum_{t=T}^{\infty} \frac{W_t}{|a_t|} \\
 &\quad - \sum_{s=T}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} [(t-s+1)Q_t + P_t] \\
 &\quad - \sum_{j=T}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} (R_t + |r_t|) \\
 &> L - cM - \min \{M - L, L - cM - N\} \\
 &\geq N; \tag{34}
 \end{aligned}$$

that is,  $U_L x + S_L y \in \Omega_1(N, M)$  for any  $x, y \in \Omega_1(N, M)$ . The rest of the proof is similar to that of Theorem 6 and is omitted. This completes the proof.  $\square$

**Theorem 8.** Assume that there exist constants  $n_1 \in \mathbb{N}_{n_0}$ ,  $M$ ,  $N$ , and  $c$  with  $M > N > 0$  and  $c \in [0, (M - N)/M)$  and four nonnegative sequences  $\{W_n\}_{n \in \mathbb{N}_{n_0}}$ ,  $\{P_n\}_{n \in \mathbb{N}_{n_0}}$ ,  $\{Q_n\}_{n \in \mathbb{N}_{n_0}}$ , and  $\{R_n\}_{n \in \mathbb{N}_{n_0}}$  satisfying (12), (13), and

$$-c \leq \gamma_n \leq 0, \quad \forall n \geq n_1 \tag{35}$$

Then (2) possesses uncountably many bounded positive solutions in  $\Omega_1(N, M)$ .

*Proof.* Let  $L \in (N, (1 - c)M)$ . It follows from (13) that there exists  $T \geq n_0 + n_1 + \tau + |\beta|$  sufficiently large such that

$$\begin{aligned}
 &\sum_{t=T}^{\infty} \frac{W_t}{|a_t|} + \sum_{s=T}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} [(t-s+1)Q_t + P_t] \\
 &\quad + \sum_{j=T}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} (R_t + |r_t|) \tag{36} \\
 &< \min \{(1 - c)M - L, L - N\}.
 \end{aligned}$$

Let the mappings  $U_L$  and  $S_L : \Omega_1(N, M) \rightarrow l_{\beta}^{\infty}$  be defined by (15) and (16), respectively. Making use of (12), (15), (16), (35), and (36), we derive that for any  $x = \{x_n\}_{n \in \mathbb{Z}_{\beta}}$ ,  $y = \{y_n\}_{n \in \mathbb{Z}_{\beta}} \in \Omega_1(N, M)$  and  $n \geq T$ ,

$$\begin{aligned}
 (U_L x)_n + (S_L y)_n &= L - \gamma_n x_{n-\tau} + \sum_{t=n}^{\infty} \frac{1}{a_t} f(t, y_{b_{1t}}, \dots, y_{b_{kt}}) \\
 &\quad + \sum_{s=n}^{\infty} \sum_{t=s}^{\infty} \frac{1}{a_s} [(t-s+1)h(t, y_{d_{1t}}, \dots, y_{d_{kt}}) \\
 &\quad \quad - g(t, y_{c_{1t}}, \dots, y_{c_{kt}})] \\
 &\quad - \sum_{j=n}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{a_j} [p(t, y_{o_{1t}}, \dots, y_{o_{kt}}) - r_t] \\
 &\leq L + cM + \sum_{t=T}^{\infty} \frac{W_t}{|a_t|} \\
 &\quad + \sum_{s=T}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} [(t-s+1)Q_t + P_t] \\
 &\quad + \sum_{j=T}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} (R_t + |r_t|) \\
 &< L + cM + \min \{(1 - c)M - L, L - N\} \\
 &\leq M, \\
 (U_L x)_n + (S_L y)_n &= L - \gamma_n x_{n-\tau} + \sum_{t=n}^{\infty} \frac{1}{a_t} f(t, y_{b_{1t}}, \dots, y_{b_{kt}})
 \end{aligned}$$

$$\begin{aligned}
 & + \sum_{s=n}^{\infty} \sum_{t=s}^{\infty} \frac{1}{a_s} \left[ (t-s+1) h(t, y_{d_{1t}}, \dots, y_{d_{kt}}) \right. \\
 & \quad \left. - g(t, y_{c_{1t}}, \dots, y_{c_{kt}}) \right] \\
 & - \sum_{j=n}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{a_j} \left[ p(t, y_{o_{1t}}, \dots, y_{o_{kt}}) - r_t \right] \\
 & \geq L - \sum_{t=T}^{\infty} \frac{W_t}{|a_t|} - \sum_{s=T}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} \left[ (t-s+1) Q_t + P_t \right] \\
 & - \sum_{j=T}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} (R_t + |r_t|) \\
 & > L - \min \{ (1-c)M - L, L - N \} \\
 & \geq N;
 \end{aligned} \tag{37}$$

that is,  $U_L x + S_L y \in \Omega_1(N, M)$  for any  $x, y \in \Omega_1(N, M)$ . The rest of the proof is similar to that of Theorem 6 and is omitted. This completes the proof.  $\square$

**Theorem 9.** Assume that there exist constants  $n_1 \in \mathbb{N}_{n_0}$ ,  $M, N$ , and  $c$  with  $M > N > 0$  and  $c > M/(M - N)$  and nonnegative sequences  $\{W_n\}_{n \in \mathbb{N}_{n_0}}$ ,  $\{P_n\}_{n \in \mathbb{N}_{n_0}}$ ,  $\{Q_n\}_{n \in \mathbb{N}_{n_0}}$ , and  $\{R_n\}_{n \in \mathbb{N}_{n_0}}$  satisfying (13):

$$\gamma_n \geq c, \quad \forall n \geq n_1, \tag{38}$$

$$|f(n, u_1, u_2, \dots, u_k)| \leq W_n, \quad |g(n, u_1, u_2, \dots, u_k)| \leq P_n,$$

$$|h(n, u_1, u_2, \dots, u_k)| \leq Q_n, \quad |p(n, u_1, u_2, \dots, u_k)| \leq R_n,$$

$$\forall (n, u_1, u_2, \dots, u_k) \in \mathbb{N}_{n_0} \times \left[ 0, \frac{M}{c} \right]^k. \tag{39}$$

Then (2) possesses uncountably many bounded positive solutions in  $l_{\beta}^{\infty}$ .

*Proof.* Let  $L \in (N + M/c, M)$ . It follows from (13) that there exists an integer  $T \geq n_0 + n_1 + \tau + |\beta|$  satisfying

$$\begin{aligned}
 & \sum_{t=T}^{\infty} \frac{W_t}{|a_t|} + \sum_{s=T}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} \left[ (t-s+1) Q_t + P_t \right] \\
 & + \sum_{j=T}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} (R_t + |r_t|) \\
 & < \min \left\{ M - L, L - N - \frac{M}{c} \right\}.
 \end{aligned} \tag{40}$$

Define two mappings  $U_L$  and  $S_L : \Omega_{2T}(N, M) \rightarrow l_{\beta}^{\infty}$  by

$$(U_L x)_n = \begin{cases} \frac{1}{\gamma_{n+\tau}} (L - x_{n+\tau}), & n \geq T, \\ (U_L x)_T, & \beta \leq n < T \end{cases} \tag{41}$$

$$(S_L x)_n = \begin{cases} \frac{1}{\gamma_{n+\tau}} \left\{ \sum_{t=n+\tau}^{\infty} \frac{1}{a_t} f(t, x_{b_{1t}}, \dots, x_{b_{kt}}) \right. \\ \quad \left. + \sum_{s=n+\tau}^{\infty} \sum_{t=s}^{\infty} \frac{1}{a_s} \left[ (t-s+1) \right. \right. \\ \quad \quad \left. \left. \times h(t, x_{d_{1t}}, \dots, x_{d_{kt}}) \right. \right. \\ \quad \quad \left. \left. - g(t, x_{c_{1t}}, \dots, x_{c_{kt}}) \right] \right. \\ \quad \left. - \sum_{j=n+\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{a_j} \right. \\ \quad \quad \left. \times \left[ p(t, x_{o_{1t}}, \dots, x_{o_{kt}}) \right. \right. \\ \quad \quad \left. \left. - r_t \right] \right\}, & n \geq T, \\ (S_L x)_T, & \beta \leq n < T, \end{cases} \tag{42}$$

for each  $x = \{x_n\}_{n \in \mathbb{Z}_{\beta}} \in \Omega_{2T}(N, M)$ . By means of (13) and (39)–(42), we get that for any  $x = \{x_n\}_{n \in \mathbb{Z}_{\beta}}, y = \{y_n\}_{n \in \mathbb{Z}_{\beta}} \in \Omega_{2T}(N, M)$ , and  $n \geq T$ ,

$$\begin{aligned}
 & (U_L x)_n + (S_L y)_n \\
 & = \frac{1}{\gamma_{n+\tau}} \left( L - x_{n+\tau} + \sum_{t=n+\tau}^{\infty} \frac{1}{a_t} f(t, y_{b_{1t}}, \dots, y_{b_{kt}}) \right. \\
 & \quad \left. + \sum_{s=n+\tau}^{\infty} \sum_{t=s}^{\infty} \frac{1}{a_s} \left[ (t-s+1) h(t, y_{d_{1t}}, \dots, y_{d_{kt}}) \right. \right. \\
 & \quad \quad \left. \left. - g(t, y_{c_{1t}}, \dots, y_{c_{kt}}) \right] \right. \\
 & \quad \left. - \sum_{j=n+\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{a_j} \left[ p(t, y_{o_{1t}}, \dots, y_{o_{kt}}) \right. \right. \\
 & \quad \quad \left. \left. - r_t \right] \right)
 \end{aligned}$$

$$\begin{aligned}
 & \leq \frac{1}{\gamma_{n+\tau}} \left( L - \frac{N}{\gamma_{n+\tau}} + \sum_{t=T+\tau}^{\infty} \frac{W_t}{|a_t|} \right. \\
 & \quad \left. + \sum_{s=T+\tau}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} \left[ (t-s+1) Q_t + P_t \right] \right. \\
 & \quad \left. + \sum_{j=T+\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{a_j} (R_t + |r_t|) \right)
 \end{aligned}$$

$$\begin{aligned}
 &< \frac{1}{\gamma_{n+\tau}} \left( L + \min \left\{ M - L, L - N - \frac{M}{c} \right\} \right) \\
 &\leq \frac{M}{\gamma_{n+\tau}}, \\
 (U_L x)_n + (S_L y)_n &= \frac{1}{\gamma_{n+\tau}} \left( L - x_{n+\tau} + \sum_{t=n+\tau}^{\infty} \frac{1}{a_t} f(t, y_{b_t}, \dots, y_{b_{kt}}) \right. \\
 &\quad + \sum_{s=n+\tau}^{\infty} \sum_{t=s}^{\infty} \frac{1}{a_s} [(t-s+1)h(t, y_{d_{1t}}, \dots, y_{d_{kt}}) \\
 &\quad \quad \quad \left. - g(t, y_{c_{1t}}, \dots, y_{c_{kt}})] \right. \\
 &\quad \left. - \sum_{j=n+\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{a_j} [p(t, y_{o_{1t}}, \dots, y_{o_{kt}}) - r_t] \right) \\
 &\geq \frac{1}{\gamma_{n+\tau}} \left( L - \frac{M}{\gamma_{n+\tau}} - \sum_{t=T+\tau}^{\infty} \frac{W_t}{|a_t|} \right. \\
 &\quad \left. - \sum_{s=T+\tau}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} [(t-s+1)Q_t + P_t] \right. \\
 &\quad \left. - \sum_{j=T+\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{a_j} (R_t + |r_t|) \right) \\
 &> \frac{1}{\gamma_{n+\tau}} \left( L - \frac{M}{c} - \min \left\{ M - L, L - N - \frac{M}{c} \right\} \right) \\
 &\geq \frac{N}{\gamma_{n+\tau}},
 \end{aligned} \tag{43}$$

which yield that  $U_L x + S_L y \in \Omega_{2T}(N, M)$  for any  $x, y \in \Omega_{2T}(N, M)$ . By virtue of (38) and (41), we infer that

$$\begin{aligned}
 \|U_L x - U_L y\| &= \sup_{n \in \mathbb{Z}_\beta} |(U_L x)_n - (U_L y)_n| \\
 &= \max \left\{ \sup_{T > n \geq \beta} |(U_L x)_n - (U_L y)_n|, \right. \\
 &\quad \left. \sup_{n \geq T} |(U_L x)_n - (U_L y)_n| \right\} \tag{44} \\
 &= \sup_{n \geq T} \left( \frac{1}{|\gamma_{n+\tau}|} |x_{n+\tau} - y_{n+\tau}| \right) \\
 &\leq \frac{1}{c} \|x - y\|;
 \end{aligned}$$

that is,  $U_L$  is a contraction in  $\Omega_{2T}(N, M)$  because  $c > M/(M - N) > 1$ .

In order to prove that  $S_L$  is completely continuous in  $\Omega_{2T}(N, M)$ , we have to show that  $S_L$  is continuous in  $\Omega_{2T}(N, M)$  and  $S_L(\Omega_{2T}(N, M))$  is relatively compact. Suppose

that  $\{x^m\}_{m \in \mathbb{N}}$  is an arbitrary sequence in  $\Omega_{2T}(N, M)$  and  $x \in \Omega_{2T}(N, M)$  with  $\lim_{m \rightarrow \infty} x^m = x$ , where  $x^m = \{x_n^m\}_{n \in \mathbb{Z}_\beta}$  for each  $m \in \mathbb{N}$  and  $x = \{x_n\}_{n \in \mathbb{Z}_\beta}$ . On account of (13), (39),  $\lim_{m \rightarrow \infty} x^m = x$ , and the continuity of  $f, g, h$ , and  $p$ , we obtain that for given  $\varepsilon > 0$ , there exist  $T_1, T_2, T_3$ , and  $T_4 \in \mathbb{N}$  with  $T_4 > T_3 > T_2 > T_1 > T$  satisfying

$$\begin{aligned}
 &\sum_{t=T_1+\tau}^{\infty} \frac{W_t}{|a_t|} + \sum_{s=T_1+\tau}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} [(t-s+1)Q_t + P_t] \\
 &\quad + \sum_{j=T_1+\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} R_t < \frac{\varepsilon c}{18}; \\
 &\max \left\{ \sum_{t=T_3+\tau}^{\infty} \frac{t-s+1}{|a_j|} R_t : T \leq j \leq T_1 + \tau, j \leq s \leq T_2 + \tau \right\} \\
 &< \frac{\varepsilon c}{18T_1 T_2}; \\
 &\max \left\{ \sum_{s=T_2+\tau}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} R_t : T \leq j \leq T_1 + \tau \right\} \\
 &< \frac{\varepsilon c}{18T_1}; \\
 &\max \left\{ \sum_{t=T_2+\tau}^{\infty} \frac{1}{|a_s|} [(t-s+1)Q_t + P_t] \right. \\
 &\quad \left. : T \leq s \leq T_1 + \tau \right\} < \frac{\varepsilon c}{18T_1}; \\
 &\max \left\{ |q(t, x_{b_{1t}}^m, \dots, x_{b_{kt}}^m) \right. \\
 &\quad \left. - q(t, x_{b_{1t}}, \dots, x_{b_{kt}}) \right| : q \in \{f, g, h, p\} \right\} \\
 &< \frac{\varepsilon c}{18T_1 T_2 T_3 (A + B + E)}, \quad \forall m \geq T_4, T \leq t \leq T_3,
 \end{aligned} \tag{45}$$

where

$$\begin{aligned}
 A &= \max \left\{ \frac{1}{|a_s|} : T + \tau \leq s \leq T_1 + \tau \right\}, \\
 B &= \max \left\{ \frac{t-s+1}{|a_s|} : T + \tau \leq s \leq T_1 + \tau, s \leq t \leq T_2 + \tau \right\}, \\
 E &= \max \left\{ \frac{t-s+1}{|a_j|} : T + \tau \leq j \leq T_1 + \tau, \right. \\
 &\quad \left. j \leq s \leq T_2 + \tau, s \leq t \leq T_3 + \tau \right\}.
 \end{aligned} \tag{46}$$

It follows from (42)–(46) that

$$\begin{aligned}
& \|S_L x^m - S_L x\| \\
&= \sup_{n \in \mathbb{Z}_\beta} |(S_L x^m)_n - (S_L x)_n| \\
&= \max \left\{ \sup_{T > n \geq \beta} |(S_L x^m)_n - (S_L x)_n|, \right. \\
&\quad \left. \sup_{n \geq T} |(S_L x^m)_n - (S_L x)_n| \right\} \\
&= \sup_{n \geq T} \left| \frac{1}{\gamma_{n+\tau}} \left( \sum_{t=n+\tau}^{\infty} \frac{1}{a_t} f(t, x_{b_{1t}}^m, \dots, x_{b_{kt}}^m) \right. \right. \\
&\quad \left. \left. + \sum_{s=n+\tau}^{\infty} \sum_{t=s}^{\infty} \frac{1}{a_s} [(t-s+1)h(t, x_{d_{1t}}^m, \dots, x_{d_{kt}}^m) \right. \right. \\
&\quad \quad \left. \left. - g(t, x_{c_{1t}}^m, \dots, x_{c_{kt}}^m)] \right. \right. \\
&\quad \left. \left. - \sum_{j=n+\tau}^{\infty} \sum_{t=j}^{\infty} \sum_{s=t}^{\infty} \frac{t-s+1}{a_j} [p(t, x_{o_{1t}}^m, \dots, x_{o_{kt}}^m) - r_t] \right) \right. \\
&\quad \left. - \frac{1}{\gamma_{n+\tau}} \left( \sum_{t=n+\tau}^{\infty} \frac{1}{a_t} f(t, x_{b_{1t}}, \dots, x_{b_{kt}}) \right. \right. \\
&\quad \left. \left. + \sum_{s=n+\tau}^{\infty} \sum_{t=s}^{\infty} \frac{1}{a_s} [(t-s+1)h(t, x_{d_{1t}}, \dots, x_{d_{kt}}) \right. \right. \\
&\quad \quad \left. \left. - g(t, x_{c_{1t}}, \dots, x_{c_{kt}})] \right. \right. \\
&\quad \left. \left. - \sum_{j=n+\tau}^{\infty} \sum_{t=j}^{\infty} \sum_{s=t}^{\infty} \frac{t-s+1}{a_j} \right. \right. \\
&\quad \quad \left. \left. \times [p(t, x_{o_{1t}}, \dots, x_{o_{kt}}) - r_t] \right) \right| \\
&\leq \frac{1}{c} \left( \sum_{t=T+\tau}^{\infty} \frac{1}{|a_t|} |f(t, x_{b_{1t}}^m, \dots, x_{b_{kt}}^m) \right. \\
&\quad \left. - f(t, x_{b_{1t}}, \dots, x_{b_{kt}}) \right| \\
&\quad + \sum_{s=T+\tau}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} [(t-s+1)|h(t, x_{d_{1t}}^m, \dots, x_{d_{kt}}^m) \\
&\quad \quad - h(t, x_{d_{1t}}, \dots, x_{d_{kt}})| \\
&\quad \quad + |g(t, x_{c_{1t}}^m, \dots, x_{c_{kt}}^m) \\
&\quad \quad - g(t, x_{c_{1t}}, \dots, x_{c_{kt}})|] \\
&\quad + \sum_{j=T+\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} |p(t, x_{o_{1t}}^m, \dots, x_{o_{kt}}^m) \\
&\quad \quad - p(t, x_{o_{1t}}, \dots, x_{o_{kt}})|
\end{aligned}$$

$$\begin{aligned}
& \left. - p(t, x_{o_{1t}}, \dots, x_{o_{kt}}) \right| \\
&\leq \frac{1}{c} \left( \sum_{t=T+\tau}^{T_1+\tau-1} \frac{1}{|a_t|} |f(t, x_{b_{1t}}^m, \dots, x_{b_{kt}}^m) - f(t, x_{b_{1t}}, \dots, x_{b_{kt}})| \right. \\
&\quad \left. + \sum_{t=T_1+\tau}^{\infty} \frac{1}{|a_t|} |f(t, x_{b_{1t}}^m, \dots, x_{b_{kt}}^m) \right. \\
&\quad \quad \left. - f(t, x_{b_{1t}}, \dots, x_{b_{kt}}) \right| \\
&\quad + \sum_{s=T+\tau}^{T_1+\tau-1} \sum_{t=s}^{T_2+\tau-1} \frac{1}{|a_s|} [(t-s+1)|h(t, x_{d_{1t}}^m, \dots, x_{d_{kt}}^m) \\
&\quad \quad - h(t, x_{d_{1t}}, \dots, x_{d_{kt}})| \\
&\quad \quad + |g(t, x_{c_{1t}}^m, \dots, x_{c_{kt}}^m) \\
&\quad \quad - g(t, x_{c_{1t}}, \dots, x_{c_{kt}})|] \\
&\quad + \sum_{s=T+\tau}^{T_1+\tau-1} \sum_{t=T_2+\tau}^{\infty} \frac{1}{|a_s|} [(t-s+1)|h(t, x_{d_{1t}}^m, \dots, x_{d_{kt}}^m) \\
&\quad \quad - h(t, x_{d_{1t}}, \dots, x_{d_{kt}})| \\
&\quad \quad + |g(t, x_{c_{1t}}^m, \dots, x_{c_{kt}}^m) \\
&\quad \quad - g(t, x_{c_{1t}}, \dots, x_{c_{kt}})|] \\
&\quad + \sum_{s=T_1+\tau}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} [(t-s+1)|h(t, x_{d_{1t}}^m, \dots, x_{d_{kt}}^m) \\
&\quad \quad - h(t, x_{d_{1t}}, \dots, x_{d_{kt}})| \\
&\quad \quad + |g(t, x_{c_{1t}}^m, \dots, x_{c_{kt}}^m) \\
&\quad \quad - g(t, x_{c_{1t}}, \dots, x_{c_{kt}})|] \\
&\quad + \sum_{j=T+\tau}^{T_1+\tau-1} \sum_{s=j}^{T_2+\tau-1} \sum_{t=s}^{T_3+\tau-1} \frac{t-s+1}{|a_j|} |p(t, x_{o_{1t}}^m, \dots, x_{o_{kt}}^m) \\
&\quad \quad - p(t, x_{o_{1t}}, \dots, x_{o_{kt}})| \\
&\quad + \sum_{j=T+\tau}^{T_1+\tau-1} \sum_{s=j}^{T_2+\tau-1} \sum_{t=T_3+\tau}^{\infty} \frac{t-s+1}{|a_j|} |p(t, x_{o_{1t}}^m, \dots, x_{o_{kt}}^m) \\
&\quad \quad - p(t, x_{o_{1t}}, \dots, x_{o_{kt}})| \\
&\quad + \sum_{j=T+\tau}^{T_1+\tau-1} \sum_{s=T_2+\tau}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} |p(t, x_{o_{1t}}^m, \dots, x_{o_{kt}}^m) \\
&\quad \quad - p(t, x_{o_{1t}}, \dots, x_{o_{kt}})|
\end{aligned}$$

$$+ \sum_{j=T_1+\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} \left| p(t, x_{o_{1t}}^m, \dots, x_{o_{kt}}^m) - p(t, x_{o_{1t}}, \dots, x_{o_{kt}}) \right|$$

$$\leq \frac{1}{c} \left( \sum_{t=T+\tau}^{T_1+\tau-1} A \left| f(t, x_{b_{1t}}^m, \dots, x_{b_{kt}}^m) - f(t, x_{b_{1t}}, \dots, x_{b_{kt}}) \right| \right.$$

$$+ \sum_{s=T+\tau}^{T_1+\tau-1} \sum_{t=s}^{T_2+\tau-1} \left[ B \left| h(t, x_{d_{1t}}^m, \dots, x_{d_{kt}}^m) - h(t, x_{d_{1t}}, \dots, x_{d_{kt}}) \right| \right. \\ \left. + A \left| g(t, x_{c_{1t}}^m, \dots, x_{c_{kt}}^m) - g(t, x_{c_{1t}}, \dots, x_{c_{kt}}) \right| \right]$$

$$+ \sum_{j=T+\tau}^{T_1+\tau-1} \sum_{s=j}^{T_2+\tau-1} \sum_{t=s}^{T_3+\tau-1} E \left| p(t, x_{o_{1t}}^m, \dots, x_{o_{kt}}^m) - p(t, x_{o_{1t}}, \dots, x_{o_{kt}}) \right|$$

$$+ 2 \sum_{t=T_1+\tau}^{\infty} \frac{W_t}{|a_t|}$$

$$+ 2 \sum_{s=T+\tau}^{T_1+\tau-1} \sum_{t=T_2+\tau}^{\infty} \frac{1}{|a_s|} [(t-s+1)Q_t + P_t]$$

$$+ 2 \sum_{s=T_1+\tau}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} [(t-s+1)Q_t + P_t]$$

$$+ 2 \sum_{j=T+\tau}^{T_1+\tau-1} \sum_{s=j}^{T_2+\tau-1} \sum_{t=T_3+\tau}^{\infty} \frac{t-s+1}{|a_j|} R_t$$

$$+ 2 \sum_{j=T+\tau}^{T_1+\tau-1} \sum_{s=T_2+\tau}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} R_t$$

$$+ 2 \sum_{j=T_1+\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} R_t$$

$$< \frac{\varepsilon A (T_1 - T)}{18T_1 T_2 T_3 (A + B + E)}$$

$$+ \frac{\varepsilon (B + A) (T_1 - T) (T_2 - T)}{18T_1 T_2 T_3 (A + B + E)}$$

$$+ \frac{\varepsilon E (T_1 - T) (T_2 - T) (T_3 - T)}{18T_1 T_2 T_3 (A + B + E)}$$

$$+ \frac{2\varepsilon (T_1 - T)}{18T_1} + \frac{2\varepsilon (T_1 - T) (T_2 - T)}{18T_1 T_2}$$

$$+ \frac{2\varepsilon (T_1 - T)}{18T_1} + \frac{\varepsilon}{18} < \varepsilon, \quad \forall m \geq T_4,$$

(47)

which means that  $S_L$  is continuous in  $\Omega_{2T}(N, M)$ . It follows from (13) that for given  $\varepsilon > 0$ , there exists  $V > T$  satisfying

$$\sum_{t=V+\tau}^{\infty} \frac{W_t}{|a_t|}$$

$$+ \sum_{s=V+\tau}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} [(t-s+1)Q_t + P_t] \quad (48)$$

$$+ \sum_{j=V+\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} (R_t + |r_t|) < \frac{\varepsilon}{4}.$$

Next we prove that  $S_L(\Omega_{2T}(N, M))$  is uniformly Cauchy. In view of (42) and (48), we infer that for any  $x = \{x_n\}_{n \in \mathbb{Z}_\beta} \in \Omega_{2T}(N, M)$  and  $m, n \geq V$ ,

$$|(S_L x)_m - (S_L x)_n|$$

$$= \left| \frac{1}{\gamma_{m+\tau}} \sum_{t=m+\tau}^{\infty} \frac{1}{a_t} f(t, x_{b_{1t}}, \dots, x_{b_{kt}}) \right.$$

$$\left. - \frac{1}{\gamma_{n+\tau}} \sum_{t=n+\tau}^{\infty} \frac{1}{a_t} f(t, x_{b_{1t}}, \dots, x_{b_{kt}}) \right.$$

$$+ \frac{1}{\gamma_{m+\tau}} \sum_{s=m+\tau}^{\infty} \sum_{t=s}^{\infty} \frac{1}{a_s} [(t-s+1)h(t, x_{d_{1t}}, \dots, x_{d_{kt}})$$

$$\left. - g(t, x_{c_{1t}}, \dots, x_{c_{kt}}) \right]$$

$$- \frac{1}{\gamma_{n+\tau}} \sum_{s=n+\tau}^{\infty} \sum_{t=s}^{\infty} \frac{1}{a_s} [(t-s+1)h(t, x_{d_{1t}}, \dots, x_{d_{kt}})$$

$$\left. - g(t, x_{c_{1t}}, \dots, x_{c_{kt}}) \right]$$

$$+ \frac{1}{\gamma_{n+\tau}} \sum_{j=n+\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{a_j} [p(t, x_{o_{1t}}, \dots, x_{o_{kt}}) - r_t]$$

$$- \frac{1}{\gamma_{m+\tau}} \sum_{j=m+\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{a_j} [p(t, x_{o_{1t}}, \dots, x_{o_{kt}}) - r_t]$$

$$\begin{aligned}
 &\leq \frac{1}{c} \left( \sum_{t=m+\tau}^{\infty} \frac{W_t}{|a_t|} + \sum_{t=n+\tau}^{\infty} \frac{W_t}{|a_t|} \right. \\
 &\quad + \sum_{s=m+\tau}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} [(t-s+1)Q_t + P_t] \\
 &\quad + \sum_{s=n+\tau}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} [(t-s+1)Q_t + P_t] \\
 &\quad + \sum_{j=n+\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} (R_t + |r_t|) \\
 &\quad \left. + \sum_{j=m+\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} (R_t + |r_t|) \right) \\
 &\leq \frac{2}{c} \left( \sum_{t=V+\tau}^{\infty} \frac{W_t}{|a_t|} \right. \\
 &\quad + \sum_{s=V+\tau}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} [(t-s+1)Q_t + P_t] \\
 &\quad \left. + \sum_{j=V+\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} (R_t + |r_t|) \right) \\
 &< \varepsilon.
 \end{aligned}$$

Note that (40) and (42) yield that

$$\begin{aligned}
 &\|S_L x\| \\
 &= \sup_{n \in \mathbb{Z}_\beta} |(S_L x)_n| \\
 &= \max \left\{ \sup_{T > n \geq \beta} |(S_L x)_n|, \sup_{n \geq T} |(S_L x)_n| \right\} \\
 &= \sup_{n \geq T} \left| \frac{1}{\gamma_{n+\tau}} \left( \sum_{t=n+\tau}^{\infty} \frac{1}{a_t} f(t, x_{b_{1t}}, \dots, x_{b_{kt}}) \right. \right. \\
 &\quad + \sum_{s=n+\tau}^{\infty} \sum_{t=s}^{\infty} \frac{1}{a_s} [(t-s+1)h(t, x_{d_{1t}}, \dots, x_{d_{kt}}) \\
 &\quad \quad \left. - g(t, x_{c_{1t}}, \dots, x_{c_{kt}})] \right. \\
 &\quad \left. \left. - \sum_{j=n+\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{a_j} [p(t, x_{o_{1t}}, \dots, x_{o_{kt}}) - r_t] \right) \right|
 \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{1}{c} \left( \sum_{t=T+\tau}^{\infty} \frac{W_t}{|a_t|} \right. \\
 &\quad + \sum_{s=T+\tau}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} [(t-s+1)Q_t + P_t] \\
 &\quad \left. + \sum_{j=T+\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} (R_t + |r_t|) \right) \\
 &< \frac{1}{c} \min \left\{ M - L, L - N - \frac{M}{c} \right\} \\
 &< \frac{M}{c}, \quad \forall x = \{x_n\}_{n \in \mathbb{Z}_\beta} \in \Omega_{2T}(N, M),
 \end{aligned} \tag{50}$$

which gives that  $S_L(\Omega_{2T}(N, M))$  is bounded. Thus Lemma 3 means that there exists  $x = \{x_n\}_{n \in \mathbb{Z}_\beta} \in \Omega_{2T}(N, M)$  such that  $U_L x + S_L x = x$ , which is a bounded positive solution of (2) in  $\Omega_{2T}(N, M) \subset I_\beta^\infty$ .

Let  $L_1, L_2 \in (N + M/c, M)$  and  $L_1 \neq L_2$ . For any  $j \in \{1, 2\}$ , we deduce similarly that there exist a positive integer  $T_j \geq n_0 + n_1 + \tau + |\beta|$ , a closed bounded and convex subset  $\Omega_{2T_j}(N, M)$  of  $I_\beta^\infty$ , and two mappings  $U_{L_j}$  and  $S_{L_j}$  satisfying (40)–(42), where  $L$  and  $T$  are replaced by  $L_j$  and  $T_j$ , respectively, and  $U_{L_j} + S_{L_j}$  possesses a fixed point  $z_j = \{z_{jn}\}_{n \in \mathbb{Z}_\beta} \in \Omega_{2T_j}(N, M)$ , which is a bounded positive solution of (2); that is,

$$\begin{aligned}
 &z_{jn} \\
 &= \frac{1}{\gamma_{n+\tau}} \left( L_j - z_{j(n+\tau)} + \sum_{t=n+\tau}^{\infty} \frac{1}{a_t} f(t, z_{j b_{1t}}, \dots, z_{j b_{kt}}) \right. \\
 &\quad + \sum_{s=n+\tau}^{\infty} \sum_{t=s}^{\infty} \frac{1}{a_s} [(t-s+1)h(t, z_{j d_{1t}}, \dots, z_{j d_{kt}}) \\
 &\quad \quad \left. - g(t, z_{j c_{1t}}, \dots, z_{j c_{kt}})] \right. \\
 &\quad \left. - \sum_{j=n+\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{a_j} [p(t, z_{j o_{1t}}, \dots, z_{j o_{kt}}) - r_t] \right), \\
 &\quad \forall n \geq T_j, \quad j \in \{1, 2\}.
 \end{aligned} \tag{51}$$

Observe that (13) implies that there exists  $T_3 \in \mathbb{N}$  with  $T_3 > \max\{T_1, T_2\}$  satisfying

$$\begin{aligned}
 &\sum_{t=T_3+\tau}^{\infty} \frac{W_t}{|a_t|} \\
 &\quad + \sum_{s=T_3+\tau}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} [(t-s+1)Q_t + P_t] \\
 &\quad + \sum_{j=T_3+\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} R_t < \frac{|L_1 - L_2|}{4},
 \end{aligned} \tag{52}$$

which together with (51) yields that for each  $n \geq T_3$

$$\begin{aligned}
 & \left| z_{1n} - z_{2n} + \frac{z_{1n+\tau} - z_{2n+\tau}}{\gamma_{n+\tau}} \right| \\
 &= \left| \frac{1}{\gamma_{n+\tau}} \left( L_1 - L_2 + \sum_{t=n+\tau}^{\infty} \frac{1}{a_t} f(t, z_{1b_{1t}}, \dots, z_{1b_{kt}}) \right. \right. \\
 &\quad \left. \left. - \sum_{t=n+\tau}^{\infty} \frac{1}{a_t} f(t, z_{2b_{1t}}, \dots, z_{2b_{kt}}) \right. \right. \\
 &\quad \left. \left. + \sum_{s=n+\tau}^{\infty} \sum_{t=s}^{\infty} \frac{1}{a_s} \left[ (t-s+1) h(t, z_{1d_{1t}}, \dots, z_{1d_{kt}}) \right. \right. \right. \\
 &\quad \quad \left. \left. - g(t, z_{1c_{1t}}, \dots, z_{1c_{kt}}) \right] \right. \\
 &\quad \left. \left. - \sum_{s=n+\tau}^{\infty} \sum_{t=s}^{\infty} \frac{1}{a_s} \left[ (t-s+1) h(t, z_{2d_{1t}}, \dots, z_{2d_{kt}}) \right. \right. \right. \\
 &\quad \quad \left. \left. - g(t, z_{2c_{1t}}, \dots, z_{2c_{kt}}) \right] \right. \\
 &\quad \left. \left. - \sum_{j=n+\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{a_j} \left[ p(t, z_{1o_{1t}}, \dots, z_{1o_{kt}}) - r_t \right] \right. \right. \\
 &\quad \left. \left. + \sum_{j=n+\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{a_j} \left[ p(t, z_{2o_{1t}}, \dots, z_{2o_{kt}}) - r_t \right] \right. \right. \Bigg| \\
 &\geq \frac{1}{\gamma_{n+\tau}} \left( \left| L_1 - L_2 \right| - \sum_{t=n+\tau}^{\infty} \frac{1}{a_t} \left| f(t, z_{1b_{1t}}, \dots, z_{1b_{kt}}) \right. \right. \\
 &\quad \left. \left. - f(t, z_{2b_{1t}}, \dots, z_{2b_{kt}}) \right| \right. \\
 &\quad \left. - \sum_{s=n+\tau}^{\infty} \sum_{t=s}^{\infty} \frac{1}{a_s} \left[ (t-s+1) \left| h(t, z_{1d_{1t}}, \dots, z_{1d_{kt}}) \right. \right. \right. \\
 &\quad \quad \left. \left. - h(t, z_{2d_{1t}}, \dots, z_{2d_{kt}}) \right| \right. \\
 &\quad \left. \left. + \left| g(t, z_{1c_{1t}}, \dots, z_{1c_{kt}}) \right. \right. \right. \\
 &\quad \quad \left. \left. - g(t, z_{2c_{1t}}, \dots, z_{2c_{kt}}) \right| \right] \right. \\
 &\quad \left. \left. - \sum_{j=n+\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{a_j} \left| p(t, z_{1o_{1t}}, \dots, z_{1o_{kt}}) \right. \right. \right. \\
 &\quad \quad \left. \left. - p(t, z_{2o_{1t}}, \dots, z_{2o_{kt}}) \right| \right. \Bigg) \\
 &\geq \frac{1}{\gamma_{n+\tau}} \left( \left| L_1 - L_2 \right| - 2 \sum_{t=T_3+\tau}^{\infty} \frac{W_t}{|a_t|} \right. \\
 &\quad \left. - 2 \sum_{s=T_3+\tau}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} \left[ (t-s+1) Q_t + P_t \right] \right. \\
 &\quad \left. - 2 \sum_{j=T_3+\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} R_t \right)
 \end{aligned}$$

$$\begin{aligned}
 &\geq \frac{1}{\gamma_{n+\tau}} \left( \left| L_1 - L_2 \right| - \frac{\left| L_1 - L_2 \right|}{2} \right) \\
 &> \frac{\left| L_1 - L_2 \right|}{2\gamma_{n+\tau}},
 \end{aligned} \tag{53}$$

which implies that

$$\left| z_{1n} - z_{2n} + \frac{z_{1n+\tau} - z_{2n+\tau}}{\gamma_{n+\tau}} \right| > 0, \quad \forall n \geq T_3; \tag{54}$$

that is,  $z_1 \neq z_2$ . Consequently, (2) has uncountably many bounded positive solutions in  $l_\beta^\infty$ . This completes the proof.  $\square$

**Theorem 10.** Assume that there exist constants  $n_1 \in \mathbb{N}_{n_0}$ ,  $M$ ,  $N$ ,  $c$ , and  $d$  with  $d > c > 1$  and  $M(1 - 1/c) > N(1 - 1/d)$  and nonnegative sequences  $\{W_n\}_{n \in \mathbb{N}_{n_0}}$ ,  $\{P_n\}_{n \in \mathbb{N}_{n_0}}$ ,  $\{Q_n\}_{n \in \mathbb{N}_{n_0}}$ , and  $\{R_n\}_{n \in \mathbb{N}_{n_0}}$  satisfying (13):

$$c \leq \gamma_n \leq d, \quad \forall n \geq n_1, \tag{55}$$

$$\begin{aligned}
 &|f(n, u_1, u_2, \dots, u_k)| \leq W_n, \quad |g(n, u_1, u_2, \dots, u_k)| \leq P_n, \\
 &|h(n, u_1, u_2, \dots, u_k)| \leq Q_n, \quad |p(n, u_1, u_2, \dots, u_k)| \leq R_n,
 \end{aligned}$$

$$\forall (n, u_1, u_2, \dots, u_k) \in \mathbb{N}_{n_0} \times \left[ \frac{N}{d}, \frac{M}{c} \right]^k. \tag{56}$$

Then (2) possesses uncountably many bounded positive solutions in  $l_\beta^\infty$ .

*Proof.* Let  $L \in (N + M/c, M + N/d)$ . It follows from (13) that there exists an integer  $T \geq n_0 + n_1 + \tau + |\beta|$  satisfying

$$\begin{aligned}
 &\sum_{t=T}^{\infty} \frac{W_t}{|a_t|} + \sum_{s=T}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} \left[ (t-s+1) Q_t + P_t \right] \\
 &+ \sum_{j=T}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} (R_t + |r_t|) \\
 &< \min \left\{ M + \frac{N}{d} - L, L - N - \frac{M}{c} \right\}.
 \end{aligned} \tag{57}$$

Let the mappings  $U_L$  and  $S_L : \Omega_{2T}(N, M) \rightarrow l_\beta^\infty$  be defined by (41) and (42), respectively. Using (13) and (41), (42) and

(57), we obtain that for any  $x = \{x_n\}_{n \in \mathbb{Z}_\beta}$ ,  $y = \{y_n\}_{n \in \mathbb{Z}_\beta} \in \Omega_{2T}(N, M)$  and  $n \geq T$ ,

$$\begin{aligned} & (U_L x)_n + (S_L y)_n \\ &= \frac{1}{\gamma_{n+\tau}} \left( L - x_{n+\tau} + \sum_{t=n+\tau}^{\infty} \frac{1}{a_t} f(t, y_{b_{1t}}, \dots, y_{b_{kt}}) \right. \\ &\quad + \sum_{s=n+\tau}^{\infty} \sum_{t=s}^{\infty} \frac{1}{a_s} [(t-s+1)h(t, y_{d_{1t}}, \dots, y_{d_{kt}}) \\ &\quad\quad - g(t, y_{c_{1t}}, \dots, y_{c_{kt}})] \\ &\quad \left. - \sum_{j=n+\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{a_j} [p(t, y_{o_{1t}}, \dots, y_{o_{kt}}) - r_t] \right) \\ &\leq \frac{1}{\gamma_{n+\tau}} \left( L - \frac{N}{\gamma_{n+\tau}} + \sum_{t=T+\tau}^{\infty} \frac{W_t}{|a_t|} \right. \\ &\quad + \sum_{s=T+\tau}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} [(t-s+1)Q_t + P_t] \\ &\quad \left. + \sum_{j=T+\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{a_j} (R_t + |r_t|) \right) \\ &< \frac{1}{\gamma_{n+\tau}} \left( L - \frac{N}{d} + \min \left\{ M + \frac{N}{d} - L, L - N - \frac{M}{c} \right\} \right) \\ &\leq \frac{M}{\gamma_{n+\tau}}, \end{aligned}$$

$$\begin{aligned} & (U_L x)_n + (S_L y)_n \\ &= \frac{1}{\gamma_{n+\tau}} \left( L - x_{n+\tau} + \sum_{t=n+\tau}^{\infty} \frac{1}{a_t} f(t, y_{b_{1t}}, \dots, y_{b_{kt}}) \right. \\ &\quad + \sum_{s=n+\tau}^{\infty} \sum_{t=s}^{\infty} \frac{1}{a_s} [(t-s+1)h(t, y_{d_{1t}}, \dots, y_{d_{kt}}) \\ &\quad\quad - g(t, y_{c_{1t}}, \dots, y_{c_{kt}})] \\ &\quad \left. - \sum_{j=n+\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{a_j} [p(t, y_{o_{1t}}, \dots, y_{o_{kt}}) - r_t] \right) \\ &\geq \frac{1}{\gamma_{n+\tau}} \left( L - \frac{M}{\gamma_{n+\tau}} - \sum_{t=T+\tau}^{\infty} \frac{W_t}{|a_t|} \right. \\ &\quad - \sum_{s=T+\tau}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} [(t-s+1)Q_t + P_t] \\ &\quad \left. - \sum_{j=T+\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{a_j} (R_t + |r_t|) \right) \end{aligned}$$

$$\begin{aligned} & > \frac{1}{\gamma_{n+\tau}} \left( L - \frac{M}{c} - \min \left\{ M + \frac{N}{d} - L, L - N - \frac{M}{c} \right\} \right) \\ & \geq \frac{N}{\gamma_{n+\tau}}, \end{aligned} \tag{58}$$

which imply that  $U_L x + S_L y \in \Omega_{2T}(N, M)$  for any  $x, y \in \Omega_{2T}(N, M)$ . The rest of the proof is similar to that of Theorem 9 and is omitted. This completes the proof.  $\square$

**Theorem 11.** Assume that there exist constants  $n_1 \in \mathbb{N}_{n_0}$ ,  $M, N$ , and  $c$  with  $M > N > 0$  and  $c > M/(M - N)$  and nonnegative sequences  $\{W_n\}_{n \in \mathbb{N}_{n_0}}$ ,  $\{P_n\}_{n \in \mathbb{N}_{n_0}}$ ,  $\{Q_n\}_{n \in \mathbb{N}_{n_0}}$ , and  $\{R_n\}_{n \in \mathbb{N}_{n_0}}$  satisfying (13), (39), and

$$\gamma_n \leq -c, \quad \forall n \geq n_1. \tag{59}$$

Then (2) possesses uncountably many bounded positive solutions in  $l_\beta^\infty$ .

*Proof.* Let  $L \in (N, M(1 - 1/c))$ . It follows from (13) that there exists an integer  $T \geq n_0 + n_1 + \tau + |\beta|$  satisfying

$$\begin{aligned} & \sum_{t=T}^{\infty} \frac{W_t}{|a_t|} + \sum_{s=T}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} [(t-s+1)Q_t + P_t] \\ & + \sum_{j=T}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} (R_t + |r_t|) \\ & < \min \left\{ L - N, M - \frac{M}{c} - L \right\}. \end{aligned} \tag{60}$$

Define two mappings  $U_L$  and  $S_L : \Omega_{3T}(N, M) \rightarrow l_\beta^\infty$  by (42) and

$$(U_L x)_n = \begin{cases} \frac{1}{\gamma_{n+\tau}} (-L - x_{n+\tau}), & n \geq T, \\ (U_L x)_T, & \beta \leq n < T, \end{cases} \tag{61}$$

for each  $x = \{x_n\}_{n \in \mathbb{Z}_\beta} \in \Omega_{3T}(N, M)$ . It follows from (42) and (59)–(61) that for any  $x = \{x_n\}_{n \in \mathbb{Z}_\beta}$ ,  $y = \{y_n\}_{n \in \mathbb{Z}_\beta} \in \Omega_{3T}(N, M)$ , and  $n \geq T$ ,

$$\begin{aligned} & (U_L x)_n + (S_L y)_n \\ &= \frac{1}{\gamma_{n+\tau}} \left( -L - x_{n+\tau} + \sum_{t=n+\tau}^{\infty} \frac{1}{a_t} f(t, y_{b_{1t}}, \dots, y_{b_{kt}}) \right. \\ &\quad + \sum_{s=n+\tau}^{\infty} \sum_{t=s}^{\infty} \frac{1}{a_s} [(t-s+1)h(t, y_{d_{1t}}, \dots, y_{d_{kt}}) \\ &\quad\quad - g(t, y_{c_{1t}}, \dots, y_{c_{kt}})] \\ &\quad \left. - \sum_{j=n+\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{a_j} [p(t, y_{o_{1t}}, \dots, y_{o_{kt}}) - r_t] \right) \end{aligned}$$



$$\begin{aligned}
 &\leq \frac{1}{\gamma_{n+\tau}} \left( -L + \frac{M}{\gamma_{n+\tau}} - \sum_{t=T+\tau}^{\infty} \frac{W_t}{|a_t|} \right. \\
 &\quad \left. - \sum_{s=T+\tau}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} [(t-s+1)Q_t + P_t] \right. \\
 &\quad \left. - \sum_{j=T+\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{a_j} (R_t + |r_t|) \right) \\
 &< \frac{1}{\gamma_{n+\tau}} \left( -L - \frac{M}{c} - \min \left\{ L - N, M - \frac{M}{c} - L \right\} \right) \\
 &\leq \frac{-M}{\gamma_{n+\tau}}, \\
 (U_L x)_n + (S_L y)_n &= \frac{1}{\gamma_{n+\tau}} \left( -L - x_{n+\tau} + \sum_{t=n+\tau}^{\infty} \frac{1}{a_t} f(t, y_{b_{1t}}, \dots, y_{b_{kt}}) \right. \\
 &\quad \left. + \sum_{s=n+\tau}^{\infty} \sum_{t=s}^{\infty} \frac{1}{a_s} [(t-s+1)h(t, y_{d_{1t}}, \dots, y_{d_{kt}}) \right. \\
 &\quad \quad \left. - g(t, y_{c_{1t}}, \dots, y_{c_{kt}})] \right. \\
 &\quad \left. - \sum_{j=n+\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{a_j} [p(t, y_{o_{1t}}, \dots, y_{o_{kt}}) \right. \\
 &\quad \quad \left. - r_t] \right) \\
 &\geq \frac{1}{\gamma_{n+\tau}} \left( -L - x_{n+\tau} + \sum_{t=T+\tau}^{\infty} \frac{W_t}{|a_t|} \right. \\
 &\quad \left. + \sum_{s=T+\tau}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} [(t-s+1)Q_t + P_t] \right. \\
 &\quad \left. + \sum_{j=T+\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{a_j} (R_t + |r_t|) \right) \\
 &> \frac{1}{\gamma_{n+\tau}} \left( -L + \min \left\{ L - N, M - \frac{M}{c} - L \right\} \right) \\
 &\geq \frac{-N}{\gamma_{n+\tau}},
 \end{aligned} \tag{62}$$

which yield that  $U_L x + S_L y \in \Omega_{3T}(N, M)$ . The rest of the proof is similar to that of Theorem 9 and hence is omitted. This completes the proof.  $\square$

**Theorem 12.** Assume that there exist constants  $n_1 \in \mathbb{N}_{n_0}$ ,  $M$ ,  $N$ ,  $c$ , and  $d$  with  $M > N > 0$  and  $M(1 - 1/c) > N(1 + 1/d)$ ,  $d > c > 1$ , and nonnegative sequences  $\{W_n\}_{n \in \mathbb{N}_{n_0}}$ ,  $\{P_n\}_{n \in \mathbb{N}_{n_0}}$ ,  $\{Q_n\}_{n \in \mathbb{N}_{n_0}}$ , and  $\{R_n\}_{n \in \mathbb{N}_{n_0}}$  satisfying (13), (56), and

$$-d \leq \gamma_n \leq -c, \quad \forall n \geq n_1. \tag{63}$$

Then (2) possesses uncountably many bounded positive solutions in  $l_\beta^\infty$ .

*Proof.* Let  $L \in (N(1 + 1/d), M(1 - 1/c))$ . It follows from (13) that there exists an integer  $T \geq n_0 + n_1 + \tau + |\beta|$  satisfying

$$\begin{aligned}
 &\sum_{t=T}^{\infty} \frac{W_t}{|a_t|} \\
 &+ \sum_{s=T}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} [(t-s+1)Q_t + P_t] \\
 &+ \sum_{j=T}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} (R_t + |r_t|) \\
 &< \min \left\{ L - \frac{N}{d} - N, M - \frac{M}{c} - L \right\}.
 \end{aligned} \tag{64}$$

Let the mappings  $U_L$  and  $S_L : \Omega_{3T}(N, M) \rightarrow l_\beta^\infty$  be defined by (42) and (61), respectively. By means of (42), (56), (61), and (64), we deduce that for any  $x = \{x_n\}_{n \in \mathbb{Z}_\beta}$ ,  $y = \{y_n\}_{n \in \mathbb{Z}_\beta} \in \Omega_{3T}(N, M)$ , and  $n \geq T$ ,

$$\begin{aligned}
 (U_L x)_n + (S_L y)_n &= \frac{1}{\gamma_{n+\tau}} \left( -L - x_{n+\tau} \right. \\
 &\quad \left. + \sum_{t=n+\tau}^{\infty} \frac{1}{a_t} f(t, y_{b_{1t}}, \dots, y_{b_{kt}}) \right. \\
 &\quad \left. + \sum_{s=n+\tau}^{\infty} \sum_{t=s}^{\infty} \frac{1}{a_s} [(t-s+1)h(t, y_{d_{1t}}, \dots, y_{d_{kt}}) \right. \\
 &\quad \quad \left. - g(t, y_{c_{1t}}, \dots, y_{c_{kt}})] \right. \\
 &\quad \left. - \sum_{j=n+\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{a_j} [p(t, y_{o_{1t}}, \dots, y_{o_{kt}}) - r_t] \right) \\
 &\leq \frac{1}{\gamma_{n+\tau}} \left( -L + \frac{M}{\gamma_{n+\tau}} - \sum_{t=T+\tau}^{\infty} \frac{W_t}{|a_t|} \right. \\
 &\quad \left. - \sum_{s=T+\tau}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} [(t-s+1)Q_t + P_t] \right. \\
 &\quad \left. - \sum_{j=T+\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{a_j} (R_t + |r_t|) \right) \\
 &< \frac{1}{\gamma_{n+\tau}} \left( -L - \frac{M}{c} - \min \left\{ L - \frac{N}{d} - N, M - \frac{M}{c} - L \right\} \right) \\
 &\leq \frac{-M}{\gamma_{n+\tau}},
 \end{aligned}$$

$$\begin{aligned}
 & (U_L x)_n + (S_L y)_n \\
 &= \frac{1}{\gamma_{n+\tau}} \left( -L - x_{n+\tau} + \sum_{t=n+\tau}^{\infty} \frac{1}{a_t} f(t, y_{b_t}, \dots, y_{b_{kt}}) \right. \\
 &\quad + \sum_{s=n+\tau}^{\infty} \sum_{t=s}^{\infty} \frac{1}{a_s} [(t-s+1)h(t, y_{d_{1t}}, \dots, y_{d_{kt}}) \\
 &\quad\quad\quad - g(t, y_{c_{1t}}, \dots, y_{c_{kt}})] \\
 &\quad \left. - \sum_{j=n+\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{a_j} [p(t, y_{o_{1t}}, \dots, y_{o_{kt}}) - r_t] \right) \\
 &\geq \frac{1}{\gamma_{n+\tau}} \left( -L + \frac{N}{d} - x_{n+\tau} + \sum_{t=T+\tau}^{\infty} \frac{W_t}{|a_t|} \right. \\
 &\quad + \sum_{s=T+\tau}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} [(t-s+1)Q_t + P_t] \\
 &\quad \left. + \sum_{j=T+\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{a_j} (R_t + |r_t|) \right) \\
 &> \frac{1}{\gamma_{n+\tau}} \left( -L + \frac{N}{d} + \min \left\{ L - \frac{N}{d} - N, M - \frac{M}{c} - L \right\} \right) \\
 &\geq \frac{-N}{\gamma_{n+\tau}},
 \end{aligned} \tag{65}$$

which mean that  $U_L x + S_L y \in \Omega_{3T}(N, M)$ . The rest of the proof is similar to that of Theorem 9 and hence is omitted. This completes the proof.  $\square$

**Theorem 13.** Assume that there exist constants  $n_1 \in \mathbb{N}_{n_0}$ ,  $M$ , and  $N$  with  $M > N > 0$  and nonnegative sequences  $\{W_n\}_{n \in \mathbb{N}_{n_0}}$ ,  $\{P_n\}_{n \in \mathbb{N}_{n_0}}$ ,  $\{Q_n\}_{n \in \mathbb{N}_{n_0}}$ , and  $\{R_n\}_{n \in \mathbb{N}_{n_0}}$  satisfying (12), (13), and

$$\gamma_n = 1, \quad \forall n \geq n_1. \tag{66}$$

Then (2) possesses uncountably many bounded positive solutions in  $\Omega_1(N, M)$

*Proof.* Let  $L \in (N, M)$ . Equation (13) ensures that there exists  $T \geq n_0 + n_1 + \tau + |\beta|$  sufficiently large such that

$$\begin{aligned}
 & \sum_{t=T}^{\infty} \frac{W_t}{|a_t|} + \sum_{s=T}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} [(t-s+1)Q_t + P_t] \\
 & + \sum_{j=T}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} (R_t + |r_t|) \\
 & < \min \{M - L, L - N\}.
 \end{aligned} \tag{67}$$

Define a mapping  $S_L : \Omega_1(N, M) \rightarrow I_{\beta}^{\infty}$  by

$$(S_L x)_n = \begin{cases} L + \sum_{i=1}^{\infty} \sum_{t=n+(2i-1)\tau}^{n+2i\tau-1} \frac{1}{a_t} f(t, x_{b_{1t}}, \dots, x_{b_{kt}}) \\ + \sum_{i=1}^{\infty} \sum_{s=n+(2i-1)\tau}^{n+2i\tau-1} \sum_{t=s}^{\infty} \frac{1}{a_s} [(t-s+1) \\ \times h(t, x_{d_{1t}}, \dots, x_{d_{kt}}) \\ - g(t, x_{c_{1t}}, \dots, x_{c_{kt}})] \\ - \sum_{i=1}^{\infty} \sum_{j=n+(2i-1)\tau}^{n+2i\tau-1} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{a_j} \\ \times [p(t, x_{o_{1t}}, \dots, x_{o_{kt}}) \\ - r_t], & n \geq T, \\ (S_L x)_T, & \beta \leq n < T, \end{cases} \tag{68}$$

for each  $x = \{x_n\}_{n \in \mathbb{Z}_{\beta}} \in \Omega_1(N, M)$ . In view of (12), (67), and (68), we deduce that for every  $x = \{x_n\}_{n \in \mathbb{Z}_{\beta}} \in \Omega_1(N, M)$  and  $n \geq T$ ,

$$\begin{aligned}
 & |(S_L x)_n - L| \\
 &= \left| \sum_{i=1}^{\infty} \sum_{t=n+(2i-1)\tau}^{n+2i\tau-1} \frac{1}{a_t} f(t, x_{b_{1t}}, \dots, x_{b_{kt}}) \right. \\
 &\quad + \sum_{i=1}^{\infty} \sum_{s=n+(2i-1)\tau}^{n+2i\tau-1} \sum_{t=s}^{\infty} \frac{1}{a_s} [(t-s+1)h(t, x_{d_{1t}}, \dots, x_{d_{kt}}) \\
 &\quad\quad\quad - g(t, x_{c_{1t}}, \dots, x_{c_{kt}})] \\
 &\quad \left. - \sum_{i=1}^{\infty} \sum_{j=n+(2i-1)\tau}^{n+2i\tau-1} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{a_j} \right. \\
 &\quad\quad\quad \left. \times [p(t, x_{o_{1t}}, \dots, x_{o_{kt}}) - r_t] \right| \\
 &\leq \sum_{i=1}^{\infty} \sum_{t=n+(2i-1)\tau}^{n+2i\tau-1} \frac{W_t}{|a_t|} \\
 &\quad + \sum_{i=1}^{\infty} \sum_{s=n+(2i-1)\tau}^{n+2i\tau-1} \sum_{t=s}^{\infty} \frac{1}{|a_s|} [(t-s+1)Q_t + P_t] \\
 &\quad + \sum_{i=1}^{\infty} \sum_{j=n+(2i-1)\tau}^{n+2i\tau-1} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} (R_t + |r_t|) \\
 &\leq \sum_{t=T}^{\infty} \frac{W_t}{|a_t|} + \sum_{s=T}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} [(t-s+1)Q_t + P_t] \\
 &\quad + \sum_{j=T}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} (R_t + |r_t|) \\
 &< \min \{M - L, L - N\},
 \end{aligned} \tag{69}$$

which means that  $S_L(\Omega_1(N, M)) \subseteq \Omega_1(N, M)$  and  $\|S_L x\| \leq M$  for all  $x \in \Omega_1(N, M)$ . It follows from (13) that for each  $\varepsilon > 0$ , there exists  $V > T$  satisfying (22). Using (22) and (68), we obtain that for any  $x = \{x_n\}_{n \in \mathbb{Z}_\beta} \in \Omega_1(N, M)$  and  $m, n > V$ ,

$$\begin{aligned} & |(S_L x)_m - (S_L x)_n| \\ &= \left| \sum_{i=1}^{\infty} \sum_{t=m+(2i-1)\tau}^{m+2i\tau-1} \frac{1}{a_t} f(t, x_{b_{1t}}, \dots, x_{b_{kt}}) \right. \\ &\quad - \sum_{i=1}^{\infty} \sum_{t=n+(2i-1)\tau}^{n+2i\tau-1} \frac{1}{a_t} f(t, x_{b_{1t}}, \dots, x_{b_{kt}}) \\ &\quad + \sum_{i=1}^{\infty} \sum_{s=m+(2i-1)\tau}^{m+2i\tau-1} \sum_{t=s}^{\infty} \frac{1}{a_s} \left[ (t-s+1) h(t, x_{d_{1t}}, \dots, x_{d_{kt}}) \right. \\ &\quad \quad \left. - g(t, x_{c_{1t}}, \dots, x_{c_{kt}}) \right] \\ &\quad - \sum_{i=1}^{\infty} \sum_{s=n+(2i-1)\tau}^{n+2i\tau-1} \sum_{t=s}^{\infty} \frac{1}{a_s} \left[ (t-s+1) h(t, x_{d_{1t}}, \dots, x_{d_{kt}}) \right. \\ &\quad \quad \left. - g(t, x_{c_{1t}}, \dots, x_{c_{kt}}) \right] \\ &\quad + \sum_{i=1}^{\infty} \sum_{j=n+(2i-1)\tau}^{n+2i\tau-1} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{a_j} \\ &\quad \quad \times [p(t, x_{o_{1t}}, \dots, x_{o_{kt}}) - r_t] \\ &\quad - \sum_{i=1}^{\infty} \sum_{j=m+(2i-1)\tau}^{m+2i\tau-1} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{a_j} \\ &\quad \quad \times [p(t, x_{o_{1t}}, \dots, x_{o_{kt}}) - r_t] \Big| \\ &\leq \sum_{i=1}^{\infty} \sum_{t=m+(2i-1)\tau}^{m+2i\tau-1} \frac{W_t}{|a_t|} + \sum_{i=1}^{\infty} \sum_{t=n+(2i-1)\tau}^{n+2i\tau-1} \frac{W_t}{|a_t|} \\ &\quad + \sum_{i=1}^{\infty} \sum_{s=m+(2i-1)\tau}^{m+2i\tau-1} \sum_{t=s}^{\infty} \frac{1}{|a_s|} [(t-s+1)Q_t + P_t] \\ &\quad + \sum_{i=1}^{\infty} \sum_{s=n+(2i-1)\tau}^{n+2i\tau-1} \sum_{t=s}^{\infty} \frac{1}{|a_s|} [(t-s+1)Q_t + P_t] \\ &\quad + \sum_{i=1}^{\infty} \sum_{j=n+(2i-1)\tau}^{n+2i\tau-1} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} (R_t + |r_t|) \\ &\quad + \sum_{i=1}^{\infty} \sum_{j=m+(2i-1)\tau}^{m+2i\tau-1} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} (R_t + |r_t|) \end{aligned}$$

$$\begin{aligned} &\leq 2 \sum_{t=V}^{\infty} \frac{W_t}{|a_t|} + 2 \sum_{s=V}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} [(t-s+1)Q_t + P_t] \\ &\quad + 2 \sum_{j=V}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} (R_t + |r_t|) \\ &< \varepsilon, \end{aligned} \tag{70}$$

which yields that  $S_L(\Omega_1(N, M))$  is uniformly Cauchy.

Now we prove that  $S_L$  is continuous in  $\Omega_1(N, M)$ . Suppose that  $\{x^m\}_{m \in \mathbb{N}}$  is an arbitrary sequence in  $\Omega_1(N, M)$  and  $x \in \Omega_1(N, M)$  with  $\lim_{m \rightarrow \infty} x^m = x$ , where  $x^m = \{x_n^m\}_{n \in \mathbb{Z}_\beta}$  for each  $m \in \mathbb{N}$  and  $x = \{x_n\}_{n \in \mathbb{Z}_\beta}$ . Using (12), (13),  $\lim_{m \rightarrow \infty} x^m = x$ , and the continuity of  $f, g, h$ , and  $p$ , we conclude that for given  $\varepsilon > 0$ , there exist  $T_1, T_2, T_3$ , and  $T_4 \in \mathbb{N}$  with  $T_4 > T_3 > T_2 > T_1 > T$  satisfying (19) and (20). It follows from (19), (20), and (68) that

$$\begin{aligned} &\|S_L x^m - S_L x\| \\ &= \sup_{n \in \mathbb{Z}_\beta} |(S_L x^m)_n - (S_L x)_n| \\ &= \max \left\{ \sup_{T > n \geq \beta} |(S_L x^m)_n - (S_L x)_n|, \right. \\ &\quad \left. \sup_{n \geq T} |(S_L x^m)_n - (S_L x)_n| \right\} \\ &= \sup_{n \geq T} \left| L + \sum_{i=1}^{\infty} \sum_{t=n+(2i-1)\tau}^{n+2i\tau-1} \frac{1}{a_t} f(t, x_{b_{1t}}^m, \dots, x_{b_{kt}}^m) \right. \\ &\quad + \sum_{i=1}^{\infty} \sum_{s=n+(2i-1)\tau}^{n+2i\tau-1} \sum_{t=s}^{\infty} \frac{1}{a_s} [(t-s+1)h(t, x_{d_{1t}}^m, \dots, x_{d_{kt}}^m) \\ &\quad \quad \left. - g(t, x_{c_{1t}}^m, \dots, x_{c_{kt}}^m)] \right. \\ &\quad - \sum_{i=1}^{\infty} \sum_{j=n+(2i-1)\tau}^{n+2i\tau-1} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{a_j} [p(t, x_{o_{1t}}^m, \dots, x_{o_{kt}}^m) - r_t] \\ &\quad \left. - \left( L + \sum_{i=1}^{\infty} \sum_{t=n+(2i-1)\tau}^{n+2i\tau-1} \frac{1}{a_t} f(t, x_{b_{1t}}, \dots, x_{b_{kt}}) \right) \right. \\ &\quad \left. + \sum_{i=1}^{\infty} \sum_{s=n+(2i-1)\tau}^{n+2i\tau-1} \sum_{t=s}^{\infty} \frac{1}{a_s} [(t-s+1) \right. \\ &\quad \quad \left. \times h(t, x_{d_{1t}}, \dots, x_{d_{kt}}) \right. \\ &\quad \quad \left. - g(t, x_{c_{1t}}, \dots, x_{c_{kt}}) \right] \\ &\quad - \sum_{i=1}^{\infty} \sum_{j=n+(2i-1)\tau}^{n+2i\tau-1} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{a_j} \end{aligned}$$

$$\begin{aligned}
 & \left. \times [p(t, x_{o_{1t}}, \dots, x_{o_{kt}}) - r_t] \right) \Bigg| \\
 \leq & \sum_{i=1}^{\infty} \sum_{t=n+(2i-1)\tau}^{n+2i\tau-1} \frac{1}{|a_t|} \left| f(t, x_{b_{1t}}^m, \dots, x_{b_{kt}}^m) \right. \\
 & \left. - f(t, x_{b_{1t}}, \dots, x_{b_{kt}}) \right| \\
 + & \sum_{i=1}^{\infty} \sum_{s=n+(2i-1)\tau}^{n+2i\tau-1} \sum_{t=s}^{\infty} \frac{1}{|a_s|} \left[ (t-s+1) |h(t, x_{d_{1t}}^m, \dots, x_{d_{kt}}^m) \right. \\
 & \left. - h(t, x_{d_{1t}}, \dots, x_{d_{kt}}) \right] \\
 & + |g(t, x_{c_{1t}}^m, \dots, x_{c_{kt}}^x) \\
 & \left. - g(t, x_{c_{1t}}, \dots, x_{c_{kt}}) \right| \\
 + & \sum_{i=1}^{\infty} \sum_{j=n+(2i-1)\tau}^{n+2i\tau-1} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} \\
 & \times |p(t, x_{o_{1t}}^m, \dots, x_{o_{kt}}^m) \\
 & \left. - p(t, x_{o_{1t}}, \dots, x_{o_{kt}}) \right| \\
 \leq & \sum_{t=T}^{\infty} \frac{1}{|a_t|} |f(t, x_{b_{1t}}^m, \dots, x_{b_{kt}}^m) - f(t, x_{b_{1t}}, \dots, x_{b_{kt}})| \\
 + & \sum_{s=T}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} \left[ (t-s+1) |h(t, x_{d_{1t}}^m, \dots, x_{d_{kt}}^m) \right. \\
 & \left. - h(t, x_{d_{1t}}, \dots, x_{d_{kt}}) \right] \\
 & + |g(t, x_{c_{1t}}^m, \dots, x_{c_{kt}}^x) - g(t, x_{c_{1t}}, \dots, x_{c_{kt}})| \\
 + & \sum_{j=T}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} |p(t, x_{o_{1t}}^m, \dots, x_{o_{kt}}^m) \\
 & \left. - p(t, x_{o_{1t}}, \dots, x_{o_{kt}}) \right| \\
 < & \varepsilon,
 \end{aligned} \tag{71}$$

which implies that  $S_L$  is continuous in  $\Omega_1(N, M)$ . Thus Lemma 4 means that  $S_L$  possesses a fixed point  $x = \{x_n\}_{n \in \mathbb{Z}_\beta} \in \Omega_1(N, M)$ ; that is,

$$x_n = L + \sum_{i=1}^{\infty} \sum_{t=n+(2i-1)\tau}^{n+2i\tau-1} \frac{1}{a_t} f(t, x_{b_{1t}}, \dots, x_{b_{kt}})$$

$$\begin{aligned}
 & + \sum_{i=1}^{\infty} \sum_{s=n+(2i-1)\tau}^{n+2i\tau-1} \sum_{t=s}^{\infty} \frac{1}{a_s} \left[ (t-s+1) \right. \\
 & \left. \times h(t, x_{d_{1t}}, \dots, x_{d_{kt}}) \right. \\
 & \left. - g(t, x_{c_{1t}}, \dots, x_{c_{kt}}) \right] \\
 - & \sum_{i=1}^{\infty} \sum_{j=n+(2i-1)\tau}^{n+2i\tau-1} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{a_j} \\
 & \times [p(t, x_{o_{1t}}, \dots, x_{o_{kt}}) - r_t], \\
 & \forall n \geq T + \tau, \\
 x_{n-\tau} = & L + \sum_{i=1}^{\infty} \sum_{t=n+(2i-2)\tau}^{n+(2i-1)\tau-1} \frac{1}{a_t} f(t, x_{b_{1t}}, \dots, x_{b_{kt}}) \\
 + & \sum_{i=1}^{\infty} \sum_{s=n+(2i-2)\tau}^{n+(2i-1)\tau-1} \sum_{t=s}^{\infty} \frac{1}{a_s} \left[ (t-s+1) \right. \\
 & \left. \times h(t, x_{d_{1t}}, \dots, x_{d_{kt}}) \right. \\
 & \left. - g(t, x_{c_{1t}}, \dots, x_{c_{kt}}) \right] \\
 - & \sum_{i=1}^{\infty} \sum_{j=n+(2i-2)\tau}^{n+(2i-1)\tau-1} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{a_j} \\
 & \times [p(t, x_{o_{1t}}, \dots, x_{o_{kt}}) - r_t], \\
 & \forall n \geq T + \tau,
 \end{aligned} \tag{72}$$

which imply that

$$\begin{aligned}
 x_n + x_{n-\tau} & = 2L + \sum_{t=n}^{\infty} \frac{1}{a_t} f(t, x_{b_{1t}}, \dots, x_{b_{kt}}) \\
 & + \sum_{s=n}^{\infty} \sum_{t=s}^{\infty} \frac{1}{a_s} \left[ (t-s+1) h(t, x_{d_{1t}}, \dots, x_{d_{kt}}) \right. \\
 & \left. - g(t, x_{c_{1t}}, \dots, x_{c_{kt}}) \right] \\
 - & \sum_{j=n}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{a_j} [p(t, x_{o_{1t}}, \dots, x_{o_{kt}}) - r_t],
 \end{aligned}$$

$$\forall n \geq T + \tau, \tag{73}$$

which means that

$$\begin{aligned} &\Delta(a_n \Delta(x_n + x_{n-\tau})) + \Delta f(n, x_{b_{1n}}, \dots, x_{b_{kn}}) \\ &\quad + g(n, x_{c_{1n}}, \dots, x_{c_{kn}}) \\ &= \sum_{t=n}^{\infty} h(n, x_{d_{1n}}, \dots, x_{d_{kn}}) \\ &\quad - \sum_{t=n}^{\infty} (t - n + 1) [p(t, x_{o_{1t}}, \dots, x_{o_{kt}}) - r_t], \end{aligned} \tag{74}$$

$$\forall n \geq T + \tau,$$

$$\begin{aligned} &\Delta^3(a_n \Delta(x_n + x_{n-\tau})) + \Delta^3 f(n, x_{b_{1n}}, \dots, x_{b_{kn}}) \\ &\quad + \Delta^2 g(n, x_{c_{1n}}, \dots, x_{c_{kn}}) + \Delta h(n, x_{d_{1n}}, \dots, x_{d_{kn}}) \\ &\quad + p(n, x_{o_{1n}}, \dots, x_{o_{kn}}) = r_n, \quad \forall n \geq T + \tau, \end{aligned}$$

which yields that  $x = \{x_n\}_{n \in \mathbb{Z}_\beta} \in \Omega_1(N, M)$  is bounded positive solution of (2). The rest of the proof is similar to that of Theorem 6 and is omitted. This completes the proof.  $\square$

**Theorem 14.** Assume that there exist constants  $n_1 \in \mathbb{N}_{n_0}$ ,  $M$ , and  $N$  with  $M > N > 0$  and nonnegative sequences  $\{W_n\}_{n \in \mathbb{N}_{n_0}}$ ,  $\{P_n\}_{n \in \mathbb{N}_{n_0}}$ ,  $\{Q_n\}_{n \in \mathbb{N}_{n_0}}$ , and  $\{R_n\}_{n \in \mathbb{N}_{n_0}}$  satisfying (12) and

$$\gamma_n = -1, \quad \forall n \geq n_1; \tag{75}$$

$$\begin{aligned} &\max \left\{ \sum_{i=1}^{\infty} \sum_{t=n_0+i\tau}^{\infty} \frac{W_t}{|a_t|}, \right. \\ &\quad \sum_{i=1}^{\infty} \sum_{s=n_0+i\tau}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} [P_t + (t - s + 1) Q_t], \\ &\quad \sum_{i=1}^{\infty} \sum_{j=n_0+i\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t - s + 1}{|a_j|} \\ &\quad \left. \times (R_t + |r_t|) \right\} < +\infty. \end{aligned} \tag{76}$$

Then (2) possesses uncountably many bounded positive solution in  $\Omega_1(N, M)$ .

*Proof.* Let  $L \in (N, M)$ . It follows from (76) that there exists  $T \geq n_0 + n_1 + \tau + |\beta|$  sufficiently large such that

$$\begin{aligned} &\sum_{i=1}^{\infty} \sum_{t=T+i\tau}^{\infty} \frac{W_t}{|a_t|} \\ &\quad + \sum_{i=1}^{\infty} \sum_{s=T+i\tau}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} [(t - s + 1) Q_t + P_t] \\ &\quad + \sum_{i=1}^{\infty} \sum_{j=T+i\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t - s + 1}{|a_j|} (R_t + |r_t|) \\ &\quad < \min \{M - L, L - N\}. \end{aligned} \tag{77}$$

Define a mapping  $S_L : \Omega_1(N, M) \rightarrow l_\beta^\infty$  by

$$\begin{aligned} &(S_L x)_n \\ &= \begin{cases} L - \sum_{i=1}^{\infty} \sum_{t=n+i\tau}^{\infty} \frac{1}{a_t} f(t, x_{b_{1t}}, \dots, x_{b_{kt}}) \\ \quad - \sum_{i=1}^{\infty} \sum_{s=n+i\tau}^{\infty} \sum_{t=s}^{\infty} \frac{1}{a_s} \\ \quad \quad \times [(t - s + 1) h(t, x_{d_{1t}}, \dots, x_{d_{kt}}) \\ \quad \quad \quad - g(t, x_{c_{1t}}, \dots, x_{c_{kt}})] \\ \quad - \sum_{i=1}^{\infty} \sum_{j=n+i\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t - s + 1}{a_j} \\ \quad \quad \quad \times [p(t, x_{o_{1t}}, \dots, x_{o_{kt}}) - r_t], \\ \quad \quad \quad \quad \quad \quad \quad \quad n \geq T, \\ (S_L x)_T, \quad \quad \quad \quad \quad \quad \quad \beta \leq n < T, \end{cases} \end{aligned} \tag{78}$$

for all  $x = \{x_n\}_{n \in \mathbb{Z}_\beta} \in \Omega_1(N, M)$ . By virtue of (12), (77), and (78), we know that for every  $x = \{x_n\}_{n \in \mathbb{Z}_\beta} \in \Omega_1(N, M)$  and  $n \geq T$ ,

$$\begin{aligned} &|(S_L x)_n - L| \\ &= \left| - \sum_{i=1}^{\infty} \sum_{t=n+i\tau}^{\infty} \frac{1}{a_t} f(t, x_{b_{1t}}, \dots, x_{b_{kt}}) \right. \\ &\quad - \sum_{i=1}^{\infty} \sum_{s=n+i\tau}^{\infty} \sum_{t=s}^{\infty} \frac{1}{a_s} [(t - s + 1) h(t, x_{d_{1t}}, \dots, x_{d_{kt}}) \\ &\quad \quad \quad - g(t, x_{c_{1t}}, \dots, x_{c_{kt}})] \\ &\quad - \sum_{i=1}^{\infty} \sum_{j=n+i\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t - s + 1}{a_j} \\ &\quad \quad \quad \left. \times [p(t, x_{o_{1t}}, \dots, x_{o_{kt}}) - r_t] \right| \\ &\leq \sum_{i=1}^{\infty} \sum_{t=n+i\tau}^{\infty} \frac{W_t}{|a_t|} \\ &\quad + \sum_{i=1}^{\infty} \sum_{s=n+i\tau}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} [(t - s + 1) Q_t + P_t] \end{aligned}$$

$$\begin{aligned}
 & + \sum_{i=1}^{\infty} \sum_{j=n+ir}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} (R_t + |r_t|) \\
 & \leq \sum_{i=1}^{\infty} \sum_{t=T+ir}^{\infty} \frac{W_t}{|a_t|} \\
 & + \sum_{i=1}^{\infty} \sum_{s=T+ir}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} [(t-s+1)Q_t + P_t] \\
 & + \sum_{i=1}^{\infty} \sum_{j=T+ir}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} (R_t + |r_t|) \\
 & < \min \{M - L, L - N\},
 \end{aligned} \tag{79}$$

which means that  $S_L(\Omega_1(N, M)) \subseteq \Omega_1(N, M)$  and  $\|S_L x\| \leq M$  for each  $x \in \Omega_1(N, M)$ .

Next we prove that  $S_L(\Omega_1(N, M))$  is uniformly Cauchy. It follows from (76) that for any given  $\varepsilon > 0$  there exists  $T^* \geq T$  with

$$\begin{aligned}
 & \sum_{i=1}^{\infty} \sum_{t=T^*+ir}^{\infty} \frac{W_t}{|a_t|} \\
 & + \sum_{i=1}^{\infty} \sum_{s=T^*+ir}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} [(t-s+1)Q_t + P_t] \\
 & + \sum_{i=1}^{\infty} \sum_{j=T^*+ir}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} (R_t + |r_t|) \\
 & < \frac{\varepsilon}{2}.
 \end{aligned} \tag{80}$$

It follows from (12), (78), and (80) that for any  $x = \{x_n\}_{n \in \mathbb{Z}_\beta} \in \Omega_1(N, M)$  and  $m, n \geq T^*$ ,

$$\begin{aligned}
 & |(S_L x)_m - (S_L x)_n| \\
 & = \left| \sum_{i=1}^{\infty} \sum_{t=n+ir}^{\infty} \frac{1}{a_t} f(t, x_{b_{1t}}, \dots, x_{b_{kt}}) \right. \\
 & - \sum_{i=1}^{\infty} \sum_{t=m+ir}^{\infty} \frac{1}{a_t} f(t, x_{b_{1t}}, \dots, x_{b_{kt}}) \\
 & + \sum_{i=1}^{\infty} \sum_{s=n+ir}^{\infty} \sum_{t=s}^{\infty} \frac{1}{a_s} [(t-s+1)h(t, x_{d_{1t}}, \dots, x_{d_{kt}}) \\
 & \quad \quad \quad \left. - g(t, x_{c_{1t}}, \dots, x_{c_{kt}})] \right. \\
 & - \sum_{i=1}^{\infty} \sum_{s=m+ir}^{\infty} \sum_{t=s}^{\infty} \frac{1}{a_s} [(t-s+1)h(t, x_{d_{1t}}, \dots, x_{d_{kt}}) \\
 & \quad \quad \quad \left. - g(t, y_{c_{1t}}, \dots, y_{c_{kt}})] \right|
 \end{aligned}$$

$$\begin{aligned}
 & + \sum_{i=1}^{\infty} \sum_{j=n+ir}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{a_j} [p(t, x_{o_{1t}}, \dots, x_{o_{kt}}) - r_t] \\
 & - \sum_{i=1}^{\infty} \sum_{j=m+ir}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{a_j} \\
 & \quad \quad \quad \times [p(t, x_{o_{1t}}, \dots, x_{o_{kt}}) - r_t] \Big| \\
 & \leq \sum_{i=1}^{\infty} \sum_{t=n+ir}^{\infty} \frac{W_t}{|a_t|} + \sum_{i=1}^{\infty} \sum_{t=m+ir}^{\infty} \frac{W_t}{|a_t|} \\
 & + \sum_{i=1}^{\infty} \sum_{s=n+ir}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} [(t-s+1)Q_t + P_t] \\
 & + \sum_{i=1}^{\infty} \sum_{s=m+ir}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} [(t-s+1)Q_t + P_t] \\
 & + \sum_{i=1}^{\infty} \sum_{j=n+ir}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} (R_t + |r_t|) \\
 & + \sum_{i=1}^{\infty} \sum_{j=m+ir}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} (R_t + |r_t|) \\
 & \leq 2 \sum_{i=1}^{\infty} \sum_{t=T^*+ir}^{\infty} \frac{W_t}{|a_t|} \\
 & + 2 \sum_{i=1}^{\infty} \sum_{s=T^*+ir}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} [(t-s+1)Q_t + P_t] \\
 & + 2 \sum_{i=1}^{\infty} \sum_{j=T^*+ir}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} (R_t + |r_t|) \\
 & < \varepsilon,
 \end{aligned} \tag{81}$$

which yields that  $S_L(\Omega_1(N, M))$  is uniformly Cauchy.

Now we prove that  $S_L$  is continuous in  $\Omega_1(N, M)$ . Suppose that  $\{x^m\}_{m \in \mathbb{N}}$  is an arbitrary sequence in  $\Omega_1(N, M)$  and  $x \in \Omega_1(N, M)$  with  $\lim_{m \rightarrow \infty} x^m = x$ , where  $x^m = \{x_n^m\}_{n \in \mathbb{Z}_\beta}$  for each  $m \in \mathbb{N}$  and  $x = \{x_n\}_{n \in \mathbb{Z}_\beta}$ . By (12), (76),  $\lim_{m \rightarrow \infty} x^m = x$ , and the continuity of  $f, g, h$ , and  $p$ , we get that for given  $\varepsilon > 0$ , there exist  $T_1, T_2, T_3, T_4$ , and  $T_5 \in \mathbb{N}$  with  $T_5 > T_4 > T_3 > T_2 > T_1 > T$  satisfying

$$\begin{aligned}
 & \sum_{i=T_1}^{\infty} \sum_{t=T+ir}^{\infty} \frac{W_t}{|a_t|} \\
 & + \sum_{i=T_1}^{\infty} \sum_{s=T+ir}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} [(t-s+1)Q_t + P_t]
 \end{aligned}$$

$$\begin{aligned}
 & + \sum_{i=T_1}^{\infty} \sum_{j=T+i\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} R_t < \frac{\varepsilon}{18}; \\
 & \sum_{t=T+T_2\tau}^{\infty} \frac{W_t}{|a_t|} + \sum_{s=T+T_2\tau}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} [(t-s+1)Q_t + P_t] \\
 & + \sum_{j=T+T_2\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} R_t < \frac{\varepsilon}{18T_1}; \\
 & \max \left\{ \sum_{t=T+T_3\tau}^{\infty} \frac{1}{|a_s|} [(t-s+1)Q_t + P_t] \right. \\
 & \quad \left. + \sum_{s=T+T_3\tau}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} R_t \right. \\
 & \quad \left. : T+\tau \leq s \leq T+T_2\tau, T+\tau \leq j \leq T+T_2\tau \right\} \\
 & < \frac{\varepsilon}{18T_1T_2\tau}; \\
 & \max \left\{ \sum_{t=T+T_4\tau}^{\infty} \frac{t-s+1}{|a_j|} R_t \right. \\
 & \quad \left. : T+\tau \leq j \leq T+T_2\tau, j \leq s \leq T+T_3\tau \right\} \\
 & < \frac{\varepsilon}{18T_1T_2T_3\tau^2};
 \end{aligned}$$

$$\begin{aligned}
 & \sum_{i=1}^{T_1-1} \sum_{t=T+i\tau}^{T+T_2\tau-1} A \left| f(t, x_{b_{it}}^m, \dots, x_{b_{kt}}^m) - f(t, x_{b_{it}}^m, \dots, x_{b_{kt}}^m) \right| \\
 & + \sum_{i=1}^{T_1-1} \sum_{s=T+i\tau}^{T+T_2\tau-1} \sum_{t=s}^{T+T_3\tau-1} \left[ B \left| h(t, x_{d_{it}}^m, \dots, x_{d_{kt}}^m) \right. \right. \\
 & \quad \left. \left. - h(t, x_{d_{it}}^m, \dots, x_{d_{kt}}^m) \right| \right. \\
 & \quad \left. + A \left| g(t, x_{c_{it}}^m, \dots, x_{c_{kt}}^m) \right. \right. \\
 & \quad \left. \left. - g(t, x_{c_{it}}^m, \dots, x_{c_{kt}}^m) \right| \right] \\
 & + \sum_{i=1}^{T_1-1} \sum_{j=T+i\tau}^{T+T_2\tau-1} \sum_{s=j}^{T+T_3\tau-1} \sum_{t=s}^{T+T_4\tau-1} E \left| p(t, x_{o_{it}}^m, \dots, x_{o_{kt}}^m) \right. \\
 & \quad \left. - p(t, x_{o_{it}}^m, \dots, x_{o_{kt}}^m) \right| < \frac{\varepsilon}{18}, \tag{82}
 \end{aligned}$$

where

$$A = \max \left\{ \frac{1}{|a_s|} : T \leq s \leq T+T_2\tau \right\},$$

$$\begin{aligned}
 B & = \max \left\{ \frac{t-s+1}{|a_s|} : T \leq s \leq T+T_2\tau, \right. \\
 & \quad \left. s \leq t \leq T+T_3\tau \right\}, \\
 E & = \max \left\{ \frac{t-s+1}{|a_j|} : T \leq j \leq T+T_2\tau, \right. \\
 & \quad \left. j \leq s \leq T+T_3\tau, s \leq t \leq T+T_4\tau \right\}. \tag{83}
 \end{aligned}$$

In terms of (12), (78), and (83), we have

$$\begin{aligned}
 & \|S_L x^m - S_L x\| \\
 & = \sup_{n \in \mathbb{Z}_\beta} |(S_L x^m)_n - (S_L x)_n| \\
 & = \max \left\{ \sup_{\beta \geq n > T} |(S_L x^m)_n - (S_L x)_n|, \right. \\
 & \quad \left. \sup_{n \geq T} |(S_L x^m)_n - (S_L x)_n| \right\} \\
 & = \sup_{n \geq T} \left| L - \sum_{i=1}^{\infty} \sum_{t=n+i\tau}^{\infty} \frac{1}{a_t} f(t, x_{b_{it}}^m, \dots, x_{b_{kt}}^m) \right. \\
 & \quad \left. - \sum_{i=1}^{\infty} \sum_{s=n+i\tau}^{\infty} \sum_{t=s}^{\infty} \frac{1}{a_s} [(t-s+1)h(t, x_{d_{it}}^m, \dots, x_{d_{kt}}^m) \right. \\
 & \quad \quad \left. - g(t, x_{c_{it}}^m, \dots, x_{c_{kt}}^m)] \right. \\
 & \quad \left. - \sum_{i=1}^{\infty} \sum_{j=n+i\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{a_j} [p(t, x_{o_{it}}^m, \dots, x_{o_{kt}}^m) - r_t] \right. \\
 & \quad \left. - \left( L - \sum_{i=1}^{\infty} \sum_{t=n+i\tau}^{\infty} \frac{1}{a_t} f(t, x_{b_{it}}^m, \dots, x_{b_{kt}}^m) \right. \right. \\
 & \quad \left. \left. - \sum_{i=1}^{\infty} \sum_{s=n+i\tau}^{\infty} \sum_{t=s}^{\infty} \frac{1}{a_s} [(t-s+1)h(t, x_{d_{it}}^m, \dots, x_{d_{kt}}^m) \right. \right. \\
 & \quad \quad \left. \left. - g(t, x_{c_{it}}^m, \dots, x_{c_{kt}}^m)] \right. \right. \\
 & \quad \left. \left. - \sum_{i=1}^{\infty} \sum_{j=n+i\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{a_j} \right. \right. \\
 & \quad \left. \left. \times [p(t, x_{o_{it}}^m, \dots, x_{o_{kt}}^m) - r_t] \right) \right|
 \end{aligned}$$

$$\begin{aligned}
 & \leq \sum_{i=1}^{\infty} \sum_{t=T+i\tau}^{\infty} \frac{1}{|a_t|} \left| f(t, x_{b_{it}}^m, \dots, x_{b_{kt}}^m) - f(t, x_{b_{it}}^m, \dots, x_{b_{kt}}^m) \right| \\
 & \quad + \sum_{i=1}^{\infty} \sum_{s=T+i\tau}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} [(t-s+1)|h(t, x_{d_{it}}^m, \dots, x_{d_{kt}}^m)
 \end{aligned}$$





$$\begin{aligned}
 & + A \left| g \left( t, x_{c_{1t}}^m, \dots, x_{c_{kt}}^m \right) \right. \\
 & \quad \left. - g \left( t, x_{c_{1t}}, \dots, x_{c_{kt}} \right) \right] \\
 & + \sum_{i=1}^{T_1-1} \sum_{j=T+i\tau}^{T_2\tau-1} \sum_{s=j}^{T_3\tau-1} \sum_{t=s}^{T_4\tau-1} E \left| p \left( t, x_{o_{1t}}^m, \dots, x_{o_{kt}}^m \right) \right. \\
 & \quad \left. - p \left( t, x_{o_{1t}}, \dots, x_{o_{kt}} \right) \right] \\
 & + 2 \sum_{i=1}^{T_1-1} \sum_{t=T+T_2\tau}^{\infty} \frac{W_t}{|a_t|} + 2 \sum_{i=T_1}^{\infty} \sum_{t=T+i\tau}^{\infty} \frac{W_t}{|a_t|} \\
 & + 2 \sum_{i=1}^{T_1-1} \sum_{s=T+i\tau}^{T_2\tau-1} \sum_{t=T+T_3\tau}^{\infty} \frac{1}{|a_s|} \left[ (t-s+1) Q_t + P_t \right] \\
 & + 2 \sum_{i=1}^{T_1-1} \sum_{s=T+T_2\tau}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} \left[ (t-s+1) Q_t + P_t \right] \\
 & + 2 \sum_{i=T_1}^{\infty} \sum_{s=T+i\tau}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} \left[ (t-s+1) Q_t + P_t \right] \\
 & + 2 \sum_{i=1}^{T_1-1} \sum_{j=T+i\tau}^{T_2\tau-1} \sum_{s=j}^{T_3\tau-1} \sum_{t=T+T_4\tau}^{\infty} \frac{t-s+1}{|a_j|} R_t \\
 & + 2 \sum_{i=1}^{T_1-1} \sum_{j=T+i\tau}^{T_2\tau-1} \sum_{s=T+T_3\tau}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} R_t \\
 & + 2 \sum_{i=1}^{T_1-1} \sum_{j=T+T_2\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} R_t \\
 & + 2 \sum_{i=T_1}^{\infty} \sum_{j=T+i\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} R_t \\
 & < \frac{\varepsilon}{18} + \frac{2(T_1-1)(T_2-1)\tau\varepsilon}{18T_1T_2\tau} \\
 & + \frac{2(T_1-1)\varepsilon}{18T_1} \\
 & + \frac{2(T_1-1)(T_2-1)(T_3-1)\tau^2\varepsilon}{18T_1T_2T_3\tau^2} \\
 & + \frac{2\varepsilon}{18} < \varepsilon, \quad \forall m \geq T_5,
 \end{aligned}
 \tag{84}$$

which yields that  $S_L$  is continuous in  $\Omega_1(N, M)$ . It follows from Lemmas 2 and 4 that  $S_L$  has a fixed point  $x = \{x_n\}_{n \in \mathbb{Z}_\beta} \in \Omega_1(N, M)$ ; that is,

$$\begin{aligned}
 x_n & = L - \sum_{i=1}^{\infty} \sum_{t=n+i\tau}^{\infty} \frac{1}{a_t} f \left( t, x_{b_{1t}}, \dots, x_{b_{kt}} \right) \\
 & \quad - \sum_{i=1}^{\infty} \sum_{s=n+i\tau}^{\infty} \sum_{t=s}^{\infty} \frac{1}{a_s} \left[ (t-s+1) \right.
 \end{aligned}$$

$$\begin{aligned}
 & \quad \left. \times h \left( t, x_{d_{1t}}, \dots, x_{d_{kt}} \right) \right. \\
 & \quad \left. - g \left( t, x_{c_{1t}}, \dots, x_{c_{kt}} \right) \right] \\
 & + \sum_{i=1}^{\infty} \sum_{j=n+i\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{a_j} \\
 & \quad \times \left[ p \left( t, x_{o_{1t}}, \dots, x_{o_{kt}} \right) - r_t \right], \\
 & \quad \forall n \geq T + \tau, \\
 x_{n-\tau} & = L - \sum_{i=1}^{\infty} \sum_{t=n+(i-1)\tau}^{\infty} \frac{1}{a_t} f \left( t, x_{b_{1t}}, \dots, x_{b_{kt}} \right) \\
 & \quad - \sum_{i=1}^{\infty} \sum_{s=n+(i-1)\tau}^{\infty} \sum_{t=s}^{\infty} \frac{1}{a_s} \\
 & \quad \times \left[ (t-s+1) h \left( t, x_{d_{1t}}, \dots, x_{d_{kt}} \right) \right. \\
 & \quad \left. - g \left( t, x_{c_{1t}}, \dots, x_{c_{kt}} \right) \right] \\
 & + \sum_{i=1}^{\infty} \sum_{j=n+(i-1)\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{a_j} \\
 & \quad \times \left[ p \left( t, x_{o_{1t}}, \dots, x_{o_{kt}} \right) - r_t \right], \\
 & \quad \forall n \geq T + \tau,
 \end{aligned}
 \tag{85}$$

which imply that

$$\begin{aligned}
 \Delta \left( x_n - x_{n-\tau} \right) & = -\frac{1}{a_n} f \left( n, x_{b_{1n}}, \dots, x_{b_{kn}} \right) \\
 & \quad - \sum_{t=n}^{\infty} \frac{1}{a_n} \left[ (t-n+1) h \left( t, x_{d_{1t}}, \dots, x_{d_{kt}} \right) \right. \\
 & \quad \left. - g \left( t, x_{c_{1t}}, \dots, x_{c_{kt}} \right) \right] \\
 & \quad + \sum_{s=n}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{a_n} \left[ p \left( t, x_{o_{1t}}, \dots, x_{o_{kt}} \right) - r_t \right], \\
 & \quad \forall n \geq T + \tau,
 \end{aligned}
 \tag{86}$$

which yields that

$$\begin{aligned}
 \Delta^3 \left( a_n \Delta \left( x_n - x_{n-\tau} \right) \right) & + \Delta^3 f \left( n, x_{b_{1n}}, \dots, x_{b_{kn}} \right) \\
 & + \Delta^2 g \left( n, x_{c_{1n}}, \dots, x_{c_{kn}} \right) + \Delta h \left( n, x_{d_{1n}}, \dots, x_{d_{kn}} \right) \\
 & + p \left( n, x_{o_{1n}}, \dots, x_{o_{kn}} \right) = r_n, \quad \forall n \geq T + \tau;
 \end{aligned}
 \tag{87}$$

that is,  $x = \{x_n\}_{n \in \mathbb{Z}_\beta} \in \Omega_1(N, M)$  is a bounded positive solution of (2). The rest of the proof is similar to that of Theorem 6 and is omitted. This is completes proof.  $\square$

### 4. Applications

Now we display nine examples as applications of the results presented in Section 3.

*Example 1.* Consider the fourth order nonlinear neutral delay difference equation:

$$\begin{aligned} &\Delta^3 \left( (n^5 + 2n) \Delta \left( x_n + \frac{(-1)^n n^2}{4n^2 + n + 3} x_{n-\tau} \right) \right) \\ &+ \Delta^3 \left( nx_{n^2-n}^{17} + \sqrt{n}x_{n^2-n}^5 - \sqrt{n-3} \right) \\ &+ \Delta^2 \left( \frac{(-1)^n x_{n-3}^{21} + 5n^2 + 1}{\sqrt{n^7 - 17} (x_{n-3}^4 + 3n^2 + 2)} \right) \\ &+ \Delta \left( \frac{12\sqrt{n} - x_{2n-1}^{15} \cos^2(n^3 x_{2n-1}^5)}{2 + n^5 |x_{2n-1}|} \right) \\ &+ \frac{(-1)^{n-1} n \sin^{11}(nx_{n^3-3n}^9)}{\sqrt{n^2 + 2n + 5} (1 + x_{n^3-3n}^{22}) + n^5} \\ &= \frac{4n^3 + 2n - 1}{(n + 1)^{12} \sqrt{3n^2 + 7}}, \end{aligned} \tag{88}$$

$\forall n \geq 7.$

It follows from Theorem 6 that (88) possesses uncountably many bounded positive solutions in  $\Omega_1(N, M)$ .

*Example 2.* Consider the fourth order nonlinear neutral delay difference equation:

$$\begin{aligned} &\Delta^3 \left( (-1)^{n+1} n^{5/2} \Delta \left( x_n + \frac{6n - 3}{10n + 2} x_{n-\tau} \right) \right) \\ &+ \Delta^3 \left( \frac{x_{n^3-5}^5 + 2n^3}{x_{n^3-5}^8 + (n + 3)^4} \right) \\ &+ \Delta^2 \left( \frac{(-1)^{n-1} n^2}{x_{2n-1}^{14} + n^5 + |\cos(x_{2n-1}^2 n^3)|} \right) \\ &+ \Delta \left( \frac{x_{n-1}^4 \sin \sqrt{n + x_{n-1}^2 - n^4}}{n^7 + x_{n-1}^6 \cos^4 \sqrt{|x_{n-1}^3|} + n} \right) \\ &+ \frac{x_{5n-9}^8 n^6}{n^9 + |nx_{5n-9} \sin^3(n^2 + 1)|} \\ &= \frac{\cos(n^3 - 2n + 3)}{n^4 + \sqrt{3n - 1}}, \end{aligned} \tag{89}$$

$\forall n \geq 1.$

It follows from Theorem 7 that (89) possesses uncountably many bounded positive solutions in  $\Omega_1(N, M)$ .

*Example 3.* Consider the fourth order nonlinear neutral delay difference equation:

$$\begin{aligned} &\Delta^3 \left( (n^7 - n^3 + 2) \right. \\ &\quad \left. \times \Delta \left( x_n - \frac{4n^2 \cos^4(3n^2 + 2)}{5n^2 + 2n} x_{n-\tau} \right) \right) \\ &+ \Delta^3 \left( \sqrt{n-9} x_{n^3-5}^3 x_{n+11}^5 \right) \\ &+ \Delta^2 \left( \frac{\sqrt{n+3} x_{n^3-4}^{21} - \sqrt{2n+5} x_{n^2-1}^{12}}{n^5 + 3nx_{n^3-4}^2 x_{n^2-1}^2} \right) \\ &+ \Delta \left( \frac{(x_{n-7} + x_{n^4-1} + n)^4}{|(n + x_{n-7})^5 (n + x_{n^4-1})^3| + 1} \right) \\ &+ \frac{n^2 x_{n^2-3}^3 x_{n-4}^5 + \ln n}{n^2 + x_{n-4}^4 + (n + x_{n^2-3})^6} \\ &= \frac{(-1)^{n+1} (4n^5 - 3)}{n^9 + 4n^2 + 5}, \end{aligned} \tag{90}$$

$\forall n \geq 9.$

Theorem 8 implies that (90) possesses uncountably many bounded positive solutions in  $\Omega_1(N, M)$ .

*Example 4.* Consider the fourth order nonlinear neutral delay difference equation:

$$\begin{aligned} &\Delta^3 \left( n^3 \Delta \left( x_n + \sqrt{\ln(3n^3 - 134n)} x_{n-\tau} \right) \right) \\ &+ \Delta^3 \left( \frac{2n + x_{2n-3}^3 \cos \sqrt{n}}{(n + x_{2n-3})^4 (n + 1)^5 + 72} \right) \\ &+ \Delta^2 \left( \frac{nx_{n^5}^7 \sin^3(n^2 x_{n^5}^3)}{(\sqrt{n+2} + 3)^6} \right) \\ &+ \Delta \left( \frac{x_{n+6}^2 - (-1)^{n+1} n}{\sqrt{n^2 + 1} + (x_{n+6} + \sqrt{n})^{10}} \right) \\ &+ \frac{n^2 + x_{2n-18}^{27}}{n^{13} + x_{2n-18}^2 |\cos^3(nx_{2n-18}^4)|} \\ &= \frac{n^7 - n^{11} - 2}{n^{15} + 36}, \end{aligned} \tag{91}$$

$\forall n \geq 15.$

Theorem 9 yields that (91) has uncountably many bounded positive solutions in  $I_\beta^\infty$ .

*Example 5.* Consider the fourth order nonlinear neutral delay difference equation:

$$\begin{aligned} &\Delta^3 \left( (-1)^n n \ln^2 n \Delta \left( x_n + \left( 1 + \frac{1}{n} \right)^n x_{n-\tau} \right) \right) \\ &+ \Delta^3 \left( \frac{\sqrt{n} x_{n-4}^3 - x_{n-4}^2 + 1}{n^5 + \ln(1 + x_{n-4}^2)} \right) \\ &+ \Delta^2 \left( \frac{n^2 x_{n-9}^3 - n x_{n-9} - \sqrt{n}}{n^8 + (x_{n-9}^4 - n)^4} \right) \\ &+ \Delta \left( \frac{x_{n-6}^5 - (-1)^n n}{n^6 + \ln^3(n + x_{n-6}^8)} \right) \\ &+ \frac{\cos(n - \ln(1 + x_{n-7}^2))}{n^5 + \cos^3(n x_{n-7}^7) + 1} \\ &= \frac{n^2 - n + (-1)^n}{n^9 + 3n + 1}, \end{aligned} \tag{92}$$

$\forall n \geq 10.$

Theorem 10 guarantees that (92) possesses uncountably many bounded positive solutions in  $l_\beta^\infty$ .

*Example 6.* Consider the fourth order nonlinear neutral delay difference equation:

$$\begin{aligned} &\Delta^3 \left( \frac{n^4 + 2n}{\ln(n + 5)} \Delta \left( x_n - (n - 16)^3 x_{n-\tau} \right) \right) \\ &+ \Delta^3 \left( \sqrt{n^2 - 2n} \cos^2(n x_{n-5}^2) \right) \\ &+ \Delta^2 \left( \frac{x_{n^2-2n+3}^3 \ln^2(n - 13)}{n^{11} + |x_{n^2-2n+3} + 1|} \right) \\ &+ \Delta \left( \frac{n x_{n-19}^4 \sin(x_{n-19}^2 + n)}{n^9 + 2n + x_{n-19}^6} \right) \\ &+ \frac{n^2 \sin^{16}(n + x_{n-13}^4)}{(n + x_{n-13}^2)^8} \\ &= \frac{(-1)^{n+1} (n^3 + 11)}{(n + 2)^5 (n + 5)^4}, \end{aligned} \tag{93}$$

$\forall n \geq 18.$

Theorem 11 ensures that (93) possesses uncountably many bounded positive solutions in  $l_\beta^\infty$ .

*Example 7.* Consider the fourth order nonlinear neutral delay difference equation:

$$\begin{aligned} &\Delta^3 \left( -n^8 \Delta \left( x_n - (3 - \sin^9 n) x_{n-\tau} \right) \right) \\ &+ \Delta^3 \left( \frac{n^5 \ln(n + x_{n-30}^2) - x_{n-30}^3}{n^{25} + x_{n-30}^8} \right) \\ &+ \Delta^2 \left( \frac{n^3 x_{n^2-1}^3 - n x_{n^2-1}^2 - \cos(n^5)}{n^{81} + (n x_{n^2-1}^3 - 1)^2} \right) \\ &+ \Delta \left( \frac{n^6 x_{n-2}^3 - n + 3}{n^{16} + (n^3 - x_{n-2}^7)^2} \right) \\ &+ \frac{n^8 + 3n^5 x_{n-4}^2 - x_{n-4}^3}{n^{58} + n^{22} - \sin(n^5 x_{n-4}^3)} \\ &= \frac{n^{25} + (-1)^n n^{11} - 1}{n^{91} + 33n + 5}, \end{aligned} \tag{94}$$

$\forall n \geq 1.$

Theorem 12 guarantees that (94) possesses uncountably many bounded positive solutions in  $l_\beta^\infty$ .

*Example 8.* Consider the fourth order nonlinear neutral delay difference equation:

$$\begin{aligned} &\Delta^3 \left( (-1)^{n-1} (n + 2)^3 \Delta \left( x_n + x_{n-\tau} \right) \right) \\ &+ \Delta^3 \left( \frac{n^4 x_{n^2-9}^5 x_{n-4}^2}{n^3 + |x_{n^2-9}^{25} - n x_{n-4}^{37}|} \right) \\ &+ \Delta^2 \left( \frac{n^4 + x_{n-1}^2 x_{n^3-3}^3}{n^9 + \ln^3(n^2 + x_{n^3-3}^6)} \right) \\ &+ \Delta \left( \frac{\sin(n^9 x_{n-2}^8 x_{5n-6}^9)}{n^5 + (n + 2 x_{n-2}^{82} x_{5n-6}^{19})^2} \right) \\ &+ \frac{\ln(n^2 + x_{n^2-17}^6 x_{n-6}^{14})}{n^{16} + (2 x_{n^2-17} + x_{n-6})^2} \\ &= \frac{(-1)^{n(n+1)/2} \cos^7(3n^5 + 1)}{n^5 + n^3 + 3}, \end{aligned} \tag{95}$$

$\forall n \geq 8.$

It follows from Theorem 13 that (95) possesses uncountably many bounded positive solutions in  $\Omega(N, M)$ .

*Example 9.* Consider the fourth order nonlinear neutral delay difference equation:

$$\begin{aligned} & \Delta^3 \left( n^{17} \Delta (x_n - x_{n-\tau}) \right) \\ & + \Delta^3 \left( \sqrt{n} x_{9n-5}^3 x_{2n-1}^2 - \sin^3 \left( \arctan \left( n^2 + 1 \right) \right) \right) \\ & + \Delta^2 \left( \frac{n^3 + x_{n-7}^4 x_{n^2-2}^5}{(n + x_{n-7}^6)^{13} + |x_{n-7} - n x_{n^2-2}^2|} \right) \\ & + \Delta \left( \frac{x_{n-10}^3 \ln^4 \left( n^2 + x_{n^3-3}^2 \right)}{(n + x_{n-10}^2 x_{n^3-3}^2)^9 (n+2)^{31}} \right) \\ & + \frac{(3-n^2) x_{3n-2}^2 - n x_{n-9}^9}{n^{60} + |x_{3n-2} x_{n-9}|} \\ & = \frac{(-1)^n (n^2 - 1) \cos^5 (n - \sqrt{n})}{n^{19} + 15n^8 + 1}, \end{aligned} \quad (96)$$

$\forall n \geq 11.$

Theorem 14 means that (96) possesses uncountably many bounded positive solutions in  $\Omega(N, M)$ .

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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