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Research Article

On Existence and Uniqueness of g-Best Proximity Points under $(\varphi, \theta, \alpha, g)$-Contractivity Conditions and Consequences

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We collect, improve, and generalize very recent results due to Mongkolkeha et al. (2014) in three directions: firstly, we study g-best proximity points; secondly, we employ more general test functions than can be found in that paper, which lets us prove best proximity results using different kinds of control functions; thirdly, we introduce and handle a weak version of the P-property. Our results can also be applied to the study of coincidence points between two mappings as a particular case. As a consequence, the contractive condition we introduce is more general than was used in the mentioned paper.

1. Introduction

Fixed point theory is a branch of nonlinear analysis which has attracted much attention in recent times due to its possible applications. After the appearance of the pioneering *Banach contractive mapping principle* in 1922, many mathematicians have intensively investigated sufficient conditions to ensure that certain contractive mappings have a fixed point. Some of the most well-known generalizations are due to Zabreĭko and Krasnoselĭ [1], Edelstein [2], Browder [3], and Caristi [4].

When a mapping from a metric space into itself has no fixed points, it could be interesting to study the existence and uniqueness of some points that minimize the distance between an origin and its corresponding image. These points are known as *best proximity points* and they were introduced by Fan [5] and modified by Sadiq Basha in [6]. The study of this kind of points and their properties has become one of the newest branches of fixed point theory, and many interesting results, generalizing the notion of fixed point, have been presented. In fact, many theorems in fixed point theory have been very useful so as to introduce their corresponding extensions to this new field of study (see also [7–13] and references therein).

On the other hand, in the past years, fixed point theorems in partially ordered metric spaces have also attracted much attention, especially after the works of Ran and Reurings [14], Nieto and Rodríguez-López [15], Gnana Bhaskar and Lakshmikantham [16], Berinde and Borcut [17, 18], Karapınar and Berinde [19, 20], Berzig and Samet [21], and Roldán et al. [22–24], among others. Their results were extended to more general contractivity conditions in which *altering distance functions* play a key role. Very recently, Alghamdi and Karapınar [25] used a similar notion in G-metric spaces, and Berzig and Karapınar [26] also considered a more general kind of contractivity conditions using a pair of generalized altering distance functions.

In order to consider a contractive condition on the whole metric space that can be particularized to partially ordered metric spaces, some advances have been done in recent times (see, for instance, [25–27] and references therein). This subject has been extended by Mongkolkeha et al. [28] to the field of determining best proximity points, describing a wide class of contractive mappings and using very general control functions. The main aim of this paper is to collect, generalize, and improve their results using contractive conditions and control functions that can be particularized in a wide kind of different results applicable to several frameworks.

2. Preliminaries

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ denote the set of all nonnegative integers. Throughout this paper, let (X, d) be a metric space, let A and B two nonempty subsets of X, and let $T : A \to B$, $g : A \to A$, and $\alpha : X \times X \to [0, \infty)$ be three mappings. Define

$$\begin{aligned} \Delta_{AB} &= \operatorname{dist}(A, B) = \inf(\{d(a, b) : a \in A, b \in B\}), \\ A_0 &= \{a \in A : \exists b \in B \text{ such that } d(a, b) = \Delta_{AB}\}, \\ B_0 &= \{b \in B : \exists a \in A \text{ such that } d(a, b) = \Delta_{AB}\}. \end{aligned} \tag{1}$$

Notice that, if $a \in A$ and $b \in B$ verify $d(a, b) = \Delta_{AB}$, then $a \in A_0$ and $b \in B_0$. Therefore, A_0 is nonempty if, and only if, B_0 is nonempty. Thus, if A_0 is nonempty, then A, B, and B_0 are nonempty subsets of X. It is clear that, if $A \cap B \neq \emptyset$, then A_0 is nonempty. In [29], the authors discussed sufficient conditions in order to guarantee the nonemptiness of A_0. In general, if A and B are closed subsets of a normed linear space such that $\Delta_{AB} > 0$, then A_0 is contained in the boundary of A (see [6]).

The main aim of this paper is to study sufficient conditions to ensure the existence and, in some cases, the unicity of the following kind of points.

Definition 1. One will say a point $x \in A$ is a *g-best proximity point of T* if $d(gx, Tx) = \Delta_{AB}$ and x is a best *proximity point of T* if $d(x, Tx) = \Delta_{AB}$.

If $A = B$, a g-best proximity point of T is called a *coincidence point of T and g* (i.e., $Tx = gx$), and if g is the identity mapping on A, then x is a *fixed point of T* (i.e., $Tx = x$).

We describe the families of functions that we will use henceforth.

Definition 2. (i) One will denote by Ψ the family of all functions $\varphi : [0, \infty) \to [0, \infty)$ such that, for all $t > 0$, the series $\sum_{n\geq 1} \varphi^n(t)$ converges (functions in Ψ are called *(c)-comparison functions*).

(ii) One will denote by Φ the family of all functions $\phi : [0, \infty) \to [0, \infty)$ such that $\phi(t) < t$ and $\lim_{r \to t^+} \phi(r) < t$ for all $t > 0$.

(iii) One will denote by Θ the family of all continuous mappings $\theta : [0, \infty)^4 \to [0, \infty)$ such that $\theta(a, b, c, d) = 0$ if one or more arguments take the value zero (i.e., if $abcd = 0$).

(iv) One will denote by Ω the family of all mappings $\theta : [0, \infty)^4 \to [0, \infty)$ such that $\theta(a, b, c, d) = 0$ if one or more arguments take the value zero (i.e., if $abcd = 0$).

(v) One will denote by Ω' the family of all mappings $\theta : [0, \infty)^4 \to [0, \infty)$ such that $\theta(0, b, c, d) = 0$.

(vi) One will denote by Ω'' the family of all mappings $\theta : [0, \infty)^4 \to [0, \infty)$ such that $\lim_{n\to\infty} \theta(t_n^1, t_n^2, t_n^3, t_n^4) = 0$ whatever the sequences $\{t_n^1\}, \{t_n^2\}, \{t_n^3\}, \{t_n^4\} \subset [0, \infty)$ such that, at least one of them, is convergent to zero (i.e., there exists $i \in \{1, 2, 3, 4\}$ verifying $\{t_n^i\} \to 0$).

Remark 3. (1) It is easy to see that, if $\varphi \in \Psi$, then $\varphi(t) < t$ for all $t > 0$.

(2) We point out that we do not impose any monotone condition on the control function we will use.

(3) Clearly $\Theta \subset \Omega \subset \Omega'$ and $\Theta \subset \Omega \subset \Omega''$. Notice that functions in Ω, Ω' and Ω'' have not to be continuous.

Example 4. Examples of functions in Θ are the following ones (where $\lambda > 0$):

$$\begin{aligned} \theta_1(t_1, t_2, t_3, t_4) &= \lambda t_1^{\beta_1} t_2^{\beta_2} t_3^{\beta_3} t_4^{\beta_4}, \quad \text{where } \beta_1, \beta_2, \beta_3, \beta_4 > 0; \\ \theta_2(t_1, t_2, t_3, t_4) &= \lambda \ln(1 + t_1 t_2 t_3 t_4)^{\beta}, \quad \text{where } \beta > 0; \\ \theta_3(t_1, t_2, t_3, t_4) &= \lambda \min(t_1, t_2, t_3, t_4). \end{aligned} \tag{2}$$

The mappings of Φ have been very useful in the framework of fixed point theory (see [30–32]). The following lemma can be found in the literature but we recall it here for the sake of completeness.

Lemma 5. *Let $\phi \in \Phi$ be a mapping and let $\{a_m\} \subset \mathbb{R}_0^+$ be a sequence. If $a_{m+1} \leq \phi(a_m)$ and $a_m \neq 0$ for all m, then $\{a_m\} \to 0$.*

In the following result, $\mathcal{P}_4$ denotes the family of all permutations $\sigma : \{1, 2, 3, 4\} \to \{1, 2, 3, 4\}$.

Lemma 6. *Given $\lambda > 0$ and $\theta \in \Theta$, define $\theta'_\lambda : [0, \infty)^4 \to [0, \infty)$ by*

$$\theta'_\lambda(t_1, t_2, t_3, t_4) = \lambda \max(\theta(t_{\sigma(1)}, t_{\sigma(2)}, t_{\sigma(3)}, t_{\sigma(4)}) : \sigma \in \mathcal{P}_4) \quad \forall t_1, t_2, t_3, t_4 \in [0, \infty). \tag{3}$$

Then $\theta'_\lambda \in \Theta$ and θ'_λ is symmetric. Furthermore, if $\lambda \geq 1$, then $\theta \leq \theta'_\lambda$.

Definition 7. If $\mathcal{R}$ is a binary relation on X, one will consider the mapping $\alpha_{\mathcal{R}} : X \times X \to [0, \infty)$ given, for all $x, y \in X$, by

$$\alpha_{\mathcal{R}}(x, y) = \begin{cases} 1, & \text{if } x \mathcal{R} y, \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

Definition 8. A *preorder* (or a *quasiorder*) $\preccurlyeq$ on X is a binary relation on X that is *reflexive* (i.e., $x \preccurlyeq x$ for all $x \in X$) and *transitive* (if $x, y, z \in X$ verify $x \preccurlyeq y$ and $y \preccurlyeq z$, then $x \preccurlyeq z$). In such a case, we say that $(X, \preccurlyeq)$ is a *preordered space* (or a *preordered set*). If a preorder $\preccurlyeq$ is also *antisymmetric* ($x \preccurlyeq y$ and $y \preccurlyeq x$ implies $x = y$), then $\preccurlyeq$ is called a *partial order*.

Definition 9 (Raj [33]). Let A and B be two subsets of a metric space (X, d) such that A_0 is nonempty. We say that the pair (A, B) has the *P-property* if

$$\left. \begin{array}{c} a_1, a_2 \in A_0, \quad b_2, b_2 \in B_0 \\ d(a_1, b_1) = \Delta_{AB} \\ d(a_2, b_2) = \Delta_{AB} \end{array} \right\} \Longrightarrow d(a_1, a_2) = d(b_1, b_2). \tag{5}$$

In [28], the authors introduced the following find of contractive mappings and succeed in proving the following result.

Definition 10 (Mongkolkeha et al. [28], Definition 3.1). Let A and B be nonempty subsets of a metric space (X, d). A mapping $T : A \to B$ is said to be a *generalized almost* $(\varphi, \theta)_\alpha$ *contraction* if

$$\begin{aligned}\alpha(x,y)\, d(Tx,Ty) \le \varphi(M(x,y)) &+ \theta\big(d(y,Tx) - \Delta_{AB}, d(x,Ty)\\ &- \Delta_{AB}, d(x,Tx) - \Delta_{AB},\\ &d(y,Ty) - AB\big),\end{aligned} \tag{6}$$

for all $x, y \in A$, where $\alpha : A \times A \to [0, \infty)$, $\varphi \in \Psi$, $\theta \in \Theta$, and

$$M(x,y) = \max\left(d(x,y), d(x,Tx) - \Delta_{AB}, d(y,Ty) - \Delta_{AB}, \frac{d(x,Ty) + d(y,Tx)}{2} - \Delta_{AB}\right). \tag{7}$$

Theorem 11 (Mongkolkeha et al. [28], Theorem 3.2). *Let A and B be nonempty closed subsets of a complete metric space X such that A_0 is nonempty and the pair (A, B) has the P-property. Let $T : A \to B$ satisfy the following conditions:*

(a) *T is an α-proximal admissible and generalized almost $(\varphi, \theta)_\alpha$-contraction;*

(b) *T is continuous;*

(c) *there exist elements $x_0, x_1 \in A_0$ such that $d(x_1, Tx_0) = \Delta_{AB}$ and $\alpha(x_0, x_1) \ge 1$;*

(d) *$T(A_0) \subseteq B_0$.*

Then there exists an element $x \in A$ such that

$$d(x,Tx) = \Delta_{AB}. \tag{8}$$

Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$, defined by

$$d(x_{n+1}, Tx_n) = \Delta_{AB}, \tag{9}$$

converges to the element x.

3. Existence of g-Best Proximity Points under Different Conditions

The main aim of this paper is to study the following kind of mappings and to ensure that, under some conditions, they have a g-best proximity point.

Definition 12. Let $T : A \to B$, $g : A \to A$, $\varphi : [0, \infty) \to [0, \infty)$, $\theta : [0, \infty)^4 \to [0, \infty)$, and $\alpha : X \times X \to [0, \infty)$ be five mappings. One will say that T is a $(\varphi, \theta, \alpha, g)$-*contraction* if, for all $x, y \in A_0$ such that $d(gy, Tx) = \Delta_{AB}$ and $\alpha(gx, gy) \ge 1$, we have that

$$\begin{aligned}\alpha(gx,gy)\, d(Tx,Ty) \le \varphi(M^g(x,y)) &+ \theta\big(d(gy,Tx) - \Delta_{AB},\\ &d(gx,Ty) - \Delta_{AB},\\ &d(gx,Tx) - \Delta_{AB},\\ &d(gy,Ty) - \Delta_{AB}\big),\end{aligned} \tag{10}$$

where

$$M^g(x,y) = \max\left(d(gx,gy), d(gx,Tx) - \Delta_{AB}, d(gy,Ty) - \Delta_{AB}, \frac{d(gx,Ty) + d(gy,Tx)}{2} - \Delta_{AB}\right). \tag{11}$$

In the previous definition, we have not supposed that $\varphi \in \Psi$ or $\theta \in \Theta$ because the main aim of the present paper is to introduce sufficient conditions on the involved mappings (φ, θ, α, and g) and on the ambient space to ensure the existence and, in some cases, the unicity of g-best proximity points of T.

Remark 13. (1) Some other authors used to impose that their contractive condition must be verified for all $x, y \in A$. However, our condition (10) must only be satisfied for all $x, y \in A_0$. Later, we will discuss when it is necessary to assume that this property holds for all $x, y \in A$.

(2) The mapping θ need not be symmetric. However, if $\theta \in \Theta$ and T is a $(\varphi, \theta, \alpha, g)$-contraction, then T is also a $(\varphi, \theta_1', \alpha, g)$-contraction, where θ_1' is defined as in Lemma 6. In such a case, when $\theta \in \Theta$, without loss of generality, we can consider that θ is symmetric; that is, in this case, the order of the arguments of θ in (10) is not important.

The following definitions are very useful in order to establish weaker conditions than the P-property (see also [34]) or the notion of α-proximal admissible mapping.

Definition 14. Let A and B be two subsets of a metric space (X, d) such that A_0 is nonempty, and let $T : A \to B$ and $g : A \to A$ be two mappings. One will say that the quadruple (A, B, T, g) has the following:

(i) the *weak P-property of the first kind* if

$$\left.\begin{array}{l} a_1, a_2, a_3, a_4 \in A_0 \\ d(ga_1, Ta_3) = \Delta_{AB} \\ d(ga_2, Ta_4) = \Delta_{AB} \end{array}\right\} \Longrightarrow d(ga_1, ga_2) \le d(Ta_3, Ta_4); \tag{12}$$

(ii) the *weak P-property of the second kind* if

$$\left.\begin{array}{l} a_1, a_2, a_3, a_4 \in A_0 \\ d\left(ga_1, Ta_3\right) = \Delta_{AB} \\ d\left(ga_2, Ta_4\right) = \Delta_{AB} \end{array}\right\} \Longrightarrow d\left(ga_1, ga_2\right) = d\left(Ta_3, Ta_4\right); \tag{13}$$

(iii) the *weak P-property of the third kind* if

$$\left.\begin{array}{l} a_1, a_2 \in A, \quad b_1, b_2 \in B \\ d\left(ga_1, b_1\right) = \Delta_{AB} \\ d\left(ga_2, b_2\right) = \Delta_{AB} \end{array}\right\} \Longrightarrow d\left(ga_1, ga_2\right) \leq d\left(b_1, b_2\right). \tag{14}$$

Lemma 15. *If the pair (A, B) has the P-property, then the quadruple (A, B, T, g) has the weak P-property of the first, the second, and the third kind, whatever the mappings $T : A \to B$ and $g : A \to A$.*

Remark 16. Obviously, if (X, d) is a metric space, then the pair (X, X) has the P-property. Therefore, the quadruple (X, X, T, g) has the weak P-property of the first, the second, and the third kinds whatever the mappings $T : A \to B$ and $g : A \to A$.

Definition 17. Let A and B be two subsets of a metric space (X, d) such that A_0 is nonempty, and let $T : A \to B$, $g : A \to A$, and $\alpha : X \times X \to [0, \infty)$ be three mappings. One will say that T is *(α, g)-proximal admissible* if

$$\left.\begin{array}{l} a_1, a_2, b_1, b_2 \in A_0 \\ \alpha\left(gb_1, gb_2\right) \geq 1 \\ d\left(ga_1, Tb_1\right) = \Delta_{AB} \\ d\left(ga_2, Tb_2\right) = \Delta_{AB} \end{array}\right\} \Longrightarrow \alpha\left(ga_1, ga_2\right) \geq 1. \tag{15}$$

Lemma 18. *If T is α-proximal admissible, then T is (α, g)-proximal admissible, whatever $g : A \to A$.*

Definition 19. Let $g : A \to A$ and $\alpha : X \times X \to [0, \infty)$ be two mappings and let $N \in \mathbb{N}$ and $N \geq 2$. We will say that α is *(N, g)-transitive on A_0* if

$$\begin{gathered} x_1, x_2, \ldots, x_{N+1} \in A_0 \\ \alpha\left(gx_i, gx_{i+1}\right) \geq 1, \quad \forall i \in \{1, 2, \ldots, N\} \\ \Downarrow \\ \alpha\left(gx_1, gx_{N+1}\right) \geq 1. \end{gathered} \tag{16}$$

Indeed, one will only use the notion of *$(2, g)$-transitive mapping on A_0*; that is,

$$\left.\begin{array}{l} x_1, x_2, x_3 \in A_0 \\ \alpha\left(gx_1, gx_2\right) \geq 1 \\ \alpha\left(gx_2, gx_3\right) \geq 1 \end{array}\right\} \Longrightarrow \alpha\left(gx_1, gx_3\right) \geq 1. \tag{17}$$

Next we prove our first main result.

Theorem 20. *Let A and B be two closed subsets of a complete metric space (X, d) and let $T : A \to B$, $g : A \to A$, $\varphi : [0, \infty) \to [0, \infty)$, $\theta : [0, \infty)^4 \to [0, \infty)$, and $\alpha : X \times X \to [0, \infty)$ be five mappings. Assume that the following conditions hold:*

(a) *$\emptyset \neq A_0 \subseteq gA_0$ and $T(A_0) \subseteq B_0$;*

(b) *the quadruple (A, B, T, g) has the weak P-property of the first kind;*

(c) *T is a (α, g)-proximal admissible $(\varphi, \theta, \alpha, g)$-contraction;*

(d) *if $\{z_n\} \subseteq A_0$ is a sequence such that $\{gz_n\} \subseteq A_0$ is Cauchy, then $\{z_n\}$ also is Cauchy;*

(e) *there exists $(x_0, x_1) \in A_0 \times A_0$ such that $d(gx_1, Tx_0) = \Delta_{AB}$ and $\alpha(gx_0, gx_1) \geq 1$;*

(f) *g is a continuous mapping;*

(g) *T is a continuous mapping;*

(h) *$\varphi \in \Psi$ and $\theta \in \Omega'$.*

Then there exists a convergent sequence $\{x_n\}_{n\geq 0} \subseteq A_0$ verifying

$$d\left(gx_{n+1}, Tx_n\right) = \Delta_{AB} \quad \forall n \geq 0, \tag{18}$$

whose limit is a g-best proximity point of T.

Actually, every sequence $\{x_n\}_{n\geq 0} \subseteq A_0$ verifying (18) *and $\alpha(gx_0, gx_1) \geq 1$ converges to a g-best proximity point of T.*

Remark 21. (1) Although the previous result seems to have too many hypotheses, actually, this is its best advantage. As we will see in Section 5, there are a lot of different ways to particularize this theorem which generate many independent results. For instance, our control functions do not need any kind of monotone property.

(2) This result improves the main theorem in [28] in several aspects: firstly, we introduce a mapping $g : A \to A$ which is not necessarily the identity mapping on A; secondly, (A, B) need not have the P-property; thirdly, the contractive condition on T is weaker; finally, we only suppose $\theta \in \Omega'$; that is, θ is not necessarily continuous.

(3) Taking into account the completeness of the ambient space X, the condition (d) can be interpreted as the continuity of the inverse mapping of g, if g is invertible. A simple way to guarantee this condition is to suppose that there are $\lambda, n > 0$ such that $d(x, y) \leq \lambda d(gx, gy)^n$ for all $x, y \in A$. For instance, the condition $d(x, y) \leq d(gx, gy)$ for all $x, y \in A$ can be found in [35].

(4) Notice that the second part of the thesis does not clarify whether the g-best proximity point of T is unique or not.

Proof. Given $x_1 \in A_0$, we know that $Tx_1 \in T(A_0) \subseteq B_0$. Then, there is $z_2 \in A$ such that $d(z_2, Tx_1) = \Delta_{AB}$. Therefore, $z_2 \in A_0$. Since $A_0 \subseteq gA_0$, there is $x_2 \in A_0$ such that $gx_2 = z_1$, so $d(gx_2, Tx_1) = d(z_2, Tx_1) = \Delta_{AB}$. Repeating the same argument starting from $x_2 \in A_0$, there is $x_3 \in A_0$ such that $d(gx_3, Tx_2) = \Delta_{AB}$. By induction, we can consider a sequence $\{x_n\} \subseteq A_0$ such that

$$d\left(gx_{n+1}, Tx_n\right) = \Delta_{AB} \quad \forall n \geq 0. \tag{19}$$

If there exists some $n_0 \in \mathbb{N}$ such that $gx_{n_0} = gx_{n_0+1}$, then $d(gx_{n_0}, Tx_{n_0}) = d(gx_{n_0+1}, Tx_{n_0}) = \Delta_{AB}$, so x_{n_0} is a g-best proximity point of T. In such a case, if we define $x_m = x_{n_0}$ for all $m \geq n_0$, we have that $\{x_n\}_{n\geq n_0}$ is constant, so $\{x_n\}$ converges to a g-best proximity point of T. In this case, the proof is finished.

On the contrary, suppose that

$$d(gx_n, gx_{n+1}) > 0 \quad \forall n \geq 0. \tag{20}$$

Notice that, in particular, $x_n, gx_{n+1} \in A_0$ and $Tx_n \in B_0$ for all $n \geq 0$. We claim that

$$\alpha(gx_n, gx_{n+1}) \geq 1 \quad \forall n \geq 0. \tag{21}$$

If $n = 0$, then $\alpha(gx_0, gx_1) \geq 1$ by hypothesis. Suppose that $\alpha(gx_n, gx_{n+1}) \geq 1$ for some $n \geq 0$. Hence, taking into account that T is (α, g)-proximal admissible, we have that

$$\left.\begin{array}{r} x_n, x_{n+1}, x_{n+2} \in A_0 \\ \alpha(gx_n, gx_{n+1}) \geq 1 \\ d(gx_{n+1}, Tx_n) = \Delta_{AB} \\ d(gx_{n+2}, Tx_{n+1}) = \Delta_{AB} \end{array}\right\} \Longrightarrow \alpha(gx_{n+1}, gx_{n+2}) \geq 1. \tag{22}$$

This proves that (21) holds. Moreover, using the weak P-property of the first kind, for all $n \geq 0$,

$$\begin{array}{c} x_n, x_{n+1}, x_{n+2} \in A_0 \\ d(gx_{n+1}, Tx_n) = \Delta_{AB} \\ d(gx_{n+2}, Tx_{n+1}) = \Delta_{AB} \\ \Downarrow \\ d(gx_{n+1}, gx_{n+2}) \leq d(Tx_n, Tx_{n+1}). \end{array} \tag{23}$$

Next we use (21), (23), and the $(\varphi, \theta, \alpha, g)$-contractive property of T to see that, for all $n \geq 0$,

$$\begin{aligned} d(gx_{n+1}, gx_{n+2}) &\leq d(Tx_n, Tx_{n+1}) \\ &\leq \alpha(gx_n, gx_{n+1}) d(Tx_n, Tx_{n+1}) \\ &\leq \varphi(M^g(x_n, x_{n+1})) \\ &\quad + \theta\big(d(gx_{n+1}, Tx_n) - \Delta_{AB}, \\ &\qquad d(gx_n, Tx_{n+1}) - \Delta_{AB}, \\ &\qquad d(gx_n, Tx_n) \\ &\qquad -\Delta_{AB}, d(gx_{n+1}, Tx_{n+1}) - \Delta_{AB}\big) \\ &= \varphi(M^g(x_n, x_{n+1})) \end{aligned} \tag{24}$$

(the last equality holds since the first argument of θ is zero), where

$$\begin{aligned} &M^g(x_n, x_{n+1}) \\ &= \max\Bigg(d(gx_n, gx_{n+1}), d(gx_n, Tx_n) - \Delta_{AB}, \\ &\qquad d(gx_{n+1}, Tx_{n+1}) - \Delta_{AB}, \\ &\qquad \frac{d(gx_n, Tx_{n+1}) + d(gx_{n+1}, Tx_n)}{2} - \Delta_{AB}\Bigg) \\ &\leq \max\big(d(gx_n, gx_{n+1}), d(gx_n, gx_{n+1}) \\ &\qquad + d(gx_{n+1}, Tx_n) - \Delta_{AB}, \\ &\qquad d(gx_{n+1}, gx_{n+2}) + d(gx_{n+2}, Tx_{n+1}) - \Delta_{AB}, \\ &\qquad \big(\big(d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_{n+2}) \\ &\qquad + d(gx_{n+2}, Tx_{n+1}) \\ &\qquad + d(gx_{n+1}, Tx_n)\big) \times 2^{-1}\big) - \Delta_{AB}\big) \\ &= \max\big(d(gx_n, gx_{n+1}), d(gx_n, gx_{n+1}) + \Delta_{AB} - \Delta_{AB}, \\ &\qquad d(gx_{n+1}, gx_{n+2}) + \Delta_{AB} - \Delta_{AB}, \\ &\qquad \big(\big(d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_{n+2}) \\ &\qquad + \Delta_{AB} + \Delta_{AB}\big) \times 2^{-1}\big) - \Delta_{AB}\big) \\ &= \max\Bigg(d(gx_n, gx_{n+1}), d(gx_{n+1}, gx_{n+2}), \\ &\qquad \frac{d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_{n+2})}{2}\Bigg) \\ &= \max\big(d(gx_n, gx_{n+1}), d(gx_{n+1}, gx_{n+2})\big). \end{aligned} \tag{25}$$

Joining (24) and (25), we have that

$$\begin{aligned} &d(gx_{n+1}, gx_{n+2}) \\ &\quad \leq \varphi\big(\max\big(d(gx_n, gx_{n+1}), d(gx_{n+1}, gx_{n+2})\big)\big) \quad \forall n \geq 0. \end{aligned} \tag{26}$$

Using (20) and the fact that $\varphi(t) < t$ for all $t > 0$, if there exists some $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} &\max\left(d\left(gx_{n_0}, gx_{n_0+1}\right), d\left(gx_{n_0+1}, gx_{n_0+2}\right)\right) \\ &\quad = d\left(gx_{n_0+1}, gx_{n_0+2}\right), \end{aligned} \tag{27}$$

then we have that $d(gx_{n_0+1}, gx_{n_0+2}) \leq \varphi(d(gx_{n_0+1}, gx_{n_0+2})) < d(gx_{n_0+1}, gx_{n_0+2})$, which is impossible. Then $\max(d(gx_n, gx_{n+1}), d(gx_{n+1}, gx_{n+2})) = d(gx_n, gx_{n+1})$ for all $n \geq 0$ and (26) yields to

$$d(gx_{n+1}, gx_{n+2}) \leq \varphi(d(gx_n, gx_{n+1})) \quad \forall n \geq 0. \tag{28}$$

In particular, for all $n \geq 1$,

$$\begin{aligned} d(gx_n, gx_{n+1}) &\leq \varphi(d(gx_{n-1}, gx_n)) \\ &\leq \varphi^2(d(gx_{n-2}, gx_{n-1})) \\ &\leq \cdots \leq \varphi^n(d(gx_0, gx_1)). \end{aligned} \tag{29}$$

Next we prove that $\{gx_n\}$ is a Cauchy sequence. Fix $\varepsilon > 0$ arbitrary and consider $t_0 = d(gx_0, gx_1) > 0$. Since $\varphi \in \Psi$, the series $\sum_{n\geq 1} \varphi^n(t_0)$ converges. In particular, there exists $m_0 \in \mathbb{N}$ such that

$$\sum_{k=m_0}^{\infty} \varphi^n(t_0) < \varepsilon. \tag{30}$$

Therefore, if $m > n \geq m_0$, we have that

$$\begin{aligned} d(gx_n, gx_m) &\leq \sum_{k=n}^{m-1} d(gx_k, gx_{k+1}) \\ &\leq \sum_{k=n}^{m-1} \varphi^k(d(gx_0, gx_1)) \\ &\leq \sum_{k=m_0}^{\infty} \varphi^n(t_0) < \varepsilon. \end{aligned} \tag{31}$$

This means that $\{gx_n\}$ is a Cauchy sequence. Using the hypothesis (d), $\{x_n\}$ also is a Cauchy sequence. By the completeness of (X, d), there exists $z \in X$ such that $\{x_n\} \to z$. From $x_n \in A_0 \subseteq A$ for all n, we deduce that $z \in A$ (because A is closed). Since T and g are continuous mappings, $\{Tx_n\} \to Tz$ and $\{gx_n\} \to gz$. Taking limit in (19) as $n \to \infty$, we conclude that z is a g-best proximity point of T. □

Next we change the conditions on the control functions.

Theorem 22. *Theorem 20 also holds if one replaces condition (h) by the following one:*

(h') *$\varphi \in \Phi$, $\theta \in \Omega''$, and α is $(2, g)$-transitive.*

Proof. Taking into account that $\varphi(t) < t$ for all $t > 0$ and following the lines of the proof of Theorem 20, we deduce that

$$\begin{gathered} x_n \in A_0, \quad d(gx_{n+1}, Tx_n) = \Delta_{AB}, \\ d(gx_n, gx_{n+1}) > 0, \qquad \alpha(gx_n, gx_{n+1}) \geq 1, \\ d(gx_{n+1}, gx_{n+2}) \leq \varphi(d(gx_n, gx_{n+1})) \\ \forall n \geq 0. \end{gathered} \tag{32}$$

By Lemma 5, we have that

$$\{d(gx_n, gx_{n+1})\} \longrightarrow 0. \tag{33}$$

Next, we are going to prove that $\{gx_n\}$ is a Cauchy sequence reasoning by contradiction. Assume that $\{gx_n\}$ is not Cauchy. In this case (following, for instance, [27]), there exist $\varepsilon_0 > 0$ and two subsequences $\{x_{m(k)}\}_{k\in\mathbb{N}}$ and $\{x_{n(k)}\}_{k\in\mathbb{N}}$ verifying that, for all $k \in \mathbb{N}$,

$$\begin{gathered} k \leq m(k) < n(k), \\ d(gx_{m(k)}, gx_{n(k)}) > \varepsilon_0, \\ d(gx_{m(k)}, gx_p) \leq \varepsilon_0 \\ \forall p \in \{m(k)+1, m(k)+2, \ldots, n(k)-2, n(k)-1\}, \\ \lim_{k\to\infty} d(gx_{m(k)-1}, gx_{n(k)-1}) = \varepsilon_0, \\ \lim_{k\to\infty} d(gx_{m(k)}, gx_{n(k)+p}) = \varepsilon_0 \\ \forall p \geq 0. \end{gathered} \tag{34}$$

Notice that

$$\begin{aligned} 0 &\leq d(gx_{n(k)}, Tx_{n(k)}) - \Delta_{AB} \\ &\leq d(gx_{n(k)}, gx_{n(k)+1}) + d(gx_{n(k)+1}, Tx_{n(k)}) - \Delta_{AB} \\ &= d(gx_{n(k)}, gx_{n(k)+1}). \end{aligned} \tag{35}$$

Therefore

$$\lim_{k\to\infty} [d(gx_{n(k)}, Tx_{n(k)}) - \Delta_{AB}] = 0. \tag{36}$$

Similarly,

$$\lim_{k\to\infty} [d(gx_{m(k)}, Tx_{m(k)}) - \Delta_{AB}] = 0. \tag{37}$$

Furthermore,

$$\varepsilon_0 < d(x_{m(k)}, x_{n(k)}) \leq M^g(x_{m(k)}, x_{n(k)}) \quad \forall k \geq 0, \tag{38}$$

where, for all $k \geq 0$,

$$\begin{aligned} &M^g(x_{m(k)}, x_{n(k)}) \\ &\quad = \max\Bigg(d(gx_{m(k)}, gx_{n(k)}), d(gx_{m(k)}, Tx_{m(k)}) - \Delta_{AB}, \\ &\qquad d(gx_{n(k)}, Tx_{n(k)}) - \Delta_{AB}, \\ &\qquad \frac{d(gx_{m(k)}, Tx_{n(k)}) + d(gx_{n(k)}, Tx_{m(k)})}{2} - \Delta_{AB} \Bigg). \end{aligned} \tag{39}$$

Notice that

$$\begin{aligned}
&\frac{d\left(gx_{m(k)}, Tx_{n(k)}\right)+d\left(gx_{n(k)}, Tx_{m(k)}\right)}{2}-\Delta_{AB} \\
&\leq\Big(\big(d\left(gx_{m(k)}, gx_{n(k)+1}\right)+d\left(gx_{n(k)+1}, Tx_{n(k)}\right) \\
&\qquad+d\left(gx_{n(k)}, gx_{m(k)+1}\right)+d\left(gx_{m(k)+1}, Tx_{m(k)}\right)\big) \\
&\qquad\times 2^{-1}\Big)-\Delta_{AB} \\
&=\Big(\big(d\left(gx_{m(k)}, gx_{n(k)+1}\right)+\Delta_{AB} \\
&\qquad+d\left(gx_{n(k)}, gx_{m(k)+1}\right)+\Delta_{AB}\big)\times 2^{-1}\Big)-\Delta_{AB} \\
&=\frac{d\left(gx_{m(k)}, gx_{n(k)+1}\right)+d\left(gx_{n(k)}, gx_{m(k)+1}\right)}{2}.
\end{aligned} \tag{40}$$

Taking limit as $k \rightarrow \infty$ and using (34),

$$\begin{aligned}
&\lim_{k\rightarrow\infty}\left(\frac{d\left(gx_{m(k)}, Tx_{n(k)}\right)+d\left(gx_{n(k)}, Tx_{m(k)}\right)}{2}-\Delta_{AB}\right) \\
&\leq\frac{\varepsilon_0+\varepsilon_0}{2}=\varepsilon_0.
\end{aligned} \tag{41}$$

Taking limit as $k \rightarrow \infty$ in (39) and using (34), (36), (37), and (41), we deduce that

$$\lim_{k\rightarrow\infty} M^g\left(x_{m(k)}, x_{n(k)}\right)=\max\left(\varepsilon_0, 0, 0, \varepsilon_0\right)=\varepsilon_0. \tag{42}$$

This means that $\{M^g(x_{m(k)}, x_{n(k)})\}_{k\in\mathbb{N}}$ is a sequence of real numbers converging to ε_0 and whose terms are strictly greater than ε_0. In particular, since $\varphi \in \Phi$,

$$\lim_{k\rightarrow\infty}\varphi\left(M^g\left(x_{m(k)}, x_{n(k)}\right)\right)=\lim_{t\rightarrow\varepsilon_0^+}\varphi(t)<\varepsilon_0. \tag{43}$$

From the fact that $\alpha(gx_n, gx_{n+1}) \geq 1$ for all $n \geq 0$ and using that α is $(2, g)$-transitive, we deduce that

$$\alpha\left(gx_{m(k)}, gx_{n(k)}\right)\geq 1 \qquad \forall k\geq 0. \tag{44}$$

Since (A, B, T, g) has the weak P-property of the first kind, for all $k \geq 0$,

$$\begin{gathered}
x_{m(k)}, x_{m(k)+1}, x_{n(k)}, x_{n(k)+1}\in A_0 \\
d\left(gx_{m(k)+1}, Tx_{m(k)}\right)=\Delta_{AB} \\
d\left(gx_{n(k)+1}, Tx_{n(k)}\right)=\Delta_{AB} \\
\Downarrow \\
d\left(gx_{m(k)+1}, gx_{n(k)+1}\right)\leq d\left(Tx_{m(k)}, Tx_{n(k)}\right).
\end{gathered} \tag{45}$$

Therefore, from the $(\varphi, \theta, \alpha, g)$-contractivity condition on T, it follows that, for all $k \geq 0$,

$$\begin{aligned}
d\left(gx_{m(k)+1}, gx_{n(k)+1}\right)&\leq d\left(Tx_{m(k)}, Tx_{n(k)}\right) \\
&\leq\alpha\left(Tx_{m(k)}, Tx_{n(k)}\right)d\left(Tx_{m(k)}, Tx_{n(k)}\right) \\
&\leq\varphi\left(M^g\left(x_{m(k)}, x_{n(k)}\right)\right) \\
&\quad+\theta\big(d\left(gx_{n(k)}, Tx_{m(k)}\right)-\Delta_{AB}, \\
&\qquad d\left(gx_{m(k)}, Tx_{n(k)}\right)-\Delta_{AB}, \\
&\qquad d\left(gx_{m(k)}, Tx_{m(k)}\right)-\Delta_{AB}, \\
&\qquad d\left(gx_{n(k)}, Tx_{n(k)}\right)-\Delta_{AB}\big).
\end{aligned} \tag{46}$$

Using (36), the third and the fourth arguments of θ converge to zero as $k \rightarrow \infty$. Since $\theta \in \Omega''$, all the terms tend to zero as $k \rightarrow \infty$. Hence, letting $k \rightarrow \infty$ in (46) and using (34) and (43), we conclude that

$$\begin{aligned}
\varepsilon_0&=\lim_{k\rightarrow\infty}d\left(gx_{m(k)+1}, gx_{n(k)+1}\right) \\
&\leq\lim_{k\rightarrow\infty}\varphi\left(M^g\left(x_{m(k)}, x_{n(k)}\right)\right)<\varepsilon_0,
\end{aligned} \tag{47}$$

which is impossible. This contradiction proves that $\{gx_n\}$ is a Cauchy sequence. Then, the rest of the proof is similar to the proof of Theorem 20. □

In the following theorem, we replace the continuity of T by another condition.

Theorem 23. *Theorem 20 also holds if one supposes that the contractive condition* (10) *is valid for all* $x \in A_0$ *and all* $y \in A$, *and one replaces conditions* (*b*) *and* (*g*) *by the following ones:*

(b') *the quadruple* (A, B, T, g) *has the weak P-property of the second kind;*

(g') *If* $\{x_n\} \subseteq A_0$ *is a sequence verifying* $\{x_n\} \rightarrow x \in A$ *and* $\alpha(gx_n, gx_{n+1}) \geq 1$ *for all* $n \geq 0$, *then there exists a partial subsequence* $\{x_{n(k)}\}$ *of* $\{x_n\}$ *such that* $\alpha(gx_{n(k)}, gx) \geq 1$ *for all* $k \geq 0$.

Proof. Following the lines of the proof of Theorem 20, we deduce that $\{gx_n\}$ and $\{x_n\}$ are Cauchy sequences, contained in the closed subset A, of the complete metric space (X, d). Then, there is $x \in A$ such that $\{x_n\} \rightarrow x$ and, using that g is continuous, $\{gx_n\} \rightarrow gx$. We are going to prove that x is a g-best proximity point of T.

Since (A, B, T, g) has the weak P-property of the second kind, for all $n, m \in \mathbb{N}$,

$$\begin{gathered}
x_m, x_{m+1}, x_n, x_{n+1}\in A_0 \\
d\left(gx_{m+1}, Tx_m\right)=\Delta_{AB} \\
d\left(gx_{n+1}, Tx_n\right)=\Delta_{AB} \\
\Downarrow \\
d\left(gx_{m+1}, gx_{n+1}\right)=d\left(Tx_m, Tx_n\right).
\end{gathered} \tag{48}$$

It follows that $\{Tx_n\}$ is also a Cauchy sequence in the closed subset B. Hence, there is $z \in B$ such that $\{Tx_n\} \to z$. This means that

$$\{d(gx_n, gx)\} \longrightarrow 0, \qquad \{d(Tx_n, z)\} \longrightarrow 0. \tag{49}$$

Since $d(gx_{n+1}, Tx_n) = \Delta_{AB}$ for all $n \geq 0$, we deduce that

$$d(gx, z) = \Delta_{AB}; \tag{50}$$

that is, $gx \in A_0$ and $z \in B_0$. Using condition (g′), we deduce that there exists a partial subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that

$$\alpha(gx_{n(k)}, gx) \geq 1 \quad \forall k \geq 0. \tag{51}$$

Notice that

$$\begin{aligned} 0 &\leq d(gx_{n(k)}, Tx_{n(k)}) - \Delta_{AB} \\ &\leq d(gx_{n(k)}, gx_{n(k)+1}) \\ &\quad + d(gx_{n(k)+1}, Tx_{n(k)}) - \Delta_{AB} \\ &= d(gx_{n(k)}, gx_{n(k)+1}). \end{aligned} \tag{52}$$

Therefore

$$\lim_{k\to\infty} [d(gx_{n(k)}, Tx_{n(k)}) - \Delta_{AB}] = 0. \tag{53}$$

The first and the second arguments of

$$\begin{aligned} &M^g(x_{n(k)}, x) \\ &\quad = \max\Bigg(d(gx, gx_{n(k)}), d(gx_{n(k)}, Tx_{n(k)}) - \Delta_{AB}, \\ &\qquad\qquad d(gx, Tx) - \Delta_{AB}, \\ &\qquad\qquad \frac{d(gx_{n(k)}, Tx) + d(gx, Tx_{n(k)})}{2} - \Delta_{AB} \Bigg) \end{aligned} \tag{54}$$

tend to zero, and the last argument tends to

$$\begin{aligned} &\lim_{k\to\infty} \left(\frac{d(gx_{n(k)}, Tx) + d(gx, Tx_{n(k)})}{2} - \Delta_{AB} \right) \\ &\quad \leq \lim_{k\to\infty} \Big(\big((d(gx_{n(k)}, Tx) + d(gx, gx_{n(k)+1}) \\ &\qquad\qquad + d(gx_{n(k)+1}, Tx_{n(k)})) \times 2^{-1} \big) - \Delta_{AB} \Big) \\ &\quad = \lim_{k\to\infty} \left(\frac{d(gx_{n(k)}, Tx) + d(gx, gx_{n(k)+1}) + \Delta_{AB}}{2} - \Delta_{AB} \right) \\ &\quad = \frac{d(gx, Tx) + 0 + \Delta_{AB}}{2} - \Delta_{AB} \\ &\quad = \frac{d(gx, Tx) - \Delta_{AB}}{2}. \end{aligned} \tag{55}$$

Therefore,

$$\lim_{k\to\infty} M^g(x_{n(k)}, x) = d(gx, Tx) - \Delta_{AB}. \tag{56}$$

Next we are going to show that x is a g-best proximity point of T reasoning by contradiction. Suppose that $d(gx, Tx) \neq \Delta_{AB}$; that is,

$$t_0 = d(gx, Tx) - \Delta_{AB} > 0. \tag{57}$$

Since the first and the second terms in the maximum in (54) tend to zero, and the fourth term tends to $t_0/2$, then there exists $k_0 \in \mathbb{N}$ such that

$$M^g(x_{n(k)}, x) = d(gx, Tx) - \Delta_{AB} = t_0 > 0 \quad \forall k \geq k_0. \tag{58}$$

Using the contractivity condition (notice that $x_{n(k)} \in A_0$ but $x \in A$), for all $k \geq k_0$,

$$\begin{aligned} d(Tx_{n(k)}, Tx) &\leq \alpha(gx_{n(k)}, gx)\, d(Tx_{n(k)}, Tx) \\ &\leq \varphi(M^g(x_{n(k)}, x)) \\ &\quad + \theta\big(d(gx, Tx_{n(k)}) - \Delta_{AB}, \\ &\qquad d(gx_{n(k)}, Tx) - \Delta_{AB}, \\ &\qquad d(gx_{n(k)}, Tx_{n(k)}) - \Delta_{AB}, \\ &\qquad d(gx, Tx) - \Delta_{AB}\big) \\ &= \varphi(d(gx, Tx) - \Delta_{AB}) \\ &\quad + \theta\big(d(gx, Tx_{n(k)}) - \Delta_{AB}, \\ &\qquad d(gx_{n(k)}, Tx) - \Delta_{AB}, \\ &\qquad d(gx_{n(k)}, Tx_{n(k)}) - \Delta_{AB}, \\ &\qquad d(gx, Tx) - \Delta_{AB}\big). \end{aligned} \tag{59}$$

Since the third argument of θ in (59) tends to zero and $\theta \in \Omega''$, its limit as $k \to \infty$ is zero. Therefore, letting $k \to \infty$ in (59), we have that

$$d(z, Tx) = \lim_{k\to\infty} d(Tx_{n(k)}, Tx) \leq \varphi(d(gx, Tx) - \Delta_{AB}). \tag{60}$$

As $d(gx, Tx) - \Delta_{AB} > 0$, item (1) of Remark 13 guarantees that $\varphi(d(gx, Tx) - \Delta_{AB}) < d(gx, Tx) - \Delta_{AB}$. Thus,

$$\begin{aligned} d(z, Tx) &\leq \varphi(d(gx, Tx) - \Delta_{AB}) \\ &< d(gx, Tx) - \Delta_{AB} \\ &\leq d(gx, z) + d(z, Tx) - \Delta_{AB} \\ &\leq \Delta_{AB} + d(z, Tx) - \Delta_{AB} \\ &= d(z, Tx), \end{aligned} \tag{61}$$

which is impossible. This contradiction shows that x must verify $d(gx, Tx) = \Delta_{AB}$; that is, x is a g-best proximity point of T. □

Remark 24. When α is $(2, g)$-transitive, condition (g') is equivalent to the following one.

(g'') If $\{x_n\} \subseteq A_0$ is a sequence verifying $\{x_n\} \to x \in A_0$ and $\alpha(gx_n, gx_{n+1}) \geq 1$ for all $n \geq 0$, then $\alpha(gx_n, gx) \geq 1$ for all $n \geq 0$.

Remark 25. Notice that, following the same sketch of proof with appropriate changes, Theorem 23 remains true under the hypothesis of Theorem 22.

4. Uniqueness of g-Best Proximity Points

In this section, we introduce a sufficient condition in order to demonstrate that the g-best proximity point, whose existence is guaranteed by the previous results, is unique.

Definition 26. Let $T : A \to B$, $g : A \to A$, and $\alpha : X \times X \to [0, \infty)$ be three mappings. One will say that T is (α, g)-*regular* if, for all $x, y \in A_0$ such that $\alpha(gx, gy) < 1$, there exists $z \in A_0$ such that $\alpha(gx, gz) \geq 1$ and $\alpha(gy, gz) \geq 1$.

Theorem 27. *Under the hypothesis of Theorem 20, assume that $\theta \in \Theta$ and T is (α, g)-regular. Then for all g-best proximity points x and y of T in A_0 One has that $gx = gy$.*

In particular, if g is injective on the set of all g-best proximity points of T in A_0, then T has a unique g-best proximity point.

Proof. Let $x, y \in A_0$ be two g-best proximity points of T in A_0. Since $d(gx, Tx) = d(gy, Ty) = \Delta_{AB}$ and T is a (α, g)-proximal admissible, we deduce that

$$d(gx, gy) \leq d(Tx, Ty). \tag{62}$$

We distinguish whether $\alpha(gx, gy) \geq 1$ or $\alpha(gx, gy) < 1$. Firstly, assume that $\alpha(gx, gy) \geq 1$. In such a case, the contractivity condition yields to

$$\begin{aligned} d(gx, gy) &\leq d(Tx, Ty) \\ &\leq \alpha(gx, gy)\, d(Tx, Ty) \\ &\leq \varphi(M^g(x, y)) \\ &\quad + \theta\big(d(gy, Tx) - \Delta_{AB}, d(gx, Ty) - \Delta_{AB}, \\ &\qquad d(gx, Tx) - \Delta_{AB}, d(gy, Ty) - \Delta_{AB}\big) \\ &= \varphi(M^g(x, y)), \end{aligned} \tag{63}$$

where the last equality holds since $\theta \in \Theta$ and the last two arguments of θ are zero. Since

$$\begin{aligned} &\frac{d(gx, Ty) + d(gy, Tx)}{2} - \Delta_{AB} \\ &\leq \frac{d(gx, gy) + d(gy, Ty) + d(gy, gx) + d(gx, Tx)}{2} - \Delta_{AB} \\ &= \frac{d(gx, gy) + \Delta_{AB} + d(gy, gx) + \Delta_{AB}}{2} - \Delta_{AB} \\ &= \frac{d(gx, gy) + d(gy, gx)}{2} \\ &= d(gx, gy), \end{aligned} \tag{64}$$

it follows that

$$\begin{aligned} M^g(x, y) = \max\Big(& d(gx, gy), d(gx, Tx) - \Delta_{AB}, \\ & d(gy, Ty) - \Delta_{AB}, \\ & \frac{d(gx, Ty) + d(gy, Tx)}{2} - \Delta_{AB}\Big) \\ = {} & d(gx, gy). \end{aligned} \tag{65}$$

Therefore

$$d(gx, gy) \leq \varphi(M^g(x, y)) = \varphi(d(gx, gy)), \tag{66}$$

which is only possible when $d(gx, gy) = 0$; that is, $gx = gy$.

Next, suppose that $\alpha(gx, gy) < 1$. In this case, by the (α, g)-regularity of T, there exists $z_0 \in A_0$ such that $\alpha(gx, gz_0) \geq 1$ and $\alpha(gy, gz_0) \geq 1$. Based on z_0, we are going to define a sequence $\{z_n\}$ such that $\{gz_n\}$ will converge, at the same time, to gx and to gy. By the unicity of the limit, this will prove that $gx = gy$. We only reason with x, but the same argument is valid for y.

Indeed, since $Tz_0 \in TA_0 \subseteq B_0$, there is $s_0 \in A_0$ such that $d(s_0, Tz_0) = \Delta_{AB}$, and since $s_0 \in A_0 \subseteq gA_0$, there is $z_1 \in A_0$ verifying $gz_1 = s_0$. Therefore, $d(gz_1, Tz_0) = \Delta_{AB}$. Repeating this argument, there exists a sequence $\{z_n\} \subseteq A_0$ such that $d(gz_{n+1}, Tz_n) = \Delta_{AB}$ for all $n \geq 0$. In particular, $gz_{n+1} \in A_0$ and $Tz_n \in B_0$.

Now we reason using x. We claim that

$$\alpha(gx, gz_n) \geq 1 \quad \forall n \geq 0. \tag{67}$$

If $n = 0$, $\alpha(gx, gz_0) \geq 1$ by the choice of z_0. Suppose that $\alpha(gx, gz_n) \geq 1$ for some $n \geq 0$. In such a case, taking into account that T is (α, g)-proximal admissible, we have that

$$\left.\begin{array}{c} x, z_n, z_{n+1} \in A_0 \\ \alpha(gx, gz_n) \geq 1 \\ d(gx, Tx) = \Delta_{AB} \\ d(gz_{n+1}, Tz_n) = \Delta_{AB} \end{array}\right\} \Longrightarrow \alpha(gx, gz_{n+1}) \geq 1. \tag{68}$$

This concludes that (67) holds. Taking into account that, for all $n \geq 0$,

$$\begin{aligned}
&\frac{d(gx, Tz_n) + d(gz_n, Tx)}{2} - \Delta_{AB} \\
&\leq \frac{d(gx, gz_{n+1}) + d(gz_{n+1}, Tz_n) + d(gz_n, gx) + d(gx, Tx)}{2} \\
&\quad - \Delta_{AB} \\
&= \frac{d(gx, gz_{n+1}) + \Delta_{AB} + d(gz_n, gx) + \Delta_{AB}}{2} - \Delta_{AB} \\
&= \frac{d(gx, gz_{n+1}) + d(gz_n, gx)}{2} \\
&\leq \max(d(gx, gz_n), d(gx, gz_{n+1})),
\end{aligned} \tag{69}$$

it follows that, for all $n \geq 0$,

$$\begin{aligned}
&M^g(x, z_n) \\
&\quad = \max\Bigg(d(gx, gz_n), d(gx, Tx) - \Delta_{AB}, \\
&\qquad\qquad d(gz_n, Tz_n) - \Delta_{AB}, \\
&\qquad\qquad \frac{d(gx, Tz_n) + d(gz_n, Tx)}{2} - \Delta_{AB} \Bigg) \\
&\quad \leq \max(d(gx, gz_n), d(gx, gz_{n+1})).
\end{aligned} \tag{70}$$

Therefore, using the weak P-property of the first kind,

$$\left.\begin{array}{r} x, z_n, z_{n+1} \in A_0 \\ d(gx, Tx) = \Delta_{AB} \\ d(gz_{n+1}, Tz_n) = \Delta_{AB} \end{array}\right\} \Longrightarrow d(gx, gz_{n+1}) \leq d(Tx, Tz_n), \tag{71}$$

and, hence, by the contractivity condition, for all $n \geq 0$,

$$\begin{aligned}
d(gx, gz_{n+1}) &\leq d(Tx, Tz_n) \\
&\leq \varphi(M^g(x, z_n)) \\
&\quad + \theta(d(gz_n, Tx) - \Delta_{AB}, d(gx, Tz_n) - \Delta_{AB}, \\
&\qquad d(gx, Tx) - \Delta_{AB}, d(gz_n, Tz_n) - \Delta_{AB}) \\
&\leq \varphi(M^g(x, z_n)) \\
&\leq \varphi(\max(d(gx, gz_n), d(gx, gz_{n+1}))).
\end{aligned} \tag{72}$$

Suppose that there is $n_0 \in \mathbb{N}$ such that $gz_{n_0} = gx$. In this case

$$\begin{aligned}
d(gx, gz_{n_0+1}) &\leq \varphi(\max(d(gx, gz_{n_0}), d(gx, gz_{n_0+1}))) \\
&= \varphi(d(gx, gz_{n_0+1})),
\end{aligned} \tag{73}$$

but this is only possible when $d(gx, gz_{n_0+1}) = 0$; that is, $gz_{n_0+1} = gx$. Repeating this argument, we have that $gz_n = gx$ for all $n \geq n_0$, which proves that $\{gz_n\}$ is a sequence converging to gx. In this case, the proof is finished.

On the other hand, suppose that $gz_n \neq gx$ for all $n \geq 0$; that is, $d(gx, gz_n) > 0$ for all $n \geq 0$. In this case, it is impossible that $\max(d(gx, gz_n), d(gx, gz_{n+1})) = d(gx, gz_{n+1})$ for some n, since (72) would yield to

$$\begin{aligned}
d(gx, gz_{n+1}) &\leq \varphi(\max(d(gx, gz_n), d(gx, gz_{n+1}))) \\
&= \varphi(d(gx, gz_{n+1})) \\
&< d(gx, gz_{n+1}).
\end{aligned} \tag{74}$$

Therefore, $\max(d(gx, gz_n), d(gx, gz_{n+1})) = d(gx, gz_n)$; that is, for all $n \geq 0$,

$$d(gx, gz_{n+1}) \leq \varphi(M^g(x, z_n)) = \varphi(d(gx, gz_n)). \tag{75}$$

Recursively, for all $n \geq 0$,

$$\begin{aligned}
d(gx, gz_n) &\leq \varphi(d(gx, gz_{n-1})) \\
&\leq \varphi^2(d(gx, gz_{n-2})) \\
&\leq \cdots \leq \varphi^n(d(gx, gz_0)).
\end{aligned} \tag{76}$$

Next we prove that $\{gz_n\}$ converges to gx. Fix $\varepsilon > 0$ arbitrary and consider $t_0 = d(gx, gz_0) > 0$. Since $\varphi \in \Psi$, the series $\sum_{n\geq 1} \varphi^n(t_0)$ converges. In particular, there exists $m_0 \in \mathbb{N}$ such that $\sum_{k=m_0}^{\infty} \varphi^n(t_0) < \varepsilon$. More precisely, $\varphi^n(t_0) < \varepsilon$ for all $n \geq m_0$. Therefore, if $n \geq m_0$, we have that

$$d(gx, gz_n) \leq \varphi^n(d(gx, gz_0)) = \varphi^n(t_0) < \varepsilon. \tag{77}$$

This means that $\{gx_n\}$ converges to gx, and this finishes the proof. □

Notice that if the regularity condition considered in Definition 26 holds for all $x, y \in A$, then we can deduce that $gx = gy$ for all g-best proximity points x and y of T in A, but using the weak P-property of the third kind in A. We also point out that we could deduce the unicity of the g-best proximity point if g is injective on the set of all g-best proximity points of T (not necessarily on A).

5. Consequences

If g is the identity mapping on A, we deduce the following result.

Corollary 28. *Theorem 11 immediately follows from Theorem 20.*

If there is $k \in [0, 1)$ such that $\varphi(t) = kt$ for all $t > 0$, one has the following result.

Corollary 29. *Let A and B be two closed subsets of a complete metric space (X, d). Let $T : A \to B$, $g : A \to A$, $\varphi : [0, \infty) \to [0, \infty)$, $\theta : [0, \infty)^4 \to [0, \infty)$, and $\alpha : X \times X \to [0, \infty)$ be five mappings. Assume that the following conditions hold:*

(a) $\emptyset \neq A_0 \subseteq gA_0$ *and* $T(A_0) \subseteq B_0$;

(b) *the quadruple* (A, B, T, g) *has the weak P-property of the first kind;*

(c) T *is a* (α, g)*-proximal admissible and there is* $k \in [0,1)$ *verifying that for all* $x, y \in A_0$ *such that* $d(gy, Tx) = \Delta_{AB}$ *and* $\alpha(gx, gy) \geq 1$, *one has that*

$$\begin{aligned}\alpha(gx, gy)\, d(Tx, Ty) &\leq kM^g(x, y) \\ &\quad + \theta\big(d(gy, Tx) - \Delta_{AB}, d(gx, Ty) - \Delta_{AB}, \\ &\qquad d(gx, Tx) - \Delta_{AB}, d(gy, Ty) - \Delta_{AB}\big);\end{aligned} \tag{78}$$

(d) *if* $\{z_n\} \subseteq A_0$ *is a sequence such that* $\{gz_n\} \subseteq A_0$ *is Cauchy, then* $\{z_n\}$ *also is Cauchy;*

(e) *there exists* $(x_0, x_1) \in A_0 \times A_0$ *such that* $d(gx_1, Tx_0) = \Delta_{AB}$ *and* $\alpha(gx_0, gx_1) \geq 1$;

(f) g *is a continuous mapping;*

(g) T *is a continuous mapping;*

(h) $\varphi \in \Psi$ *and* $\theta \in \Omega'$.

Then there exists a convergent sequence $\{x_n\}_{n\geq 0} \subseteq A_0$ *verifying*

$$d(gx_{n+1}, Tx_n) = \Delta_{AB} \quad \forall n \geq 0, \tag{79}$$

whose limit is a g-best proximity point of T. Actually, every sequence $\{x_n\}_{n\geq 0} \subseteq A_0$ *verifying* (18) *and* $\alpha(gx_0, gx_1) \geq 1$ *converges to a g-best proximity point of T.*

If $\theta(t_1, t_2, t_3, t_4) = 0$ for all $t_1, t_2, t_3, t_4 \geq 0$, we deduce the following corollary.

Corollary 30. *Let A and B be two closed subsets of a complete metric space* (X, d). *Let* $T : A \to B$, $g : A \to A$, $\varphi : [0, \infty) \to [0, \infty)$, $\theta : [0, \infty)^4 \to [0, \infty)$, *and* $\alpha : X \times X \to [0, \infty)$ *be five mappings. Assume that the following conditions hold:*

(a) $\emptyset \neq A_0 \subseteq gA_0$ *and* $T(A_0) \subseteq B_0$;

(b) *the quadruple* (A, B, T, g) *has the weak P-property of the first kind;*

(c) T *is a* (α, g)*-proximal admissible and there is* $\varphi \in \Psi$ *verifying that for all* $x, y \in A_0$ *such that* $d(gy, Tx) = \Delta_{AB}$ *and* $\alpha(gx, gy) \geq 1$, *one has that*

$$\alpha(gx, gy)\, d(Tx, Ty) \leq \varphi(M^g(x, y)); \tag{80}$$

(d) *if* $\{z_n\} \subseteq A_0$ *is a sequence such that* $\{gz_n\} \subseteq A_0$ *is Cauchy, then* $\{z_n\}$ *also is Cauchy;*

(e) *there exists* $(x_0, x_1) \in A_0 \times A_0$ *such that* $d(gx_1, Tx_0) = \Delta_{AB}$ *and* $\alpha(gx_0, gx_1) \geq 1$;

(f) g *is a continuous mapping;*

(g) T *is a continuous mapping;*

(h) $\varphi \in \Psi$ *and* $\theta \in \Omega'$.

Then there exists a convergent sequence $\{x_n\}_{n\geq 0} \subseteq A_0$ *verifying*

$$d(gx_{n+1}, Tx_n) = \Delta_{AB} \quad \forall n \geq 0, \tag{81}$$

whose limit is a g-best proximity point of T. Actually, every sequence $\{x_n\}_{n\geq 0} \subseteq A_0$ *verifying* (18) *and* $\alpha(gx_0, gx_1) \geq 1$ *converges to a g-best proximity point of T.*

If the pair (A, B) has the P-property, we conclude the following particular version.

Corollary 31. *Let A and B be two closed subsets of a complete metric space* (X, d). *Let* $T : A \to B$, $g : A \to A$, $\varphi : [0, \infty) \to [0, \infty)$, $\theta : [0, \infty)^4 \to [0, \infty)$, *and* $\alpha : X \times X \to [0, \infty)$ *be five mappings. Assume that the following conditions hold:*

(a) $\emptyset \neq A_0 \subseteq gA_0$ *and* $T(A_0) \subseteq B_0$;

(b) *the pair* (A, B) *has the P-property;*

(c) T *is a* (α, g)*-proximal admissible* $(\varphi, \theta, \alpha, g)$*-contraction;*

(d) *if* $\{z_n\} \subseteq A_0$ *is a sequence such that* $\{gz_n\} \subseteq A_0$ *is Cauchy, then* $\{z_n\}$ *also is Cauchy;*

(e) *there exists* $(x_0, x_1) \in A_0 \times A_0$ *such that* $d(gx_1, Tx_0) = \Delta_{AB}$ *and* $\alpha(gx_0, gx_1) \geq 1$;

(f) g *is a continuous mapping;*

(g) T *is a continuous mapping;*

(h) $\varphi \in \Psi$ *and* $\theta \in \Omega'$.

Then there exists a convergent sequence $\{x_n\}_{n\geq 0} \subseteq A_0$ *verifying*

$$d(gx_{n+1}, Tx_n) = \Delta_{AB} \quad \forall n \geq 0, \tag{82}$$

whose limit is a g-best proximity point of T. Actually, every sequence $\{x_n\}_{n\geq 0} \subseteq A_0$ *verifying* (18) *and* $\alpha(gx_0, gx_1) \geq 1$ *converges to a g-best proximity point of T.*

If α is the mapping associated to a binary relation $\preccurlyeq$ (a transitive relation, a preorder, or a partial order), we have the following result.

Corollary 32. *Let A and B be two closed subsets of a complete metric space* (X, d) *provided with a binary relation* $\preccurlyeq$. *Let* $T : A \to B$, $g : A \to A$, $\varphi : [0, \infty) \to [0, \infty)$, *and* $\theta : [0, \infty)^4 \to [0, \infty)$ *be four mappings. Assume that the following conditions hold:*

(a) $\emptyset \neq A_0 \subseteq gA_0$ *and* $T(A_0) \subseteq B_0$;

(b) *the quadruple* (A, B, T, g) *has the weak P-property of the first kind;*

(c) T *verifies the following properties:*

(c.1) *if* $a_1, a_2, b_1, b_2 \in A_0$, *then*

$$\left.\begin{array}{c} gb_1 \preccurlyeq gb_2 \\ d\left(ga_1, Tb_1\right) = \Delta_{AB} \\ d\left(ga_2, Tb_2\right) = \Delta_{AB} \end{array}\right\} \Longrightarrow ga_1 \preccurlyeq ga_2; \tag{83}$$

(c.2) *for all* $x, y \in A_0$ *such that* $d(gy, Tx) = \Delta_{AB}$ *and* $gx \preccurlyeq gy$, *one has that*

$$\begin{aligned} &d\left(Tx, Ty\right) \\ &\quad \leq \varphi\left(M^g\left(x, y\right)\right) \\ &\qquad + \theta\left(d\left(gy, Tx\right) - \Delta_{AB}, d\left(gx, Ty\right) - \Delta_{AB},\right. \\ &\qquad\qquad \left. d\left(gx, Tx\right) - \Delta_{AB}, d\left(gy, Ty\right) - \Delta_{AB}\right), \end{aligned} \tag{84}$$

(d) *if* $\{z_n\} \subseteq A_0$ *is a sequence such that* $\{gz_n\} \subseteq A_0$ *is Cauchy, then* $\{z_n\}$ *also is Cauchy;*

(e) *there exists* $(x_0, x_1) \in A_0 \times A_0$ *such that* $d(gx_1, Tx_0) = \Delta_{AB}$ *and* $gx_0 \preccurlyeq gx_1$;

(f) g *is a continuous mapping;*

(g) T *is a continuous mapping;*

(h) $\varphi \in \Psi$ *and* $\theta \in \Omega'$.

Then there exists a convergent sequence $\{x_n\}_{n\geq 0} \subseteq A_0$ *verifying*

$$d\left(gx_{n+1}, Tx_n\right) = \Delta_{AB} \quad \forall n \geq 0, \tag{85}$$

whose limit is a g*-best proximity point of* T. *Actually, every sequence* $\{x_n\}_{n\geq 0} \subseteq A_0$ *verifying* (18) *and* $\alpha(gx_0, gx_1) \geq 1$ *converges to a* g*-best proximity point of* T.

If $\alpha(x, y) = 1$ for all $x, y \in X$, then we conclude the following consequence.

Corollary 33. *Let* A *and* B *be two closed subsets of a complete metric space* (X, d) *provided with a binary relationship* $\preccurlyeq$. *Let* $T : A \to B$, $g : A \to A$, $\varphi : [0, \infty) \to [0, \infty)$, *and* $\theta : [0, \infty)^4 \to [0, \infty)$ *be four mappings. Assume that the following conditions hold:*

(a) $\emptyset \neq A_0 \subseteq gA_0$ *and* $T(A_0) \subseteq B_0$;

(b) *the quadruple* (A, B, T, g) *has the weak* P*-property of the first kind;*

(c) *for all* $x, y \in A_0$ *such that* $d(gy, Tx) = \Delta_{AB}$, *one has that*

$$\begin{aligned} &d\left(Tx, Ty\right) \\ &\quad \leq \varphi\left(M^g\left(x, y\right)\right) \\ &\qquad + \theta\left(d\left(gy, Tx\right) - \Delta_{AB}, d\left(gx, Ty\right) - \Delta_{AB},\right. \\ &\qquad\qquad \left. d\left(gx, Tx\right) - \Delta_{AB}, d\left(gy, Ty\right) - \Delta_{AB}\right); \end{aligned} \tag{86}$$

(d) *if* $\{z_n\} \subseteq A_0$ *is a sequence such that* $\{gz_n\} \subseteq A_0$ *is Cauchy, then* $\{z_n\}$ *also is Cauchy;*

(e) *there exists* $(x_0, x_1) \in A_0 \times A_0$ *such that* $d(gx_1, Tx_0) = \Delta_{AB}$;

(f) g *is a continuous mapping;*

(g) T *is a continuous mapping;*

(h) $\varphi \in \Psi$ *and* $\theta \in \Omega'$.

Then there exists a convergent sequence $\{x_n\}_{n\geq 0} \subseteq A_0$ *verifying*

$$d\left(gx_{n+1}, Tx_n\right) = \Delta_{AB} \quad \forall n \geq 0, \tag{87}$$

whose limit is a g*-best proximity point of* T. *Actually, every sequence* $\{x_n\}_{n\geq 0} \subseteq A_0$ *verifying* (18) *and* $\alpha(gx_0, gx_1) \geq 1$ *converges to a* g*-best proximity point of* T.

If $A = B$, the notion of g-best proximity point is equivalent to the concept of coincidence point. In this case, the pair (A, A) has the P-property.

Corollary 34. *Let* A *be a closed subset of a complete metric space* (X, d). *Let* $T, g : A \to A$, $\varphi : [0, \infty) \to [0, \infty)$, $\theta : [0, \infty)^4 \to [0, \infty)$, *and* $\alpha : X \times X \to [0, \infty)$ *be five mappings. Assume that the following conditions hold:*

(a) $\emptyset \neq A_0 \subseteq gA_0$ *and* $T(A_0) \subseteq A_0$;

(c) T *is a* (α, g)*-proximal admissible* $(\varphi, \theta, \alpha, g)$*-contraction;*

(d) *if* $\{z_n\} \subseteq A_0$ *is a sequence such that* $\{gz_n\} \subseteq A_0$ *is Cauchy, then* $\{z_n\}$ *also is Cauchy;*

(e) *there exists* $(x_0, x_1) \in A_0 \times A_0$ *such that* $gx_1 = Tx_0$ *and* $\alpha(gx_0, gx_1) \geq 1$;

(f) g *is a continuous mapping;*

(g) T *is a continuous mapping;*

(h) $\varphi \in \Psi$ *and* $\theta \in \Omega'$.

Then there exists a convergent sequence $\{x_n\}_{n\geq 0} \subseteq A_0$ *verifying*

$$gx_{n+1} = Tx_n \quad \forall n \geq 0, \tag{88}$$

whose limit is a coincidence point of T *and* g. *Actually, every sequence* $\{x_n\}_{n\geq 0} \subseteq A_0$ *verifying* (88) *and* $\alpha(gx_0, gx_1) \geq 1$ *converges to a coincidence point of* T *and* g.

As we have just seen, combining the previous results, including the possibility of changing (h) by (h$'$), we could deduce a lot of different independent corollaries, for instance, the following well-known ones. Using $A = B = X$, g as the identity mapping on X, $\varphi(t) = kt$ for all $t \geq 0$ and $\alpha(x, y) = 1$ for all $x, y \in X$, we deduce the following property.

Corollary 35 (Banach contractive mapping principle). *Every contractive mapping from a complete metric space into itself has a unique fixed point.*

The following results are also particular cases of our main result.

Corollary 36 (Ran and Reurings [14]). *Let $(X, \preccurlyeq)$ be an ordered set endowed with a metric d and let $T : X \to X$ be a given mapping. Suppose that the following conditions hold.*

(a) *(X, d) is complete.*

(b) *T is nondecreasing (with respect to $\preccurlyeq\preccurlyeq$).*

(c) *T is continuous.*

(d) *There exists $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$.*

(e) *There exists a constant $k \in (0, 1)$ such that $d(Tx, Ty) \leq kd(x, y)$ for all $x, y \in X$ with $x \succcurlyeq y$.*

Then T has a fixed point. Moreover, if for all $(x, y) \in X^2$ there exists $z \in X$ such that $x \preccurlyeq z$ and $y \preccurlyeq z$, one obtains uniqueness of the fixed point.

Proof. Consider $A = B = X$. Therefore $A_0 = B_0 = X$. Let g be the identity mapping on X. By Remark 16, the quadruple (X, X, T, g) has the P-property. Let define $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } x \preccurlyeq y, \\ 0, & \text{otherwise.} \end{cases} \tag{89}$$

As T is $\preccurlyeq$-nondecreasing, then T is (α, g)-proximal admissible:

$$\alpha(x, y) \geq 1 \Longrightarrow x \preccurlyeq y \Longrightarrow Tx \preccurlyeq Ty \Longrightarrow \alpha(Tx, Ty) \geq 1. \tag{90}$$

Let $x, y \in X$. If $\alpha(x, y) = 0$, then the contractivity condition (10) is obvious. If $\alpha(x, y) > 0$, then $x \preccurlyeq y$. In particular, using item (e) to $y \succcurlyeq x$ and $\varphi(t) = kt$ for all $t \geq 0$, we have that

$$\alpha(x, y)\, d(T\mathrm{x}, Ty) = d(Ty, Tx) \leq kd(y, x) = \varphi(d(x, y)), \tag{91}$$

so (10) also holds (choosing whatever $\theta \in \Theta$; for instance, see Example 4). In any case, T is a (g, α)-proximal admissible $(\varphi, \theta, \alpha, g)$-contraction. Starting from $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$, let $x_1 = Tx_0$. Then $d(gx_1, Tx_0) = \Delta_{AB} = 0$ and $\alpha(x_0, x_1) \geq 1$. As g and T are continuous, all hypotheses of Theorem 20 are satisfied. Then T has a fixed point. Moreover, Theorem 27 guarantees that it is unique. □

Nieto and Rodríguez-López [15] slightly modified the hypothesis of the previous result obtaining the following theorem.

Corollary 37 (Nieto and Rodríguez-López [15]). *Let $(X, \preccurlyeq)$ be an ordered set endowed with a metric d and let $T : X \to X$ be a given mapping. Suppose that the following conditions hold.*

(a) *(X, d) is complete.*

(b) *T is nondecreasing (with respect to $\preccurlyeq$).*

(c) *If a nondecreasing sequence $\{x_m\}$ in X converges to a some point $x \in X$, then $x_m \preccurlyeq x$ for all m.*

(d) *There exists $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$.*

(e) *There exists a constant $k \in (0, 1)$ such that $d(Tx, Ty) \leq kd(x, y)$ for all $x, y \in X$ with $x \succcurlyeq y$.*

Then T has a fixed point. Moreover, if for all $(x, y) \in X^2$ there exists $z \in X$ such that $x \preccurlyeq z$ and $y \preccurlyeq z$, one obtains uniqueness of the fixed point.

Proof. We can follow point by point the proof of the previous result, but using Theorem 23 rather than Theorem 20. □

From these results, it is also possible to prove many other fixed point results (see, for instance, [36]).

The main differences between our results and Theorem 11 are the following ones. (1) Theorem 11 can only ensure the existence of fixed points; however, we study the existence and uniqueness of coincidence points, involving a mapping g. (2) Our results use more general test functions in the contractivity condition, and our results guarantee existence and uniqueness of coincidence points under different contractivity conditions.

Example 38. Let $X = \mathbb{R}$ be provided with its Euclidean metric d and consider $A = B = X$ and $T, g : X \to X$ are defined by $Tx = (x/2) + 1$ and $gx = 2x$ for all $x \in X$. Let $\alpha : X \times X \to [0, \infty)$ be the mapping given by.

$$\alpha(x, y) = \begin{cases} 1, & \text{if } x \leq y, \\ 0, & \text{otherwise.} \end{cases} \tag{92}$$

Notice that $A_0 = B_0 = X$. If we take $x_0 = -2$ and $x_1 = 0$, then all hypotheses of Theorem 20 are satisfied using $\varphi(t) = t/2$ for all $t \geq 0$. Therefore, T and g have a coincidence point, which is $x = 2/3$. However, Theorem 11 cannot be applied because it only guarantees the existence of a fixed point of F (which, in this case, is $x = 2$).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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