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Research Article

Localizing Brain Activity from Multiple Distinct Sources via EEG

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An important question arousing in the framework of electroencephalography (EEG) is the possibility to recognize, by means of a recorded surface potential, the number of activated areas in the brain. In the present paper, employing a homogeneous spherical conductor serving as an approximation of the brain, we provide a criterion which determines whether the measured surface potential is evoked by a single or multiple localized neuronal excitations. We show that the uniqueness of the inverse problem for a single dipole is closely connected with attaining certain relations connecting the measured data. Further, we present the necessary and sufficient conditions which decide whether the collected data originates from a single dipole or from numerous dipoles. In the case where the EEG data arouses from multiple parallel dipoles, an isolation of the source is, in general, not possible.

1. Introduction

The electrochemically generated neuronal current in the brain and the secondary induction current sustained by the conductive cerebral tissue give rise to an electric potential, which can be recorded on the scalps surface. The dominant generators of the electric field measured on the surface of the head are graded synaptic potentials deriving mainly from the cerebral cortex [1], originating primarily from pyramidal cells uniformly oriented with apical dendrites perpendicular to the cortical surface [2]. The surface of the human cerebral cortex, comprising the outer covering layered structure of neuronal tissue of the brain, can be visualized as a highly convoluted sheet, strongly folded, consisting of fissures, sulci, and gyri. On the other hand, the basic fundamental model for the primary current distribution is a current dipole used as an equivalent source, summarizing the net effect of all microscopic currents located in a distinct region of the brain. This is a widely used approximation concept in the framework of neuroelectromagnetism [3, 4].

Analysis of the registered surface electric potential provides the ability to examine cognitive processes via electroencephalography, a noninvasive, nonhazardous technique capable of following changes in neural activity with millisecond temporal resolution. An important aspect in the field of medical imaging and particular noninvasive brain imaging modalities such as EEG is the problem of identifying the position and moment of an equivalent dipole source. This is a crucial task in order to understand the mechanisms of brain response to various stimuli. However, there does not exist an exclusive source configuration for each set of electroencephalographic measurements. Hence, the corresponding inverse problem is nonunique.

In addition, identifying the number of simultaneously active dipoles and their impact on the electric activity recorded on the surface of the head is a substantial nonunique task as well. Depending on the orientation of the dipoles the interaction of the related electric fields results in either an amplification of the registered EEG values or, on the other hand, cancellation effects occur [5, 6].

On the computational level, detecting and localizing brain activity has been undertaken for over three decades [7–9] and various techniques and methods have been developed in order to obtain approximate solutions. These are mainly divided into two categories, namely, parametric and nonparametric methods, depending on if the number of dipole sources is *a priori* known or not. A review presenting those methods and techniques is given in [10, 11], whereas the key areas critically affecting the accuracy and precision of source localization are analyzed in [12].

From the mathematical perspective, whenever the number of simultaneously active sources is inferred, analytic expressions determining the exact positions and moments of the corresponding dipoles can be derived for the spherical case [13] as well as for the mathematical complex but more realistic ellipsoidal geometry [14]. However, if the neuronal current exhibits two or more localized centers then a method to identify this possibility is essential.

The present paper, employing a homogeneous spherical model for the brain, provides necessary and sufficient conditions for the identification of a single or multiple localized sources by reducing the problem to a set of simple algebraic equations. The simplicity of the demonstrated analysis is based on the manipulation of the exterior electric potential [15] delivering the recorded EEG data trivially as a limiting process. By linearity, the potential of every active dipole is collected and equating the matching coefficients, a set of algebraic relations are derived, establishing the basis of our approach. Finally, straightforward manipulations regarding the obtained linear system provide the conditions which have to be satisfied for the problem to be well-posed.

The paper is organized as follows. In Section 2 we reproduce the inversion algorithm for a single dipole providing uniqueness conditions interconnecting the EEG measurements. This development is then generalized to the case of several dipoles in Section 3. An explicit paradigm, investigating the conditions for the existence of two such dipoles, is also implemented. The results are collected in Section 4.

2. Inverse EEG for One Dipole

Assume a homogeneous spherical model of the brain of radius a denoted by $\Omega \subset \mathbb{R}^3$ and let $\partial\Omega$ be its boundary. The exterior to the brain region is denoted by Ω^c. Activation of a localized region inside the brain triggers a primary neuronal current $\mathbf{J}^p$ provoking an electric potential $u(\mathbf{r}, \mathbf{r}_0)$. In the case where the neuronal current is represented by a single equivalent dipole at the point $\mathbf{r}_0$ with moment $\mathbf{Q}_0$, we obtain $\mathbf{J}^p = \mathbf{Q}_0\delta(\mathbf{r} - \mathbf{r}_0)$, where δ denotes the dirac measure.

Plonsey and Heppner [16] demonstrated that the electromagnetic activity of the brain is governed by the quasistatic theory of Maxwell's equations; namely,

$$\nabla \times \mathbf{E} = \mathbf{0}, \tag{1}$$

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J}^p + \sigma \mathbf{E}\right), \tag{2}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{3}$$

where the magnetic permeability μ_0 is assumed to be constant everywhere in $\mathbb{R}^3$.

Equation (1) allows the introduction of an electric potential u such that

$$\mathbf{E} = -\nabla u. \tag{4}$$

Moreover, by taking the divergence of (2), we immediately conclude that the *interior* electric potential u^- solves the following Neumann boundary-value problem in Ω:

$$\begin{gathered} \sigma \Delta u^- (\mathbf{r}, \mathbf{r}_0) = \mathbf{Q}_0 \cdot \nabla \delta (\mathbf{r} - \mathbf{r}_0), \quad \mathbf{r} \in \Omega, \\ \partial_r u^- (\mathbf{r}, \mathbf{r}_0) = 0, \quad \mathbf{r} \in \partial\Omega, \end{gathered} \tag{5}$$

where σ denotes the conductivity of the conducting medium occupying Ω. The solution regarding (5) is obtained by a straightforward expansion in terms of Legendre polynomials [15].

Once the above problem is solved, knowledge of the solution $u^-(\mathbf{r})$ leads to the exterior electric potential $u^+(\mathbf{r})$ satisfying the Dirichlet problem

$$\begin{gathered} \Delta u^+ (\mathbf{r}, \mathbf{r}_0) = 0, \quad \mathbf{r} \in \Omega^c, \\ u^+ (\mathbf{r}, \mathbf{r}_0) = u^- (\mathbf{r}, \mathbf{r}_0), \quad \mathbf{r} \in \partial\Omega, \\ u^+ (\mathbf{r}, \mathbf{r}_0) = \mathcal{O}\left(\frac{1}{r^2}\right), \quad \mathbf{r} \longrightarrow \infty, \end{gathered} \tag{6}$$

provided as

$$u^+ (\mathbf{r}, \mathbf{r}_0) = \frac{1}{\sigma} \mathbf{Q}_0 \cdot \nabla_{\mathbf{r}_0} \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \frac{1}{n} \frac{r_0^n}{r^{n+1}} \overline{Y}_n^m (\hat{\mathbf{r}}_0) Y_n^m (\hat{\mathbf{r}}), \tag{7}$$

where $Y_n^m(\hat{\mathbf{r}})$ denote the spherical harmonics, whereas the overline symbolizes complex conjugation.

In order to pinpoint the position $\mathbf{r}_0 = (x_{01}, x_{02}, x_{03})$ and moment $\mathbf{Q}_0 = (q_{01}, q_{02}, q_{03})$ of the dipole, six equations are required. As a result, the solution (7) is expanded for $n = 1, 2$ and the resulting relation is expressed in Cartesian coordinates, giving

$$\begin{aligned} u^+ (\mathbf{r}, \mathbf{r}_0) = {} & \frac{1}{4\pi\sigma} \frac{1}{r^3} \left(A_1 x_1 + A_2 x_2 + A_3 x_3\right) \\ & + \frac{1}{4\pi\sigma} \frac{1}{r^5} \left(B_1 x_1^2 + B_2 x_2^2 + B_3 x_3^2 + B_4 x_1 x_2 \right. \\ & \left. + B_5 x_2 x_3 + B_6 x_3 x_1\right) + \mathcal{O}\left(\frac{1}{r^4}\right), \end{aligned} \tag{8}$$

where

$$(A_1, A_2, A_3) = 3 (q_{01}, q_{02}, q_{03}), \tag{9}$$

$$B_1 = \frac{15}{2} q_{01} x_{01} - \frac{5}{2} \mathbf{Q}_0 \cdot \mathbf{r}_0, \tag{10}$$

$$B_2 = \frac{15}{2} q_{02} x_{02} - \frac{5}{2} \mathbf{Q}_0 \cdot \mathbf{r}_0, \tag{11}$$

$$B_3 = \frac{15}{2} q_{03} x_{03} - \frac{5}{2} \mathbf{Q}_0 \cdot \mathbf{r}_0, \tag{12}$$

$$B_4 = \frac{15}{2}\left(q_{01}x_{02} + q_{02}x_{01}\right), \tag{13}$$

$$B_5 = \frac{15}{2}\left(q_{02}x_{03} + q_{03}x_{02}\right), \tag{14}$$

$$B_6 = \frac{15}{2}\left(q_{03}x_{01} + q_{01}x_{03}\right), \tag{15}$$

as well as the harmonicity condition

$$B_1 + B_2 + B_3 = 0. \tag{16}$$

Merging above equations into a single system yields

$$\mathbb{A}\mathbf{r}_0 = \mathbf{B} \tag{17}$$

given that

$$\mathbb{A} = \begin{pmatrix} -A_1 & 2A_2 & -A_3 \\ -A_1 & -A_2 & 2A_3 \\ A_2 & A_1 & 0 \\ A_3 & 0 & A_1 \\ 0 & A_3 & A_2 \end{pmatrix}, \qquad \mathbf{r}_0 = \begin{pmatrix} x_{01} \\ x_{02} \\ x_{03} \end{pmatrix},$$
$$\mathbf{B} = \frac{6}{5}\begin{pmatrix} B_2 \\ B_3 \\ B_4 \\ B_5 \\ B_6 \end{pmatrix}. \tag{18}$$

A simple investigation of the latter shows that

$$x_{01} = \frac{3}{2A_1A_2A_3}\left(A_1\mathbf{a}\right)\cdot\mathbf{b},$$
$$x_{02} = \frac{3}{2A_1A_2A_3}\left(A_2\mathbf{a}\right)\cdot\left(\mathbb{R}_{x_3}(\pi)\,\mathbf{b}\right), \tag{19}$$
$$x_{03} = \frac{3}{2A_1A_2A_3}\left(A_3\mathbf{a}\right)\cdot\left(\mathbb{R}_{x_2}(\pi)\,\mathbf{b}\right),$$

where $\mathbf{a} = (A_1, A_2, A_3)^\top$, $\mathbf{b} = (-B_4, B_5, B_6)^\top$, and

$$\mathbb{R}_{x_3}(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$\mathbb{R}_{x_2}(\alpha) = \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix} \tag{20}$$

are the rotation matrices about the x_3- and x_2-axis, respectively.

Importantly, the above analysis reveals that the dipoles position $\mathbf{r}_0$ specified by relations (19) is unique *only* if the recorded values for the coefficients A_ℓ, $\ell = 1,2,3$, and B_k, $k = 1,2,3,4,5,6$, satisfy the following expressions:

$$2A_1A_2A_3\left(B_3 - B_2\right) + 3A_1\left(A_2^2 - A_3^2\right)B_5 + 3\left(A_2^2 + A_3^2\right)\left(A_3B_4 - A_2B_6\right) = 0, \tag{21}$$

$$-2A_1A_2A_3\left(B_2 + 2B_3\right) + 3\left(A_1^2 + A_3^2\right)\left(A_1B_5 - A_3B_4\right) + 3A_2\left(A_3^2 - A_1^2\right)B_6 = 0. \tag{22}$$

Bearing in mind condition (16), relations (21) and (22) are easily obtained by solving any three equations of (10)–(15) with respect to $\mathbf{r}_0$ and substituting the solution into the remaining two.

Closing this section we note that a major drawback is that knowledge of the surface potential $u(a\hat{\mathbf{r}}, \mathbf{r}_0)$ does not automatically imply knowledge of the coefficients present in (8) as well. A connection to the spherical case has to be made. This is achieved in view of the following formulas [15]:

$$u\left(a\hat{\mathbf{r}}, \mathbf{r}_0\right) = \sum_{n=1}^{\infty}\sum_{m=-n}^{n}\Gamma_n^m\left(\mathbf{r}_0\right)Y_n^m\left(\hat{\mathbf{r}}\right),$$
$$\Gamma_n^m\left(\mathbf{r}_0\right) = \frac{1}{\sigma n a^{n+1}}\left(\mathbf{Q}_0\cdot\nabla_{\mathbf{r}_0}\right)\left(r_0^n\overline{Y}_n^m\left(\hat{\mathbf{r}}_0\right)\right) \tag{23}$$

in conjunction with the orthogonality condition

$$\Gamma_n^m\left(\mathbf{r}_0\right) = \oint_{S^2} u\left(a\hat{\mathbf{r}}, \mathbf{r}_0\right)\overline{Y}_n^m\left(\hat{\mathbf{r}}\right)\mathrm{d}\Omega\left(\hat{\mathbf{r}}\right), \tag{24}$$

where S^2 denotes the boundary of the unit sphere and $\mathrm{d}\Omega(\hat{\mathbf{r}})$ is the solid angle element.

Relation (24) indicates the simple fact that measuring $u(a\hat{\mathbf{r}}, \mathbf{r}_0)$ provides information about the coefficients $\Gamma_n^m(\mathbf{r}_0)$ alone. In order to obtain the values of A_ℓ, $\ell = 1,2,3$, as well as B_k, $k = 1,2,3,4,5,6$, in the limit as r tends to a, relation (23) has to be expanded for $n = 1, 2$, providing the relations

$$A_j = \frac{3}{4\pi a^2}\oint_{S^2} u\left(a\hat{\mathbf{r}}, \mathbf{r}_0\right)x_j\mathrm{d}\Omega\left(\hat{\mathbf{r}}\right), \quad j = 1,2,3,$$
$$B_1 = \frac{5}{8\pi a^4}\oint_{S^2} u\left(a\hat{\mathbf{r}}, \mathbf{r}_0\right)\left(2x_1^2 - x_2^2 - x_3^2\right)\mathrm{d}\Omega\left(\hat{\mathbf{r}}\right),$$
$$B_2 = \frac{5}{8\pi a^4}\oint_{S^2} u\left(a\hat{\mathbf{r}}, \mathbf{r}_0\right)\left(2x_2^2 - x_3^2 - x_1^2\right)\mathrm{d}\Omega\left(\hat{\mathbf{r}}\right),$$
$$B_3 = \frac{5}{8\pi a^4}\oint_{S^2} u\left(a\hat{\mathbf{r}}, \mathbf{r}_0\right)\left(2x_3^2 - x_1^2 - x_2^2\right)\mathrm{d}\Omega\left(\hat{\mathbf{r}}\right), \tag{25}$$
$$B_4 = \frac{15}{4\pi a^4}\oint_{S^2} u\left(a\hat{\mathbf{r}}, \mathbf{r}_0\right)x_1x_2\mathrm{d}\Omega\left(\hat{\mathbf{r}}\right),$$
$$B_5 = \frac{15}{4\pi a^4}\oint_{S^2} u\left(a\hat{\mathbf{r}}, \mathbf{r}_0\right)x_2x_3\mathrm{d}\Omega\left(\hat{\mathbf{r}}\right),$$
$$B_6 = \frac{15}{4\pi a^4}\oint_{S^2} u\left(a\hat{\mathbf{r}}, \mathbf{r}_0\right)x_3x_1\mathrm{d}\Omega\left(\hat{\mathbf{r}}\right).$$

3. EEG Inversion for Multiple Dipoles

In what follows, we consider a number of simultaneously active dipoles $(\mathbf{r}_j, \mathbf{Q}_j)$, $j = 1,2,\ldots,N$. Due to linearity, the electric potential generated outside the sphere is

$$u^+\left(\mathbf{r}; \mathbf{r}_0, \mathbf{Q}_0\right) = \sum_{j=1}^{N} u^+\left(\mathbf{r}; \mathbf{r}_j, \mathbf{Q}_j\right), \tag{26}$$

where the dipole $(\mathbf{r}_0, \mathbf{Q}_0)$ represents the contribution of a fictitious dipole inside the brain, providing the same potential as the dipoles $(\mathbf{r}_j, \mathbf{Q}_j)$, $j = 1,2,\ldots,N$, collectively.

Equating coefficients in (26) given that $\mathbf{r}_j = (x_{j1}, x_{j2}, x_{j3})$ and $\mathbf{Q}_j = (q_{j1}, q_{j2}, q_{j3})$ leads to an identical system as (17) and (18),

$$(A_1, A_2, A_3) = 3\left(\sum_{j=1}^{N} q_{j1}, \sum_{j=1}^{N} q_{j2}, \sum_{j=1}^{N} q_{j3}\right), \tag{27}$$

$$B_1 = \sum_{j=1}^{N}\left(\frac{15}{2} q_{j1} x_{j1} - \frac{5}{2}\mathbf{Q}_j \cdot \mathbf{r}_j\right), \tag{28}$$

$$B_2 = \sum_{j=1}^{N}\left(\frac{15}{2} q_{j2} x_{j2} - \frac{5}{2}\mathbf{Q}_j \cdot \mathbf{r}_j\right), \tag{29}$$

$$B_3 = \sum_{j=1}^{N}\left(\frac{15}{2} q_{j3} x_{j3} - \frac{5}{2}\mathbf{Q}_j \cdot \mathbf{r}_j\right), \tag{30}$$

$$B_4 = \frac{15}{2}\sum_{j=1}^{N}\left(q_{j1} x_{j2} + q_{j2} x_{j1}\right), \tag{31}$$

$$B_5 = \frac{15}{2}\sum_{j=1}^{N}\left(q_{j2} x_{j3} + q_{j3} x_{j2}\right), \tag{32}$$

$$B_6 = \frac{15}{2}\sum_{j=1}^{N}\left(q_{j3} x_{j1} + q_{j1} x_{j3}\right). \tag{33}$$

Regardless of the number of introduced dipole sources, the constraint (16) as well as the uniqueness conditions (21) and (22) are still binding.

Let us investigate in the sequel the necessary circumstances regarding the N dipoles $(\mathbf{r}_j, \mathbf{Q}_j)$ for which the uniqueness conditions (21) and (22) are satisfied. Substituting relations (28)–(33) into (21) and (22), respectively, and apprising the fact that the resulting equations are fulfilled if the coefficients of each variable vanishes lead to

$$\left(q_{02}^2 + q_{03}^2\right)\left(q_{03} q_{j2} - q_{02} q_{j3}\right) = 0, \tag{34}$$

$$\begin{aligned} &q_{01} q_{02}\left(q_{02} q_{j3} - q_{03} q_{j2}\right) + q_{02} q_{03}\left(q_{02} q_{j1} - q_{01} q_{j2}\right) \\ &\quad + q_{03}^2\left(q_{03} q_{j1} - q_{01} q_{j3}\right) = 0, \end{aligned} \tag{35}$$

$$\begin{aligned} &q_{01} q_{03}\left(q_{02} q_{j3} - q_{03} q_{j2}\right) + q_{02} q_{03}\left(q_{01} q_{j3} - q_{03} q_{j1}\right) \\ &\quad + q_{02}^2\left(q_{01} q_{j2} - q_{02} q_{j1}\right) = 0, \end{aligned} \tag{36}$$

$$\begin{aligned} &q_{01} q_{03}\left(q_{02} q_{j1} - q_{01} q_{j2}\right) + q_{01} q_{02}\left(q_{03} q_{j1} - q_{01} q_{j3}\right) \\ &\quad + q_{03}^2\left(q_{02} q_{j3} - q_{03} q_{j2}\right) = 0, \end{aligned} \tag{37}$$

$$\left(q_{01}^2 + q_{03}^2\right)\left(q_{01} q_{j3} - q_{03} q_{j1}\right) = 0, \tag{38}$$

$$\begin{aligned} &q_{01} q_{03}\left(q_{03} q_{j2} - q_{02} q_{j3}\right) + q_{02} q_{03}\left(q_{03} q_{j1} - q_{01} q_{j3}\right) \\ &\quad + q_{01}^2\left(q_{01} q_{j2} - q_{02} q_{j1}\right) = 0, \end{aligned} \tag{39}$$

for every $j = 1, 2, \ldots, N$.

Above equations are trivially satisfied in the case where $\mathbf{Q}_0 = \mathbf{0}$. However, this particular choice leads to $u^{+}(\mathbf{r}; \mathbf{r}_0, \mathbf{Q}_0) = 0$ and turns the dipoles $(\mathbf{r}_j, \mathbf{Q}_j)$, $j = 1, 2, \ldots, N$, to silent sources, that is, brain activity impossible to be recorded via electroencephalographic measurements. On the other hand, whenever $\mathbf{Q}_0 \neq \mathbf{0}$ (34) and (35) are attained if

$$\begin{aligned} q_{03} q_{j2} - q_{02} q_{j3} &= 0, \\ q_{02} q_{j1} - q_{01} q_{j2} &= 0, \\ q_{03} q_{j1} - q_{01} q_{j3} &= 0. \end{aligned} \tag{40}$$

According to (9) and (27)

$$q_{0s} = \sum_{i=1}^{N} q_{is}, \quad s = 1, 2, 3 \tag{41}$$

and thus relations (40) read

$$\begin{aligned} q_{j2}\sum_{i=1}^{N} q_{i3} - q_{j3}\sum_{i=1}^{N} q_{i2} &= 0, \quad j = 1, 2, \ldots, N, \\ q_{j1}\sum_{i=1}^{N} q_{i2} - q_{j2}\sum_{i=1}^{N} q_{i1} &= 0, \quad j = 1, 2, \ldots, N, \\ q_{j1}\sum_{i=1}^{N} q_{i3} - q_{j3}\sum_{i=1}^{N} q_{i1} &= 0, \quad j = 1, 2, \ldots, N. \end{aligned} \tag{42}$$

It is not hard to show that aforementioned formulas are simultaneously satisfied provided that

$$\hat{\mathbf{x}}_s \cdot \mathbf{Q}_j \times \sum_{\substack{i=1 \\ i \neq j}}^{N} \mathbf{Q}_i = 0, \quad s = 1, 2, 3, \; j = 1, 2, \ldots, N, \tag{43}$$

comprising $N - 1$ equations for each direction $\hat{\mathbf{x}}_s$, $s = 1, 2, 3$, or in other words, wherever the dipoles are parallel to each other; namely,

$$\mathbf{Q}_i \times \mathbf{Q}_j = \mathbf{0}, \quad i, j = 1, 2, \ldots, N, \; i \neq j. \tag{44}$$

Other occurrences, for example, $q_{01} = 0$, $q_{02}, q_{03} \neq 0$, and so forth, result in identical conclusions.

In view of Section 2, it is feasible to acquire the positions and moments of the N dipoles if at least $6N$ of the corresponding coefficients regarding the RHS of (26), that is,

$$\Gamma_n^m\left(\mathbf{r}_j\right) = \frac{1}{\sigma n a^{n+1}}\sum_{j=1}^{N}\left(\mathbf{Q}_j \cdot \nabla_{\mathbf{r}_j}\right)\left(r_j^n \overline{Y}_n^m\left(\hat{\mathbf{r}}_j\right)\right) \tag{45}$$

are identified. In the interest of completeness, we note that a simple but strictly theoretical constraint which decides whether recorded EEG data could originate from multiple dipole sources is that the remaining coefficients (45) must be compatible with the given measurements [13].

3.1. Example: Inversion Algorithm for Two Dipoles. For the sake of demonstrating details of the presented analysis we establish to the full extent the case for two concurrently active dipoles.

Equating coefficients in (26) leads to the following relations:

$$\mathbf{Q}_0 = \mathbf{Q}_1 + \mathbf{Q}_2, \tag{46}$$

$$3q_{01}x_{01} - \mathbf{Q}_0 \cdot \mathbf{r}_0 = 3q_{11}x_{11} - \mathbf{Q}_1 \cdot \mathbf{r}_1 + 3q_{21}x_{21} - \mathbf{Q}_2 \cdot \mathbf{r}_2, \tag{47}$$

$$3q_{02}x_{02} - \mathbf{Q}_0 \cdot \mathbf{r}_0 = 3q_{12}x_{12} - \mathbf{Q}_1 \cdot \mathbf{r}_1 + 3q_{22}x_{22} - \mathbf{Q}_2 \cdot \mathbf{r}_2, \tag{48}$$

$$3q_{03}x_{03} - \mathbf{Q}_0 \cdot \mathbf{r}_0 = 3q_{13}x_{13} - \mathbf{Q}_1 \cdot \mathbf{r}_1 + 3q_{23}x_{23} - \mathbf{Q}_2 \cdot \mathbf{r}_2, \tag{49}$$

$$q_{01}x_{02} + q_{02}x_{01} = q_{11}x_{12} + q_{12}x_{11} + q_{21}x_{22} + q_{22}x_{21}, \tag{50}$$

$$q_{02}x_{03} + q_{03}x_{02} = q_{12}x_{13} + q_{13}x_{12} + q_{22}x_{23} + q_{23}x_{22}, \tag{51}$$

$$q_{03}x_{01} + q_{01}x_{03} = q_{13}x_{11} + q_{11}x_{13} + q_{23}x_{21} + q_{21}x_{23}. \tag{52}$$

Solving equations (50)–(52) with respect to $\mathbf{r}_0 = (x_{01}, x_{02}, x_{03})$ we arrive at

$$\begin{aligned}
x_{01} &= \frac{1}{2q_{02}q_{03}} \times \left[(q_{12}q_{03} + q_{13}q_{02}) x_0 + (q_{22}q_{03} + q_{23}q_{02}) x_{21} \right] \\
&\quad + \frac{1}{2q_{02}q_{03}} \times \left[-\begin{vmatrix} q_{13} & q_{11} \\ q_{23} & q_{21} \end{vmatrix} (x_{12} - x_{22}) + \begin{vmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{vmatrix} (x_{13} - x_{23}) \right], \\
x_{02} &= \frac{1}{2q_{01}q_{03}} \times \left[(q_{11}q_{03} + q_{13}q_{01}) x_{12} + (q_{21}q_{03} + q_{23}q_{01}) x_{22} \right] \\
&\quad + \frac{1}{2q_{01}q_{03}} \times \left[\begin{vmatrix} q_{12} & q_{13} \\ q_{22} & q_{23} \end{vmatrix} (x_{11} - x_{21}) - \begin{vmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{vmatrix} (x_{13} - x_{23}) \right], \\
x_{03} &= \frac{1}{2q_{01}q_{02}} \times \left[(q_{11}q_{02} + q_{12}q_{01}) x_{13} + (q_{21}q_{02} + q_{22}q_{01}) x_{23} \right] \\
&\quad + \frac{1}{2q_{01}q_{02}} \times \left[-\begin{vmatrix} q_{12} & q_{13} \\ q_{22} & q_{23} \end{vmatrix} (x_{11} - x_{21}) + \begin{vmatrix} q_{13} & q_{11} \\ q_{23} & q_{21} \end{vmatrix} (x_{12} - x_{22}) \right].
\end{aligned} \tag{53}$$

On the other hand, subtracting (48) from (47) and combining the resulting equation with (50) to eliminate x_{02} we obtain

$$\begin{aligned}
\tilde{x}_{01} &= \frac{1}{q_{01}^2 + q_{02}^2} \times \left[(q_{11}q_{01} + q_{12}q_{02}) x_{11} + (q_{21}q_{01} + q_{22}q_{02}) x_{21} \right] \\
&\quad + \frac{1}{q_{01}^2 + q_{02}^2} \begin{vmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{vmatrix} (x_{12} - x_{22}).
\end{aligned} \tag{54}$$

Similarly, subtracting (49) from (48) and eliminating x_{03} between the resulting equation and (51) we obtain

$$\begin{aligned}
\tilde{x}_{02} &= \frac{1}{q_{02}^2 + q_{03}^2} \times \left[(q_{12}q_{02} + q_{13}q_{03}) x_{12} + (q_{22}q_{02} + q_{23}q_{03}) x_{22} \right] \\
&\quad + \frac{1}{q_{02}^2 + q_{03}^2} \begin{vmatrix} q_{12} & q_{13} \\ q_{22} & q_{23} \end{vmatrix} (x_{13} - x_{23}).
\end{aligned} \tag{55}$$

Finally, subtracting (47) from (49) and using the resulting equation to eliminate x_{01} from (52) we arrive at

$$\begin{aligned}
\tilde{x}_{03} &= \frac{1}{q_{01}^2 + q_{03}^2} \times \left[(q_{13}q_{03} + q_{11}q_{01}) x_{13} + (q_{23}q_{03} + q_{21}q_{01}) x_{23} \right] \\
&\quad + \frac{1}{q_{01}^2 + q_{03}^2} \begin{vmatrix} q_{13} & q_{11} \\ q_{23} & q_{21} \end{vmatrix} (x_{11} - x_{21}).
\end{aligned} \tag{56}$$

Obviously, the point $\mathbf{r}_0 = (x_{01}, x_{02}, x_{03})$ defined by relations (53) does not coincide with the point $\tilde{\mathbf{r}}_0 = (\tilde{x}_{01}, \tilde{x}_{02}, \tilde{x}_{03})$ defined from the solutions (54)–(56). This means that the system (47)–(52) is not compatible. That is, the solution we obtain depends on the choice of the equations we choose to calculate the x_{0j}'s or, in other words, if we pick up any three equations among the six equations of the system, (with the exception of the first three which are linearly dependent) to calculate x_{0j}, $j = 1, 2, 3$, the solution we obtain does not necessarily satisfy the remaining equations. This observation tells us that the system (47)–(52) "knows" that it does not represent a single dipole and therefore it provides a criterion for deciding whether there is a single point excitation or it is an excitation that is due to more than one dipole. It is easily shown that if the moments of the two dipoles are parallel, that is, if

$$\mathbf{Q}_2 = \gamma \mathbf{Q}_1, \quad \gamma \in \mathbb{R}, \tag{57}$$

then the representations for $\mathbf{r}_0$ and $\tilde{\mathbf{r}}_0$ coincide, and therefore, in this case, it is impossible to decide if the dipoles are one or two. In fact, under the condition (57), we obtain

$$\mathbf{Q}_1 \times \mathbf{Q}_2 = \mathbf{0}, \tag{58}$$

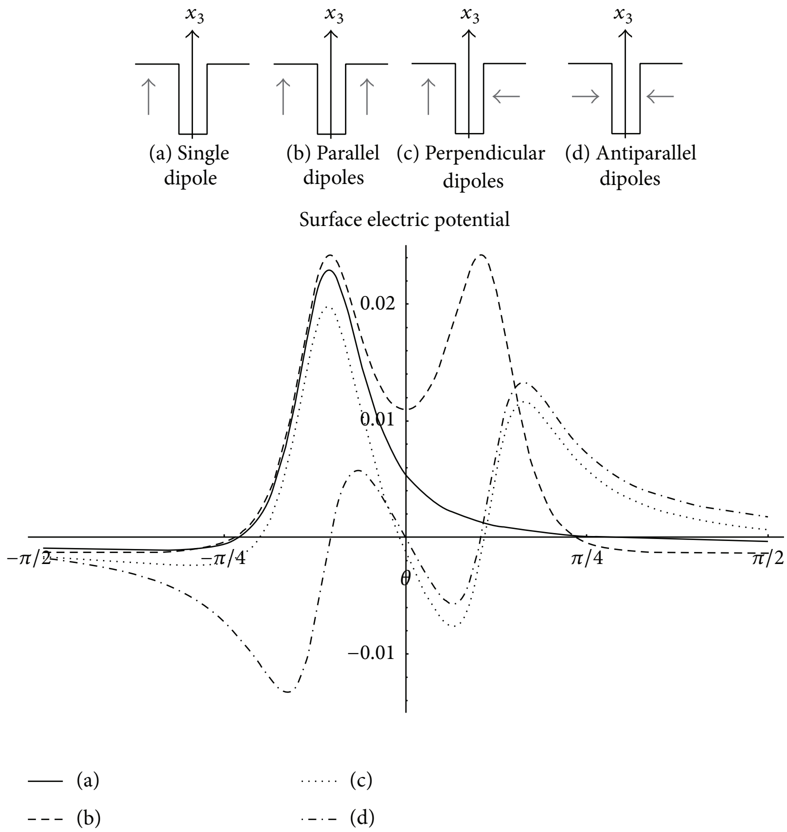

Figure 1: Electroencephalographic recordings as a result of two simultaneously active dipoles in three distinct arrangements: (b) parallel, (c) perpendicular, and (d) antiparallel. The corresponding neuronal excitation occurs on the cortical surface in neighboring regions. The presented electric potential is plotted for a fixed azimuthal angle equal to $3\pi/8$.

which means that all the 2×2 determinants in expressions (53)–(56) vanish, and the common solution is given by

$$\mathbf{r}_0 = \widetilde{\mathbf{r}}_0 = \frac{\mathbf{r}_1 + \gamma \mathbf{r}_2}{1 + \gamma} \tag{59}$$

which is a point on the line segment that connects $\mathbf{r}_1$ and $\mathbf{r}_2$. Indeed,

$$(\mathbf{r}_0 - \mathbf{r}_1) \times (\mathbf{r}_0 - \mathbf{r}_2) = \mathbf{0}, \tag{60}$$

which implies that the vectors $(\mathbf{r}_0 - \mathbf{r}_1)$ and $(\mathbf{r}_0 - \mathbf{r}_2)$ lie on the same line.

From the physical point of view, the reason why parallel dipoles are indistinguishable is due the fact that the potential lines they generate are also similarly located and therefore in the exterior space the field has the pattern of a single current dipole. In any other case, this simple pattern of the field is destroyed and it is easy to identify a more complicated source. In order to see the discrepancy between the solution $\mathbf{r}_0$ and the solution $\widetilde{\mathbf{r}}_0$ we look at the difference

$$\begin{aligned} x_{01} - \widetilde{x}_{01} = {} & \left[\frac{q_{12}q_{03} + q_{13}q_{02}}{2q_{02}q_{03}} - \frac{q_{11}q_{01} + q_{12}q_{02}}{q_{01}^2 + q_{02}^2} \right] \\ & \times (x_{11} - x_{21}) \\ & + \left[\frac{q_{23}q_{11} - q_{13}q_{21}}{2q_{02}q_{03}} - \frac{q_{12}q_{21} - q_{22}q_{11}}{q_{01}^2 + q_{02}^2} \right] \\ & \times (x_{12} - x_{22}) \\ & + \frac{q_{11}q_{22} - q_{21}q_{12}}{2q_{02}q_{03}} (x_{13} - x_{23}) . \end{aligned} \tag{61}$$

Since the moments $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are independent of the positions $\mathbf{r}_1$ and $\mathbf{r}_2$ of the dipoles, it follows that $x_{01} = \widetilde{x}_{01}$ if and only if the coefficients of $(x_{11} - x_{21})$, $(x_{12} - x_{22})$ and $(x_{13} - x_{23})$ are equal to zero. After straightforward calculations we obtain $\mathbf{Q}_1 \times \mathbf{Q}_2 = \mathbf{0}$ if $q_{02} \neq 0$ and the two dipoles are parallel. The case where $q_{02} = 0$ corresponds to $\mathbf{Q}_1 = -\mathbf{Q}_2$ and therefore to $\mathbf{Q}_0 = \mathbf{0}$, namely, a significant cancellation effect takes place.

The impact of individual dipole configurations (parallel or perpendicular) on the measured scalp potential is demonstrated in Figures 1 and 2. Two dipoles of equal strength are situated in gyri in adjacent cortical regions (configurations Figures 1(b)–1(d)) and are compared with the corresponding single dipole excitation (configuration Figure 1(a)). We observe that, in the case of synchronous activation of two dipoles (Figure 1(b)) in close proximity, compared to a single

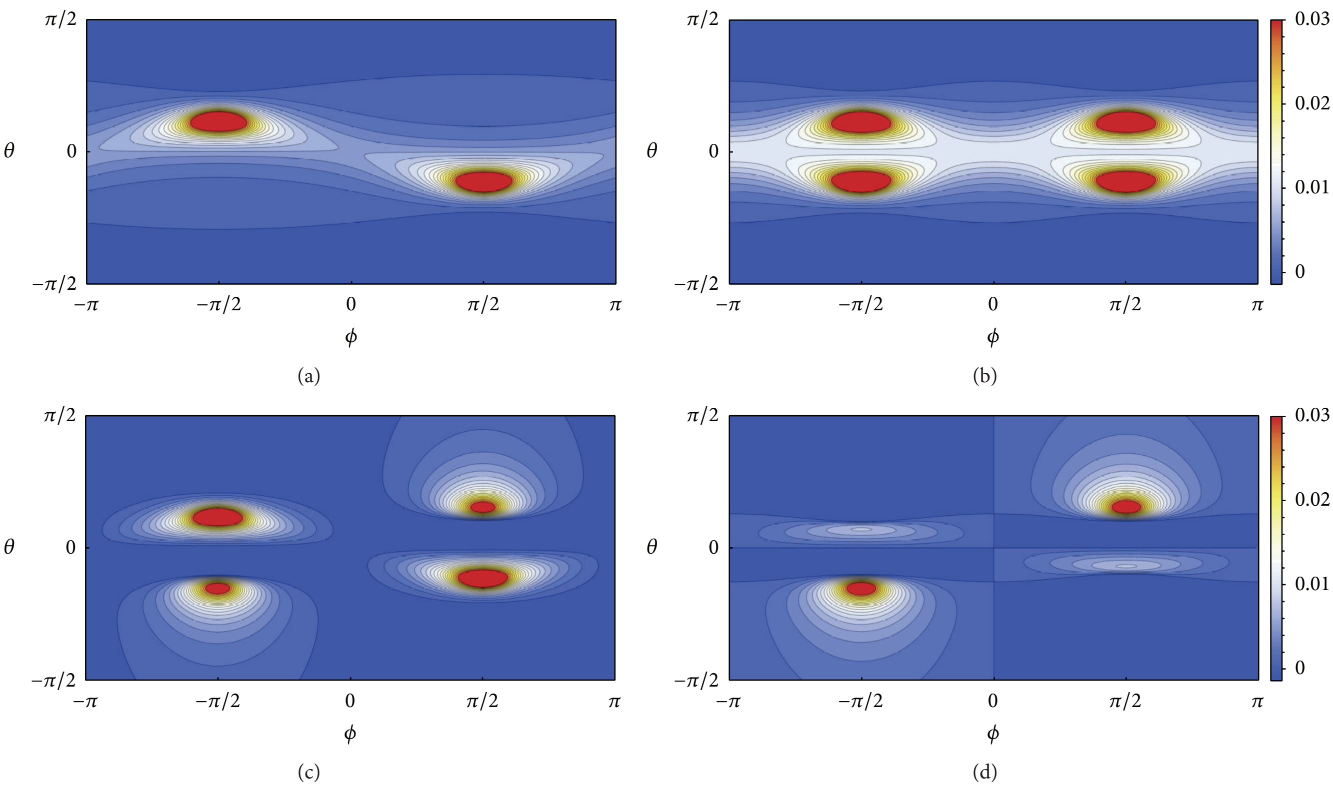

FIGURE 2: The surface electric potential as a result of the activation of (a) a single dipole, (b) two parallel dipoles, (c) two perpendicular dipoles, and (d) two antiparallel dipoles in close proximity.

excitation (Figure 1(a)), an amplification of the recorded scalp potential is detected. Locally, the increase of magnitude can be substantial (Figure 2(b)). On the contrary, instances (Figures 1(c) and 1(d)) result in a diminishing of magnitude (Figures 2(c) and 2(d)) as expected.

4. Conclusions

An analytic criterion deciding the existence of a single dipole or multiple dipoles is presented. The analytic algorithm, on which the criterion resides, is derived by expressing the generated electric surface potential in Cartesian coordinates. Then, by equating the coefficients of the Cartesian monomials with the corresponding known coefficients from the Cartesian expansion of the recorded potential, we obtain a set of algebraic equations which establish the basis of our analysis. This system, obviously overdetermined, exhibits a unique solution only if certain constraints, provided via (17) and (18), are met. In the case of a single dipole relations (17) and (18) are trivially satisfied. On the other hand, if two or more dipoles are simultaneously active relations (17) and (18), which connect the electroencephalographic recordings, are fulfilled only if the dipoles are parallel. In this case it is impossible to decide if the EEG measurements are evoked by a single dipole or by a finite number of dipoles. In any other instance, the precise number of active sources can be decided. Summarizing, the necessary and sufficient conditions which decide whether the collected EEG data originates from a single dipole or from numerous dipoles are presented. In the event where the data arouses from multiple parallel dipoles, an isolation of the source is, in general, not possible.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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