Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2014, Article ID 231892, 9 pages
http://dx.doi.org/10.1155/2014/231892

Research Article

Existence of Weak Solutions for Nonlinear Time-Fractional p-Laplace Problems

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Received 9 March 2014; Revised 5 June 2014; Accepted 10 June 2014; Published 2 July 2014

Academic Editor: Yansheng Liu

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The existence of weak solution for p-Laplace problem is studied in the paper. By exploiting the relationship between the Nehari manifold and fibering maps and combining the compact imbedding theorem and the behavior of Palais-Smale sequences in the Nehari manifold, the existence of weak solutions is established. By means of the Arzela-Ascoli fixed point theorem, some existence results of the corresponding time-fractional equations of the p-Laplace problem are obtained.

1. Introduction

Fractional calculus is a generalization of ordinary differentiation and integration on an arbitrary order that can be noninteger. The increasing interest of fractional equations is motivated by their applications in various fields of science such as physics, fluid mechanics, heat conduction with memory, chemistry, and engineering [1]. In consequence, the subject of fractional differential equations is gaining diverse and continuous attention. For example, for fractional initial value problems, the existence and multiplicity of solutions (or positive solutions) were discussed in [2–4].

In this paper, we consider the following semilinear boundary value problem:

$$\begin{aligned} &-\operatorname{div}\left(a(x)\,|\nabla u(x)|^{p-2}\nabla u(x)\right) \\ &\quad = \lambda |u(x)|^{p-2}u(x) + b(x)\,|u(x)|^{\alpha-1}u(x), \quad \text{in } \Omega, \\ &u(x) = 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where Ω is a bounded region with smooth boundary in $\mathbf{R}^N$ and $a(x)$ is a positive weight function with positive measure of the Sobolev space $W_0^{1,p}(a(x),\Omega)$, $u(x) \in W_0^{1,p}(a(x),\Omega)$, $b(x):\Omega \to R$ is a smooth function which may change sign, and λ is a real positive parameter and assume throughout that α is a fixed number such that $1<\alpha<p-1$ $(2<p\le p^*$ $(p^*=np/(n-p)))$. Thus, we will study a sublinear perturbation of a linear problem.

The problem (1) is an important and basic mathematical model, widely used in many fields. As for the specific theoretical implicity of the above model, one can see Drábek et al. [5], Adams and Fournier [6], and so on.

Similar problems have been studied by Brown and Zhang [7, 8] (when $a(x)\equiv 1$, $p\equiv 2$, with $\alpha>2$) and by Brown [9] (when $a(x)\equiv 1$, $p\equiv 2$, but with $1<\alpha<2$) by using variational viewpoint of the Nehari manifold. When $a(x)\equiv 1$, $p=2$, Amann and Lopez-Gomez [10] have studied the existence of the equation by using global bifurcation theory and Binding et al. [11, 12] used variational methods. Huang and Pu in [13] have studied the following problem:

$$\begin{aligned} &-\operatorname{div}\left(a(x)\,|\nabla u(x)|^{p-2}\nabla u(x)\right) \\ &\quad = \lambda b(x)\,|u(x)|^{p-2}u(x) + c(x)\,|u(x)|^{\alpha-1}, \quad x\in\Omega, \\ &u(x) = 0, \quad x\in\partial\Omega, \end{aligned} \tag{2}$$

where Ω is a bounded region with smooth boundary in $\mathbf{R}^N$, λ is a real positive parameter, $b(x)$ is a nonnegative function and satisfies $b(x) \in L^{q/(q-p)}$ $(p < q)$ or $b(x) \in L^{\infty}(\Omega)$, and $c(x)$ is a smooth function which may change sign in Ω, and the existence of multiple positive solutions and the properties on Nehari manifold for (2) have been established under the assumption that $p - 1 < \alpha < p_s^* - 1$, where $p_s = ps/(s+1)$, $p_s^* = Np_s/(N - p_s)$, $s \in (N/p, \infty) \cap [1/(p-1), \infty)$, $p_s < N(s+1)$.

In the above, we investigated the p-Laplace Dirichlet problem. Next, we will switch our viewpoint to consider the existence of weak solutions for the corresponding nonlinear time-fractional differential equation of the problem (1). Consider

$$\begin{aligned} & D^{\beta}u(x,t) \\ & \quad = \operatorname{div}\left(a(x)|\nabla u(x,t)|^{p-2}\nabla u(x,t)\right) \\ & \qquad + \lambda |u(x,t)|^{p-2}u(x,t) \\ & \qquad + b(x)|u(x,t)|^{\alpha-1}u(x,t), \quad \text{in } \Omega_T, \\ & \qquad u(x,t) = 0, \quad \text{on } \partial\Omega_T, \\ & \qquad u(x,0) = \phi(x), \quad \text{in } \Omega, \\ & \qquad u_t(x,0) = \psi(x), \quad \text{in } \Omega, \end{aligned} \tag{$E_{\lambda,b,t}$}$$

where $\Omega_T = \Omega \times [0, T]$, D^{β} denotes the Caputo fractional derivatives [14], $0 < \beta \le 1$ is a parameter describing the order of the fractional time, and $\phi(x), \psi(x) \in H_0^1(\Omega)$ are given real-valued functions.

Recently, the subject of fractional differential equations has emerged as an important area of investigation. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control, electromagnetic, porous media, engineering, and so forth. For some recent developments on the subject, one can see [14–21]. As far as we know, no contributions exist concerning the existence of weak solutions for the problems as we stated above.

2. Notations and Preliminary Results

Let $W_0^{1,p}(a(x),\Omega)$ be a weighted Sobolev space with a positive measurable weight function $a(x)$; its norm is defined as $\|u\| = (\int_{\Omega}(a(x)|\nabla u(x)|^p)dx)^{1/p}$, and the function $a(x)$ satisfies $\nu(x)/c \le a(x) \le c\nu(x)$, $x \in \Omega$, where $c \ge 1$, $\nu(x)$ is also a weight function (see [5, 6, 13]) and satisfies $\nu(x) \in L^1_{\text{loc}}(\Omega)$, $\nu(x)^{-1/(p-1)} \in L^1_{\text{loc}}(\Omega)$, $\nu(x)^{-s} \in L^1(\Omega)$; here, $s \in (N/p, \infty) \cap [1/(p-1), \infty)$. Throughout this paper, we denote by S_p the best Sobolev constant for the imbedding of $W_0^{1,p}(a(x),\Omega)$ in $L^p(\Omega)$. In particular, $\|u\|_{L^p(\Omega)} \le S_p\|u\|$, for all $u \in W_0^{1,p}(a(x),\Omega)$. For simplicity, we will denote $W_0^{1,p}(a(x),\Omega)$ by X and denote $\|\cdot\|_X$ by $\|\cdot\|$, and unless otherwise stated, integrals are over Ω.

Let λ_1 denote the positive principal eigenvalue of the problem:

$$\begin{aligned} & -\operatorname{div}\left(a(x)|\nabla u(x)|^{p-2}\nabla u(x)\right) \\ & \quad = \lambda |u(x)|^{p-2}u(x), \quad \text{for } x \in \Omega, \\ & u(x) = 0, \quad \text{for } x \in \partial\Omega, \end{aligned} \tag{3}$$

with corresponding positive principal eigenfunction ϕ_1. The Euler functional associated with (1) is

$$\begin{aligned} J_{\lambda}(u) = {} & \frac{1}{p}\int a|\nabla u|^p dx - \frac{\lambda}{p}\int |u|^p dx \\ & - \frac{1}{\alpha+1}\int b(x)|u|^{\alpha+1}dx, \quad \text{for } u \in X. \end{aligned} \tag{4}$$

The next lemma shows the behavior of functional J_{λ} on X.

Lemma 1. *(i) Suppose that $\lambda < \lambda_1$; then J_{λ} is bounded below on X.*

(ii) If $\lambda > \lambda_1$, then J_{λ} is no longer bounded below on X.

Proof. (i) By the spectral theorem, we have

$$\int a|\nabla u|^p dx - \lambda \int |u|^p dx \ge (\lambda_1 - \lambda)\int |u|^p dx, \quad \forall u \in X, \tag{5}$$

and so

$$\begin{aligned} J_{\lambda}(u) & \ge \frac{1}{p}(\lambda_1 - \lambda)\int |u|^p dx - \frac{\bar{b}}{\alpha+1}\int |u|^{\alpha+1}dx \\ & \ge \frac{1}{p}(\lambda_1 - \lambda)\int |u|^p dx \\ & \quad - \frac{\bar{b}}{\alpha+1}|\Omega|^{1-(\alpha+1)/p}\left(\int |u|^p dx\right)^{(\alpha+1)/p}, \end{aligned} \tag{6}$$

where $\bar{b} = \sup_{x\in\Omega} b(x)$. Hence, J_{λ} is bounded below on X when $\lambda < \lambda_1$.

(ii) If $\lambda > \lambda_1$, then $\lim_{t\to\infty} J_{\lambda}(t\phi_1) = -\infty$, so J_{λ} is unbounded below on X. □

In order to obtain existence results in the case of $\lambda < \lambda_1$, we introduce the Nehari manifold:

$$N_{\lambda} = \left\{u \in X : \left\langle J'_{\lambda}(u), u\right\rangle = 0\right\}, \tag{7}$$

where $\langle \cdot, \cdot \rangle$ denotes the usual duality. Thus, $u \in N_{\lambda}$ if and only if

$$\int a|\nabla u|^p dx - \lambda \int |u|^p dx - \int b(x)|u|^{\alpha+1}dx = 0. \tag{8}$$

Obviously, N_{λ} is a much smaller set than X and so it is easier to study J_{λ} on N_{λ}.

On N_λ, we have

$$J_\lambda(u) = \left(\frac{1}{p} - \frac{1}{\alpha+1}\right)\int\left(a|\nabla u|^p - \lambda|u|^p\right)dx = \left(\frac{1}{p} - \frac{1}{\alpha+1}\right)\int\left(b|u|^{\alpha+1}\right)dx. \quad (9)$$

The Nehari manifold is closely linked to the behavior of functions of the form $\phi_u(t) : t \to J_\lambda(tu)\,(t > 0)$. Such maps are known as fibering maps and were introduced by Drabek and Pohozaev [22] and are also discussed in Brown and Zhang [7]. If $u \in X$, we have

$$\phi_u(t) = \frac{t^p}{p}\int a|\nabla u|^p - \lambda|u|^p dx - \frac{t^{\alpha+1}}{\alpha+1}\int b|u|^{\alpha+1}dx, \quad (10)$$

$$\phi_u'(t) = t^{p-1}\int a|\nabla u|^p - \lambda|u|^p dx - t^\alpha\int b|u|^{\alpha+1}dx, \quad (11)$$

$$\phi_u''(t) = (p-1)t^{p-2}\int a|\nabla u|^p - \lambda|u|^p dx - \alpha t^{\alpha-1}\int b|u|^{\alpha+1}dx. \quad (12)$$

We can see that $u \in N_\lambda$ if and only if $\phi_u'(1) = 0$ and more generally that $\phi_u'(t) = 0$ if and only if $tu \in N_\lambda$; that is, elements in N_λ correspond to stationary points of fibering maps. It follows from (11) and (12) that if $\phi_u'(t) = 0$, then $\phi_u''(t) = t^{\alpha-1}[(p-1)-\alpha]\int b|u|^{\alpha+1}dx$. Thus, it is natural to subdivide N_λ into three parts corresponding to local minima, local maxima, and points of inflection. Consider

$$N_\lambda^+ = \left\{u \in N_\lambda : \int b|u|^{\alpha+1}dx > 0\right\},$$
$$N_\lambda^- = \left\{u \in N_\lambda : \int b|u|^{\alpha+1}dx < 0\right\}, \quad (13)$$
$$N_\lambda^0 = \left\{u \in N_\lambda : \int b|u|^{\alpha+1}dx = 0\right\}.$$

So N_λ^+, N_λ^-, and N_λ^0 correspond to minima, maxima, and points of inflection, respectively.

Let $u \in X$, then

(i) if $\int a|\nabla u|^p - \lambda|u|^p dx$ and $\int b|u|^{\alpha+1}dx$ have the same sign, then $\phi_u(t)$ has a unique turning point at

$$t(u) = \left[\frac{\int b|u|^{\alpha+1}dx}{\int\left(a|\nabla u|^p - \lambda|u|^p\right)dx}\right]^{1/((p-1)-\alpha)}. \quad (14)$$

This turning point is a local minimum (maximum) so that $t(u)u \in N_\lambda^+(N_\lambda^-)$ if and only if $\int b|u|^{\alpha+1}dx > 0$ (< 0).

(ii) If $\int a|\nabla u|^p - \lambda|u|^p dx$ and $\int b|u|^{\alpha+1}dx$ have different signs, then $\phi_u(t)$ has no turning points and so no multiples of u lie in N_λ.

Thus, if we define

$$L_+(\lambda) = \left\{u \in X : \|u\| = 1, \int a|\nabla u|^p - \lambda|u|^p dx > 0\right\},$$
$$B_+(\lambda) = \left\{u \in X : \|u\| = 1, \int b|u|^{\alpha+1}dx > 0\right\}. \quad (15)$$

Analogously, we can define $L_-(\lambda)$, $L_0(\lambda)$, $B_-(\lambda)$, and $B_0(\lambda)$ by replacing "> 0" by "< 0" or "= 0." As appropriate, we have the following:

(i) if $u \in L_+(\lambda) \cap B_+(\lambda)$, then $t \to \phi_u(t)$ has a local minimum at $t = t(u)$ and $t(u)u \in N_\lambda^+$,

(ii) if $u \in L_-(\lambda) \cap B_-(\lambda)$, then $t \to \phi_u(t)$ has a local maximum at $t = t(u)$ and $t(u)u \in N_\lambda^-$,

(iii) if $u \in L_+(\lambda) \cap B_-(\lambda)$, then $t \to \phi_u(t)$ is strictly increasing and no multiple of u lies in N_λ,

(iv) if $u \in L_-(\lambda) \cap B_+(\lambda)$, then $t \to \phi_u(t)$ is strictly decreasing and no multiple of u lies in N_λ.

Next, we will prove the existence of solutions of (1) by investigating the existence of minimizers on N_λ. The following lemma proved that minimizers on N_λ are "usually" critical points for J_λ.

Lemma 2. *Suppose that $u_0 \in N_\lambda$ is a local maximum or minimum point for J_λ on N_λ, $u_0 \notin N_\lambda^0$; then $J_\lambda'(u_0) = 0$ in $X^{-1}(\Omega)$.*

Proof. If u_0 is a local minimizer point for J_λ on N_λ, then u_0 is a solution of the optimization problem:

$$\text{Minimize } J_\lambda(u) \text{ subject to } r(u) = \left\langle J_\lambda'(u), u\right\rangle = 0. \quad (16)$$

Hence, by the theory of Lagrange multipliers, there exists $\mu \in \mathbf{R}$ such that

$$J_\lambda'(u_0) = \mu r_\lambda'(u_0), \quad \text{in } X^{-1}(\Omega). \quad (17)$$

Thus,

$$\left\langle J_\lambda'(u_0), u_0\right\rangle_X = \mu\left\langle r_\lambda'(u_0), u_0\right\rangle_X. \quad (18)$$

Since $u_0 \in N_\lambda$, $\langle J_\lambda'(u_0), u_0\rangle = 0$ and so

$$\int\left(a|\nabla u_0|^p - \lambda|u_0|^p\right)dx = \int b|u_0|^{\alpha+1}dx. \quad (19)$$

Hence,

$$\left\langle r_\lambda'(u_0), u_0\right\rangle_X = p\int\left(a|\nabla u_0|^p - \lambda|u_0|^p\right)dx - (\alpha+1)\int b|u_0|^{\alpha+1}dx = [(p-1)-\alpha]\int\left(a|\nabla u_0|^p - \lambda|u_0|^p\right)dx. \quad (20)$$

Thus, if $u_0 \notin N_\lambda^0$, $\langle r_\lambda'(u_0), u_0\rangle_X \neq 0$ and so by (18) $\mu = 0$. This completes the proof. □

3. Properties of the Nehari Manifold

In this section, we will discuss the vital role played by the condition $L_-(\lambda) \subseteq B_-(\lambda)$ in determining the nature of the Nehari manifold. When $\lambda < \lambda_1$, $\int(a|\nabla u|^p - \lambda|u|^p)dx > 0$, for all $u \in X$, and so $L_+(\lambda) = \{u \in X : \|u\| = 1\}$ and $L_-(\lambda), L_0(\lambda) = \emptyset$. When $\lambda = \lambda_1$, we have $L_-(\lambda) = \emptyset$ and $L_0(\lambda) = \{\phi_1\}$ and when λ is greater than λ_1, $L_-(\lambda)$ becomes nonempty and gets bigger as λ increases.

Theorem 3. *Suppose there exists $\tilde{\lambda}$ such that, for all $\lambda < \tilde{\lambda}$, $L_-(\lambda) \subseteq B_-(\lambda)$. Then, for $\forall\lambda < \tilde{\lambda}$,*

(i) *$L_0(\lambda) \subseteq B_-(\lambda)$ and so $L_0(\lambda) \cap B_0(\lambda) = \emptyset$,*

(ii) *N_λ^+ is bounded,*

(iii) *$0 \notin \overline{N_\lambda^-}$ and N_λ^- is closed,*

(iv) *$\overline{N_\lambda^+} \cap N_\lambda^- = \emptyset$.*

Proof. (i) Suppose that the result is false. Then there exists $u \in L_0(\lambda)$ such that $u \notin B_-(\lambda)$; if $\lambda < \mu < \tilde{\lambda}$, then $u \in L_-(\mu)$, and so $L_-(\mu) \nsubseteq B_-(\lambda)$ which is a contradiction.

(ii) Suppose that N_λ^+ is unbounded. Then there exists $\{u_n\} \subseteq N_\lambda^+$ such that $\|u_n\| \to \infty$ as $n \to \infty$. Let $v_n = u_n/\|u_n\|$; without loss of generality, we may assume that $v_n \rightharpoonup v_0$ in X and so $v_n \to v_0$ in $L^p(\Omega)$ and in $L^{\alpha+1}(\Omega)$. Since $u_n \in N_\lambda^+$, $\int(b|v_n|^{\alpha+1})dx > 0$ and so $\int(b|v_0|^{\alpha+1})dx \geq 0$. Since $u_n \in N_\lambda^+ \subseteq N_\lambda$, so, by (8), we have

$$\int\left(a|\nabla u_n|^p - \lambda|u_n|^p\right)dx = \int b|u_n|^{\alpha+1}dx, \tag{21}$$

and divide by $\|u_n\|^p$, so

$$\int\left(a|\nabla v_n|^p - \lambda|v_n|^p\right)dx = \int\left(b|v_n|^{\alpha+1}\|u_n\|^{(\alpha+1)-p}\right)dx \longrightarrow 0. \tag{22}$$

Suppose $v_n \nrightarrow v_0$ in X. Then $\int(a|\nabla v_0|^p)dx < \underline{\lim}_{n\to\infty}\int(a|\nabla v_n|^p)dx$, and so $\int(a|\nabla v_0|^p - \lambda|v_0|^p)dx < \lim_{n\to\infty}\int(a|\nabla v_n|^p - \lambda|v_n|^p)dx = 0$. Thus, $v_0/\|v_0\| \in L_-(\lambda) \subseteq B_-(\lambda)$ which is impossible as $\int(b|v_0|^{\alpha+1})dx \geq 0$. Hence, $v_n \to v_0$ in X. Thus, $\|v_0\| = 1$ and

$$\begin{aligned}&\int\left(a|\nabla v_0|^p - \lambda|v_0|^p\right)dx\\&\quad= \lim_{n\to\infty}\int\left(a|\nabla v_n|^p - \lambda|v_n|^p\right)dx = 0.\end{aligned} \tag{23}$$

Thus, $v_0 \in L_0(\lambda) \subseteq B_-(\lambda)$ which is impossible as $\int(b|v_0|^{\alpha+1})dx \geq 0$. Therefore, N_λ^+ is bounded.

(iii) Suppose $0 \in \overline{N_\lambda^-}$. Then there exists $\{u_n\} \subseteq N_\lambda^-$ such that $\lim_{n\to\infty}u_n = 0$; let $v_n = u_n/\|u_n\|$ and then we may assume that $v_n \rightharpoonup v_0$ in X and so $v_n \to v_0$ in $L^p(\Omega)$ and in $L^{\alpha+1}(\Omega)$. Since $u_n \in N_\lambda^-$, we have

$$\int\left(a|\nabla v_n|^p - \lambda|v_n|^p\right)dx = \frac{1}{\|u_n\|^{(p-1)-\alpha}}\int\left(b|v_n|^{\alpha+1}\right)dx \leq 0; \tag{24}$$

since the $\{v_n\}$ is bounded, it follows that $\lim_{n\to\infty}\int(b|v_n|^{\alpha+1})dx = 0$ and so $\int(b|v_0|^{\alpha+1})dx = 0$. Suppose $v_n \to v_0$ in X; then $\|v_0\| = 1$ and so $v_0 \in B_0(\lambda)$. Moreover

$$\begin{aligned}&\int\left(a|\nabla v_0|^p - \lambda|v_0|^p\right)dx\\&\quad= \lim_{n\to\infty}\int\left(a|\nabla v_n|^p - \lambda|v_n|^p\right)dx \leq 0,\end{aligned} \tag{25}$$

and so $v_0 \in L_0(\lambda)$ or $L_-(\lambda)$. Hence, $v_0 \in B_-(\lambda)$ and this is impossible as $\int(b|v_0|^{\alpha+1})dx = 0$. Thus, we must have that $v_n \nrightarrow v_0$ in X; then

$$\begin{aligned}&\int\left(a|\nabla v_0|^p - \lambda|v_0|^p\right)dx\\&\quad< \lim_{n\to\infty}\int\left(a|\nabla v_n|^p - \lambda|v_n|^p\right)dx \leq 0.\end{aligned} \tag{26}$$

Hence $v_0/\|v_0\| \in L_-(\lambda) \cap B_0(\lambda)$ which is impossible and so $0 \notin \overline{N_\lambda^-}$. We now prove that N_λ^- is closed. Suppose $\{u_n\} \subseteq N_\lambda^-$ and $u_n \to u$ in X. Then $u \in \overline{N_\lambda^-}$ and so $u \not\equiv 0$. Moreover,

$$\int\left(a|\nabla u|^p - \lambda|u|^p\right)dx = \int b|u|^{\alpha+1}dx \leq 0. \tag{27}$$

If both integrals equal 0, then $u/\|u\| \in L_0(\lambda) \cap B_0(\lambda)$ which contradicts (i). Hence both integrals must be negative and so $u \in N_\lambda^-$. Thus N_λ^- is closed.

(iv) Let $u \in \overline{N_\lambda^+} \cap N_\lambda^-$, as $u \in N_\lambda^-$, $u \not\equiv 0$. Moreover, it is clear that

$$\int\left(a|\nabla u|^p - \lambda|u|^p\right)dx = \int b|u|^{\alpha+1}dx = 0, \tag{28}$$

and so $u/\|u\| \in L_0(\lambda) \cap B_0(\lambda)$ which is impossible. Thus, $\overline{N_\lambda^+} \cap N_\lambda^- = \emptyset$. □

The following theorem presents $J_\lambda(u) > 0$ on N_λ^- and the behavior of $J_\lambda(u)$ on N_λ^+.

Theorem 4. *Suppose the same hypotheses are satisfied as in Theorem 3; then*

(i) *J_λ is bounded below on N_λ^+,*

(ii) *$\inf_{u\in N_\lambda^-} J_\lambda(u) > 0$ provided N_λ^- is nonempty.*

Proof. The proof of (i) is an immediate consequence of the boundedness of N_λ^+.

(ii) Suppose $\inf_{u\in N_\lambda^-} J_\lambda(u) = 0$. Then there exists $\{u_n\} \subseteq N_\lambda^-$ such that $\lim_{n\to\infty} J_\lambda(u_n) = 0$. And it is clear from (9) that $\int(a|\nabla u_n|^p - \lambda|u_n|^p)dx \to 0$ and $\int b|u_n|^{\alpha+1}dx \to 0$ as $n \to \infty$. Let $v_n = u_n/\|u_n\|$; since $0 \notin \overline{N_\lambda^-}$, $\{\|u_n\|\}$ is bounded away from 0. Hence $\lim_{n\to\infty}\int(a|\nabla v_n|^p - \lambda|v_n|^p)dx = 0$, and $\lim_{n\to\infty}\int b|v_n|^{\alpha+1}dx = 0$; we may assume that $v_n \rightharpoonup v_0$ in X and $v_n \to v_0$ in $L^p(\Omega)$ and in $L^{\alpha+1}(\Omega)$. Then $\int b|v_0|^{\alpha+1}dx = 0$. If $v_n \to v_0$ in X, we have $\|v_0\| = 1$ and $\int(a|\nabla v_0|^p - \lambda|v_0|^p)dx = 0$; that is, $v_0 \in L_0(\lambda)$, whereas if $v_n \nrightarrow v_0$ in X, $\int(a|\nabla v_0|^p - \lambda|v_0|^p)dx < 0$; that is, $v_0/\|v_0\| \in L_-(\lambda)$. In both cases, however, we must also have $v_0/\|v_0\| \in B_0(\lambda)$ and this is a contradiction. Thus, $\inf_{u\in N_\lambda^-} J_\lambda(u) > 0$. □

4. The Existence of Weak Solution

We now show that there exists a minimizer on $N_\lambda^+(N_\lambda^-)$ which is a critical point of $J_\lambda(u)$ and so a nontrivial positive solution of (1).

Theorem 5. *Suppose $L_-(\lambda) \subseteq B_-(\lambda)$ for all $\lambda < \tilde{\lambda}$; then, for $\forall \lambda < \tilde{\lambda}$,*

(i) there exists a minimizer point for J_λ on N_λ^+,

(ii) there exists a minimizer point for J_λ on N_λ^- provided that $L_-(\lambda)$ is nonempty.

Proof. (i) By Theorem 4, J_λ is bounded below on N_λ^+. Let $\{u_n\} \subseteq N_\lambda^+$ be a minimizing sequence; that is,

$$\lim_{n\to\infty} J_\lambda(u_n) = \inf_{u\in N_\lambda^+} J_\lambda(u) < 0. \tag{29}$$

Since N_λ^+ is bounded, we may assume that $u_n \rightharpoonup u_0$ in X and $u_n \to u_0$ in $L^P(\Omega)$ and in $L^{\alpha+1}(\Omega)$. Since $J_\lambda(u_n) = (1/p - 1/(\alpha+1))\int b|u_n|^{\alpha+1}dx$, it follows that

$$\int b|u_0|^{\alpha+1}dx = \lim_{n\to\infty}\int b|u_n|^{\alpha+1}dx > 0, \tag{30}$$

and so $u_0/\|u_0\| \in B_+(\lambda)$. Hence, by Theorem 3, $u_0/\|u_0\| \in L_+(\lambda)$ and so the fibering map ϕ_{u_0} has a unique minimum at $t(u_0)$, such that $t(u_0)u_0 \in N_\lambda^+$. Suppose $u_n \nrightarrow u_0$ in X; then

$$\begin{aligned}\int&\left(a|\nabla u_0|^p - \lambda|u_0|^p\right)dx\\ &< \lim_{n\to\infty}\int\left(a|\nabla u_n|^p - \lambda|u_n|^p\right)dx\\ &= \lim_{n\to\infty}\int b|u_n|^{\alpha+1}dx = \int b|u_0|^{\alpha+1}dx,\end{aligned} \tag{31}$$

and so $t(u_0) > 1$. Hence

$$J_\lambda(t(u_0)u_0) < J_\lambda(u_0) < \lim_{n\to\infty} J_\lambda(u_n) = \inf_{u\in N_\lambda^+} J_\lambda(u), \tag{32}$$

which is impossible.

Hence $u_n \to u_0$ in X and so $u_0 \in N_\lambda$. It now follows easily that u_0 is a minimizer point for J_λ on N_λ^+, since $J_\lambda(u) = J_\lambda(|u|)$ and we may assume that u_0 is a nonnegative in Ω, since $J_\lambda(u_0) < 0$, u_0 is a local minimum point for J_λ on N_λ. It follows from Lemma 2 that u_0 is a critical point of J_λ and so is a weak solution of (1).

(ii) Let $\{u_n\} \subseteq N_\lambda^-$ be a minimizing sequence. Then by Theorem 4, we must have $\lim_{n\to\infty} J_\lambda(u_n) = \inf_{u\in N_\lambda^-} J_\lambda(u) > 0$.

Suppose that $\{u_n\}$ is unbounded; we may suppose that $\|u_n\| \to \infty$ as $n \to \infty$. Let $v_n = u_n/\|u_n\|$; since $\{J_\lambda(u_n)\}$ is bounded, it follows that $\{\int(a|\nabla u_n|^p - \lambda|u_n|^p)dx\}$ and $\{\int(b|u_n|^{\alpha+1})dx\}$ are bounded and so

$$\lim_{n\to\infty}\int\left(a|\nabla v_n|^p - \lambda|v_n|^p\right)dx = \lim_{n\to\infty}\int\left(b|v_n|^{\alpha+1}\right)dx = 0; \tag{33}$$

since $\{v_n\}$ is bounded, we may assume that $v_n \rightharpoonup v_0$ in X and $v_n \to v_0$ in $L^p(\Omega)$ and in $L^{\alpha+1}(\Omega)$, so that $\int(b|v_0|^{\alpha+1})dx = 0$. If $v_n \to v_0$ in X, it is easy to see that $v_0 \in L_0(\lambda)\cap B_0(\lambda)$ which is impossible because of Theorem 3(i). Hence, $v_n \nrightarrow v_0$ in X and so

$$\begin{aligned}\int&\left(a|\nabla v_0|^p - \lambda|v_0|^p\right)dx\\ &< \lim_{n\to\infty}\int\left(a|\nabla v_n|^p - \lambda|v_n|^p\right)dx = 0;\end{aligned} \tag{34}$$

hence, $v_0 \neq 0$ and $v_0/\|v_0\| \in L_-(\lambda) \cap B_0(\lambda)$, which is again impossible. Thus, $\{u_n\}$ is bounded and so we may assume that $u_n \rightharpoonup u_0$ in X and $u_n \to u_0$ in $L^p(\Omega)$ and in $L^{\alpha+1}(\Omega)$. Suppose $u_n \nrightarrow u_0$ in X. Then

$$\begin{aligned}\int\left(b|u_0|^{\alpha+1}\right)dx &= \lim_{n\to\infty}\int\left(b|u_n|^{\alpha+1}\right)dx\\ &= \left(\frac{1}{p} - \frac{1}{\alpha+1}\right)^{-1}\lim_{n\to\infty} J_\lambda(u_n) < 0,\\ \int\left(a|\nabla u_0|^p - \lambda|u_0|^p\right)dx &< \lim_{n\to\infty}\int\left(a|\nabla u_n|^p - \lambda|u_n|^p\right)dx\\ &= \lim_{n\to\infty}\int\left(b|u_n|^{\alpha+1}\right)dx\\ &= \int\left(b|u_0|^{\alpha+1}\right)dx.\end{aligned} \tag{35}$$

Hence $u_0/\|u_0\| \in L_-(\lambda) \cap B_-(\lambda)$, and so $t(u_0)u_0 \in N_\lambda^-$, where

$$t(u_0) = \left[\frac{\int\left(b|u_0|^{\alpha+1}\right)dx}{\int\left(a|\nabla u_0|^p - \lambda|u_0|^p\right)dx}\right]^{1/((p-1)-\alpha)} < 1. \tag{36}$$

Moreover, $t(u_0)u_n \rightharpoonup t(u_0)u_0$ but $t(u_0)u_n \nrightarrow t(u_0)u_0$ and so

$$J_\lambda(t(u_0)u_0) < \varliminf_{n\to\infty} J_\lambda(t(u_0)u_n); \tag{37}$$

since the map $t \to J_\lambda(tu_n)$ attains its maximum at $t = 1$,

$$\varliminf_{n\to\infty} J_\lambda(t(u_0)u_n) \leq \lim_{n\to\infty} J_\lambda(u_n) = \inf_{u\in N_\lambda^-} J_\lambda(u). \tag{38}$$

Hence, $J_\lambda(t(u_0)u_0) < \inf_{u\in N_\lambda^-} J_\lambda(u)$ which is impossible. Thus, $u_n \to u_0$ in X and it follows that u_0 is a minimizer point for $J_\lambda(u)$ on N_λ^-. Since $J_\lambda(u) = J_\lambda(|u|)$, we may assume that u_0 is a nonnegative in Ω; since N_λ^+ is closed, u_0 is a local minimum point for J_λ on N_λ. It follows from Lemma 2 that u_0 is a critical point of J_λ and so is a weak solution of (1). □

5. The Corresponding Time-Fractional Equation

In this section, we switch our viewpoint to the fractional order equation $(E_{\lambda,b,t})$ in Sobolev space $H_0^1(\Omega)$.

To discuss the existence of the positive solution for the equation $(E_{\lambda,b,t})$, we present some basic notations, definitions, and preliminary results which will be used throughout this section.

Definition 6 (see [23]). The Caputo fractional derivative of order α of a function $f(t)$, $t > 0$, is defined as

$$D^{\alpha} f(t) = \frac{1}{\Gamma(1-\{\alpha\})} \int_0^t \frac{1}{(t-s)^{\{\alpha\}}} f^{([\alpha]+1)} ds, \tag{39}$$

where $\{\alpha\}$ and $[\alpha]$ denote the fractional and the integer part of the real number α, respectively, and $\Gamma(\cdot)$ is the Gamma function.

Definition 7 (see [23]). The Riemann-Liouville fractional integral of order α of a function $f(t)$, $t > 0$, is defined as

$$I_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds, \tag{40}$$

provided that the right side is pointwise defined on $(0, \infty)$.

Lemma 8 (see [23]). *Assume $y \in C[0,T]$, $0 < T < 1$, and $0 < \alpha \le 1$; then the problem*

$$D^{\alpha} u(t) = y(t), \quad t \in [0,T], \tag{41}$$

has the unique solution

$$u(t) = u(0) + u'(0)\, t + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, ds. \tag{42}$$

Now, we establish some results of the existence of weak solution for $(E_{\lambda,b,t})$.

By Lemma 8, we may reduce $(E_{\lambda,b,t})$ to an equivalent integral equation as follows:

$$\begin{aligned} &-\phi(x) - \psi(x)\, t + u(x,t) \\ &\quad = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \\ &\qquad \times \left(\operatorname{div}\left(a(x)\, |\nabla u(x,s)|^{p-2} \nabla u(x,s)\right)\right. \\ &\qquad + \lambda |u(x,s)|^{p-2} u(x,s) \\ &\qquad \left. + b(x)\, |u(x,s)|^{\alpha-1} u(x,s)\right) ds, \\ &\qquad\qquad \text{in } \Omega_T, \\ &u(x,t) = 0, \quad \text{on } \partial\Omega_T. \end{aligned} \tag{$E_{\lambda,b,\text{integral}}$}$$

Now we define

$$\begin{aligned} &\Phi(u) \\ &\quad = \phi(x) + \psi(x)\, t \\ &\quad + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \\ &\qquad \times \left(\operatorname{div}\left(a(x)\, |\nabla u(x,s)|^{p-2} \nabla u(x,s)\right)\right. \\ &\qquad + \lambda |u(x,s)|^{p-2} u(x,s) \\ &\qquad \left. + b(x)\, |u(x,s)|^{\alpha-1} u(x,s)\right) ds, \\ &\qquad\qquad \text{in } \Omega_T, \\ &u(x,t) = 0, \quad \text{on } \partial\Omega_T. \end{aligned} \tag{$E_{\lambda,b,\text{fixed}}$}$$

Definition 9. One calls $u \in C([0,T]; H_0^1(\Omega))$, $0 < T < 1$ to be a weak solution of the fractional order equation $(E_{\lambda,b,t})$, if $\int_{\Omega} (u - \Phi(u)) v\, dx = 0$, $\forall t \in [0,T]$ for every $v \in H_0^1(\Omega)$.

Lemma 10. *Let $\|a(x)\|_{L^{\infty}(\Omega)}$ and $\|b(x)\|_{L^{\infty}(\Omega)}$ be bounded; then the operator $\Phi(u) : H_0^1(\Omega) \to H^1(\Omega)$ is completely continuous.*

Proof. Put

$$\begin{aligned} F(u) &= \operatorname{div}\left(a(x)\, |\nabla u(x,s)|^{p-2} \nabla u(x,s)\right) \\ &\quad + \lambda |u(x,s)|^{p-2} u(x,s) + b(x)\, |u(x,s)|^{\alpha-1} u(x,s). \end{aligned} \tag{43}$$

We can rewrite

$$\Phi(u) = \phi(x) + \psi(x)\, t + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} F(u)\, ds. \tag{44}$$

For each $v \in H_0^1(\Omega)$ and $\|v\|_{H_0^1(\Omega)} = 1$ integration by parts, we can get

$$\begin{aligned} &|\langle F(u), v\rangle| \\ &\quad = \left| \int \left(a(x)\, |\nabla u|^{p-1} \nabla v \right.\right. \\ &\qquad \left.\left. + \lambda |u|^{p-2} u v + b(x)\, |u|^{\alpha-1} u v\right) dx \right|. \end{aligned} \tag{45}$$

By Lemma 2, we know $r(u) = \langle J_{\lambda}'(u), u\rangle = 0$; that is,

$$\int \left(a |\nabla u|^p\right) dx = \int \lambda |u|^p dx + \int b |u|^{\alpha+1} dx. \tag{46}$$

Since $\|u\|_{L^p(\Omega)} \le S_p \|u\|$, so

$$\int \lambda |u|^p dx \le |\lambda| \int |u|^p dx = |\lambda|\, \|u\|_{L^p(\Omega)}^p \le |\lambda|\, S_p^p \|u\|^p. \tag{47}$$

And using the same proof as above, we can get $\int b|u|^{\alpha+1} dx \le \|b\|_{L^{\infty}(\Omega)} S_{\alpha+1}^{\alpha+1} \|u\|^{\alpha+1}$. Thus, we deduce

$$\|u\|^p \le |\lambda|\, S_p^p \|u\|^p + \|b\|_{L^{\infty}(\Omega)} S_{\alpha+1}^{\alpha+1} \|u\|^{\alpha+1}. \tag{48}$$

So

$$\|u\| \le \left(\frac{\|b\|_{L^{\infty}(\Omega)} S_{\alpha+1}^{\alpha+1}}{1 - |\lambda|\, S_p^p} \right)^{1/((p-1)-\alpha)}. \tag{49}$$

And since $2 < p \le p^*$ ($p^* = np/(n-p)$), and $a(x)$ is a positive sufficiently smooth function, there exists a positive constant C, such that $\|a(x)\|_{L^2(\Omega)} \ge \|a(x)\|_{L^1(\Omega)} \ge C$. Hence

$$\begin{aligned}
&\int_\Omega \left(a(x)|\nabla u|^p\right) d_x \\
&\quad \ge \int_\Omega \left(a(x)|\nabla u|^2\right) d_x = \|a(x)\|_{L^1(\Omega)} \int_\Omega |\nabla u|^2 d_x \\
&\quad \ge C \int_\Omega |\nabla u|^2 d_x \\
&\quad \ge C \int_\Omega |u|^2 d_x, \quad \forall x \in \Omega, \text{and a.e. time } 0 \le t \le T.
\end{aligned} \tag{50}$$

We used Poincare's inequality in the last inequality above. Thus, by Sobolev imbedding theorem [11], we have

$$W_0^{1,p}(a(x),\Omega) \hookrightarrow H_0^1(a(x),\Omega) \hookrightarrow H_0^1(\Omega) \hookrightarrow L^2(\Omega), \tag{51}$$

and thus,

$$\|u\|_{H_0^1(\Omega)} \le C\|u\|_{W_0^{1,p}(a(x),\Omega)} = C\|u\|. \tag{52}$$

In the following, we denote $\|u\|_{H_0^1(\Omega)}$ and $\|u\|_{H^{-1}(\Omega)}$ by $\|u\|_{H_0^1}$ and $\|u\|_{H^{-1}}$, respectively. Hence, by Cauchy-Schwarz inequalities, Poincare inequalities, Hölder inequalities, Sobolev imbedding theorem, and (49), for $1 < \alpha < p-1$, we can get

$$\begin{aligned}
&|\langle F(u), v\rangle| \\
&\quad = \left| \iint \left(a(x)|\nabla u|^{p-1}\nabla v \right.\right. \\
&\qquad \left.\left. + \lambda|u|^{p-2}uv + b(x)|u|^{\alpha-1}uv\right) dx \right| \\
&\quad \le \left(\int \left|a(x)|\nabla u|^{p-1}\right|^2 dx\right)^{1/2} \left(\int |\nabla v|^2 dx\right)^{1/2} \\
&\qquad + \left(\int \left|\lambda|u|^{p-2}u\right|^2 dx\right)^{1/2} \left(\int |v|^2 dx\right)^{1/2} \\
&\qquad + \left(\int \left|b(x)|u|^{\alpha-1}u\right|^2 dx\right)^{1/2} \left(\int |v|^2 dx\right)^{1/2} \\
&\quad \le \|a(x)\|_{L^\infty(\Omega)} \left(\int \left||\nabla u|^{p-1}\right|^2 dx\right)^{1/2} \left(\int |\nabla v|^2 dx\right)^{1/2} \\
&\qquad + |\lambda| \left(\int \left||u|^{p-2}u\right|^2 dx\right)^{1/2} \left(\int |\nabla v|^2 dx\right)^{1/2} \\
&\qquad + \|b(x)\|_{L^\infty(\Omega)} \left(\int \left||u|^{\alpha-1}u\right|^2 dx\right)^{1/2} \left(\int |\nabla v|^2 dx\right)^{1/2} \\
&\quad \le \left[\|a(x)\|_{L^\infty(\Omega)} \left(\int \left||\nabla u|^{p-1}\right|^2 dx\right)^{1/2} \right. \\
&\qquad + |\lambda| \left(\int \left||u|^{p-2}u\right|^2 dx\right)^{1/2} \\
&\qquad \left. + \|b(x)\|_{L^\infty(\Omega)} \left(\int \left||u|^{\alpha-1}u\right|^2 dx\right)^{1/2} \right] \|v\|_{H_0^1} \\
&\quad \le \|a(x)\|_{L^\infty(\Omega)} \left(\int \left||\nabla u|^{p-1}\right|^2 dx\right)^{1/2} \\
&\qquad + |\lambda| \left(\int \left||u|^{p-2}u\right|^2 dx\right)^{1/2} \\
&\qquad + \|b(x)\|_{L^\infty(\Omega)} \left(\int \left||u|^{\alpha-1}u\right|^2 dx\right)^{1/2} \\
&\quad \le \|a(x)\|_{L^\infty(\Omega)} \left(\int |\nabla u|^{2(p-1)} dx\right)^{(1/2(p-1))(p-1)} \\
&\qquad + |\lambda| \left(\int |u|^{2(p-1)} dx\right)^{(1/2(p-1))(p-1)} \\
&\qquad + \|b(x)\|_{L^\infty(\Omega)} \left(\int |u|^{2\alpha} dx\right)^{(1/2\alpha)\alpha} \\
&\quad \le \|a(x)\|_{L^\infty(\Omega)} \left(\int |\nabla u|^{2(p-1)} dx\right)^{(1/2(p-1))(p-1)} \\
&\qquad + |\lambda| \left(\int |\nabla u|^{2(p-1)} dx\right)^{(1/2(p-1))(p-1)} \\
&\qquad + \|b(x)\|_{L^\infty(\Omega)} \left(\int |\nabla u|^{2\alpha} dx\right)^{(1/2\alpha)\alpha} \\
&\quad = \|a(x)\|_{L^\infty(\Omega)} \|\nabla u\|_{L^{2(p-1)}(\Omega)}^{p-1} + |\lambda| \|\nabla u\|_{L^{2(p-1)}(\Omega)}^{p-1} \\
&\qquad + \|b(x)\|_{L^\infty(\Omega)} \|\nabla u\|_{L^{2(\alpha)}(\Omega)}^{\alpha} \\
&\quad \le C\|u\|_{H_0^1}^{p-1} + \|b(x)\|_{L^\infty(\Omega)} \|u\|_{H_0^1}^{\alpha} \\
&\quad \le C_1\|u\|^{p-1} + C_2\|b(x)\|_{L^\infty(\Omega)} \|u\|^{\alpha} \\
&\quad \le C_1 \left(\frac{\|b(x)\|_{L^\infty(\Omega)} S_{\alpha+1}^{\alpha+1}}{1-|\lambda| S_p^p}\right)^{(p-1)/((p-1)-\alpha)} \\
&\qquad + C_2\|b(x)\|_{L^\infty(\Omega)} \left(\frac{\|b(x)\|_{L^\infty(\Omega)} S_{\alpha+1}^{\alpha+1}}{1-|\lambda| S_p^p}\right)^{\alpha/((p-1)-\alpha)} = M.
\end{aligned} \tag{53}$$

Here, $C = \max\{\|a(x)\|_{L^\infty(\Omega)}, |\lambda|\}$ and C_1, C_2 from the Sobolev imbedding theorem.

Thus, by Cauchy-Schwarz inequalities, we obtain

$$\begin{aligned}
\|\Phi(u)\|_{H^{-1}} &= \sup_{\|v\|_{H_0^1} \le 1} |\langle \Phi(u), v\rangle| \\
&= \sup_{\|v\|_{H_0^1} \le 1} \left| \langle \phi(x), v\rangle + \langle \psi(x), v\rangle t \right. \\
&\qquad \left. + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \langle F(u), v\rangle ds \right| \\
&\le \left|\langle \phi(x), v\rangle\right| + \left|\langle \psi(x), v\rangle t\right| \\
&\quad + \left| \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \langle F(u), v\rangle ds \right|
\end{aligned}$$

$$\begin{aligned}
&\leq \|\phi(x)\|_{L^\infty(\Omega)}\|v\|_{H_0^1} + \|\psi(x)\|_{L^\infty(\Omega)}\|v\|_{H_0^1}T \\
&\quad + |\langle F(u), v\rangle| \left|\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} ds\right| \\
&\leq \|\phi(x)\|_{L^\infty(\Omega)} + \|\psi(x)\|_{L^\infty(\Omega)}T \\
&\quad + \frac{M}{\Gamma(\beta)}\left|\int_0^t (t-s)^{\beta-1} ds\right| \\
&\leq \|\phi(x)\|_{L^\infty(\Omega)} + \|\psi(x)\|_{L^\infty(\Omega)}T + \frac{M}{\beta\Gamma(\beta)}t^\beta \\
&\leq \|\phi(x)\|_{L^\infty(\Omega)} + \|\psi(x)\|_{L^\infty(\Omega)}T + \frac{M}{\beta\Gamma(\beta)}T^\beta.
\end{aligned} \tag{54}$$

Hence, $\Phi(u)$ is bounded.

On the other hand, given $\epsilon > 0$, setting

$$\delta = \left\{\left(\|\psi(x)\|_{L^\infty(\Omega)} + \frac{M}{\Gamma(\beta)}\right)^{-1}\epsilon\right\}^{1/\beta}, \tag{55}$$

then, for every $v \in H_0^1(\Omega)$, $t_1 < t_2, t_1, t_2 \in [0,T]$, $0 < T < 1$, and $t_2 - t_1 < \delta$, one has $\|\Phi u(t_2) - \Phi u(t_1)\|_{H^{-1}} = \sup_{\|v\|_{H_0^1}\leq 1}|\langle \Phi u(t_2) - \Phi u(t_1), v\rangle| \leq \epsilon$. That is to say, $\Phi(u)$ is equicontinuity. In fact,

$$\begin{aligned}
&\|\Phi u(t_2) - \Phi u(t_1)\|_{H^{-1}} \\
&\quad = \sup_{\|v\|_{H_0^1}\leq 1} |\langle \Phi u(t_2) - \Phi u(t_1), v\rangle| \\
&\quad = \sup_{\|v\|_{H_0^1}\leq 1} \left| \langle \psi(x), v\rangle (t_2 - t_1) \right. \\
&\qquad + \frac{1}{\Gamma(\beta)}\int_0^{t_2} (t_2-s)^{\beta-1}\langle F(u), v\rangle ds \\
&\qquad \left. - \frac{1}{\Gamma(\beta)}\int_0^{t_1} (t_1-s)^{\beta-1}\langle F(u), v\rangle ds \right| \\
&\quad \leq \|\psi(x)\|_{L^\infty(\Omega)}\|v\|_{H_0^1}|t_2 - t_1| \\
&\qquad + \frac{1}{\Gamma(\beta)}|\langle F(u), v\rangle| \int_0^{t_2} \left|(t_2-s)^{\beta-1}\right| ds \\
&\qquad + \frac{1}{\Gamma(\beta)}|\langle F(u), v\rangle| \int_0^{t_1} \left|(t_2-s)^{\beta-1} - (t_1-s)^{\beta-1}\right| ds \\
&\quad \leq \|\psi(x)\|_{L^\infty(\Omega)}|t_2 - t_1| + \frac{M}{\beta\Gamma(\beta)}t_2^\beta - \frac{M}{\beta\Gamma(\beta)}t_1^\beta \\
&\quad = \|\psi(x)\|_{L^\infty(\Omega)}|t_2 - t_1| + \frac{M}{\beta\Gamma(\beta)}\left(t_2^\beta - t_1^\beta\right).
\end{aligned} \tag{56}$$

In the following, we divide the proof into two cases.

Case 1. $\delta \leq t_1 < t_2 < T < 1$; since $0 < \beta \leq 1$, we get

$$\begin{aligned}
&\|\Phi u(t_2) - \Phi u(t_1)\|_{H^{-1}} \\
&\quad = \sup_{\|v\|_{H_0^1}\leq 1} |\langle \Phi u(t_2) - \Phi u(t_1), v\rangle| \\
&\quad \leq \|\psi(x)\|_{L^\infty(\Omega)}|t_2 - t_1| + \frac{M}{\beta\Gamma(\beta)}\left(t_2^\beta - t_1^\beta\right) \\
&\quad = \|\psi(x)\|_{L^\infty(\Omega)}|t_2 - t_1| + \frac{M}{\beta\Gamma(\beta)}\beta t^{\beta-1}(t_2 - t_1) \\
&\quad \leq \|\psi(x)\|_{L^\infty(\Omega)}|t_2 - t_1| + \frac{M}{\Gamma(\beta)\delta^{1-\beta}}|t_2 - t_1| \\
&\quad = \|\psi(x)\|_{L^\infty(\Omega)}\delta + \frac{M}{\Gamma(\beta)}\delta^\beta \\
&\quad \leq \|\psi(x)\|_{L^\infty(\Omega)}\delta^\beta + \frac{M}{\Gamma(\beta)}\delta^\beta \\
&\quad = \left(\|\psi(x)\|_{L^\infty(\Omega)} + \frac{M}{\Gamma(\beta)}\right)\delta^\beta \leq \epsilon;
\end{aligned} \tag{57}$$

here, $t_1 < t < t_2$, and we apply the mean theorem $t_2^\beta - t_1^\beta = \beta t^{\beta-1}(t_2 - t_1)$.

Case 2. $0 \leq t_1 < \delta, t_2 < \beta^{1/\beta}\delta$,

$$\begin{aligned}
&\|\Phi u(t_2) - \Phi u(t_1)\|_{H^{-1}} \\
&\quad = \sup_{\|v\|_{H_0^1}\leq 1} |\langle \Phi u(t_2) - \Phi u(t_1), v\rangle| \\
&\quad \leq \|\psi(x)\|_{L^\infty(\Omega)}|t_2 - t_1| + \frac{M}{\beta\Gamma(\beta)}\left(t_2^\beta - t_1^\beta\right) \\
&\quad \leq \|\psi(x)\|_{L^\infty(\Omega)}\delta + \frac{M}{\beta\Gamma(\beta)}\left(\beta^{1/\beta}\delta\right)^\beta \\
&\quad \leq \|\psi(x)\|_{L^\infty(\Omega)}\delta^\beta + \frac{M}{\Gamma(\beta)}\delta^\beta \\
&\quad = \left(\|\psi(x)\|_{L^\infty(\Omega)} + \frac{M}{\Gamma(\beta)}\right)\delta^\beta \leq \epsilon.
\end{aligned} \tag{58}$$

By the means of the Arzela-Ascoli theorem, we know that $\Phi(u) : H_0^1(\Omega) \to H^1(\Omega)$ is completely continuous. This completes the proof. □

By Lemma 10, we know that $\int_\Omega (u - \Phi(u))v\,dx = 0$, $\forall t \in [0,T]$, $0 < T < 1$, for every $v \in H_0^1(\Omega)$. That is to say, the fractional order equation $(E_{\lambda,b,t})$ has a unique weak solution $u \in C([0,T]; H_0^1(\Omega))$.

6. Conclusion

In this paper, we discussed the existence of a class of semilinear equations in the case of $1 < \alpha < p - 1$ in a weighted Sobolev space. By exploiting the relationship between the Nehari manifold and fibering maps, we first analyzed the properties of the Nehari manifold in the weighted Sobolev space, using these properties and combining the compact imbedding theorem and the behavior of Palais-Smale sequences on the Nehari manifold, we obtained the existence of the positive solutions for the problem (1). Finally, by using the Arzela-Ascoli fixed point theorem, the existence of weak solution for the time-fractional equation $(E_{\lambda,b,t})$ was obtained.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The work was supported by the National Natural Science Fund of China (11371289) and the China Scholarship Council (201206285005).

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