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Research Article

A Nonlinear Weakly Singular Retarded Henry-Gronwall Type Integral Inequality and Its Application

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We establish a class of new nonlinear retarded weakly singular integral inequality. Under several practical assumptions, the inequality is solved by adopting novel analysis techniques, and explicit bounds for the unknown functions are given clearly. An application of our result to the fractional differential equations with delay is shown at the end of the paper.

1. Introduction

Integral inequalities play increasingly important roles in the study of existence, uniqueness, boundedness, oscillation, stability, invariant manifolds, and other qualitative properties of solutions of ordinary differential equations and integral equations. One of the best known and widely used inequalities in the study of nonlinear differential equations is Gronwall-Bellman inequality [1, 2], which can be stated as follows. If u and f are nonnegative continuous functions on an interval $[a,b]$ satisfying $u(t) \le c + \int_a^t f(s)u(s)ds$, $t \in [a,b]$, then $u(t) \le c\exp(\int_a^t f(s)ds)$, $t \in [a,b]$. Many papers are devoted to different generalizations of Bellman-Gronwall inequality. Very well-known generalization of Bellman-Gronwall inequality to the nonlinear case is the Bihari inequality [3]. In 1956, Bihari [3] discussed the integral inequality

$$u(t) \le a + \int_0^t f(s)\, w(u(s))\, ds, \quad t > 0, \tag{1}$$

where $a > 0$ is a constant. In recent years, many researchers have devoted much effort to investigating weakly singular integral inequalities. For example, Henry [4] proposed a linear integral inequality with singular kernel to investigate some qualitative properties for a parabolic differential equation, and Sano and Kunimatsu [5] gave a modified version of Henry type inequality. However, such results are expressed by a complicated power series which are sometimes inconvenient for their applications. To avoid the shortcomings of these results, Medved' [6] presented a new method to discuss nonlinear singular integral inequalities of Henry type and their Bihari version is as follows:

$$u(t) \le a(t) + \int_0^t (t-s)^{\beta-1} f(s)\, w(u(s))\, ds, \tag{2}$$

and the estimates of solutions are given. From then on, more attention has been paid to such inequalities with singular kernel; see [7–24] and the references cited therein. Ye and Gao [20] considered the integral inequality of Henry-Gronwall type with delay

$$\begin{aligned} u(t) &\le a(t) + \int_{t_0}^t [b(s)\, u(s) + c(s)\, u(s-r)]\, ds, \\ &\qquad t \in [t_0, T), \\ u(t) &\le \phi(t), \quad t \in [t_0 - r, t_0) \end{aligned} \tag{3}$$

and Henry-Gronwall type retarded integral inequality with singular kernel

$$u(t) \le a(t) + \int_{t_0}^{t} (t-s)^{\beta-1} [b(s) u(s) + c(s) u(s-r)] \, ds, \quad t \in [t_0, T),$$
$$u(t) \le \phi(t), \quad t \in [t_0 - r, t_0). \tag{4}$$

In this paper, motivated by [6, 20], we discuss the nonlinear integral inequality of Henry-Gronwall type with delay

$$u(t) \le a(t) + \int_{t_0}^{t} [b(s) w(u(s)) + c(s) w(u(s-r))] \, ds, \quad t \in [t_0, T),$$
$$u(t) \le \phi(t), \quad t \in [t_0 - r, t_0) \tag{5}$$

and Henry-Gronwall type nonlinear retarded integral inequality with singular kernel

$$u(t) \le a(t) + \int_{t_0}^{t} (t-s)^{\beta-1} [b(s) w(u(s)) + c(s) w(u(s-r))] \, ds, \quad t \in [t_0, T),$$
$$u(t) \le \phi(t), \quad t \in [t_0 - r, t_0). \tag{6}$$

2. Main Results

Throughout this paper, $\mathbf{R}$ denotes the set of real numbers, $\mathbf{R}_+ = [0, +\infty)$. For convenience, before giving our main results, we cite some useful lemmas and definitions in the discussion of our proof as follows.

Definition 1 (see [6]). Let $q > 0$ be a real number and $0 < T \le \infty$. We say that a function $w : \mathbf{R}_+ \to \mathbf{R}_+$ satisfies a condition (q), if

$$e^{-qt} [w(u)]^q \le R(t) w\left(e^{-qt} u^q\right), \quad \forall u \in \mathbf{R}_+, \ t \in [0, T), \tag{7}$$

where $R(t)$ is a continuous, nonnegative function.

Lemma 2 (discrete Jensen inequality [25]). *Let $A_1, A_2, \ldots, A_n$ be nonnegative real numbers, $l > 1$ is real numbers, and n is a natural number. Then*

$$\left(A_1 + A_2 + \cdots + A_n\right)^l \le n^{l-1} \left(A_1^l + A_2^l + \cdots + A_n^l\right). \tag{8}$$

Lemma 3 (see [6]). *(1) Let $\beta > 1/2$; then*

$$\int_{t_0}^{t} (t-s)^{2\beta-2} e^{2s} ds \le \frac{2e^{2t}}{4^\beta} \Gamma(2\beta - 1), \quad t_0, t \in \mathbf{R}_+, \tag{9}$$

where $\Gamma(\beta) := \int_0^\infty \tau^{\beta-1} e^{-\tau} d\tau$ is the gamma function.

(2) Let $\beta \in (0, 1/2]$, $p = 1 + \beta$; then

$$\int_{t_0}^{t} (t-s)^{p(\beta-1)} e^{ps} ds \le \frac{e^{pt}}{p^{1+p(\beta-1)}} \Gamma(1 + p(\beta - 1)), \quad t_0, t \in \mathbf{R}_+. \tag{10}$$

Proof. (1) Using a change of variables $\tau = t - s$ and $\xi = 2\tau$ successively, we have the estimate

$$\begin{aligned}
\int_{t_0}^{t} (t-s)^{2\beta-2} e^{2s} ds &= \int_0^{t-t_0} \tau^{2\beta-2} e^{2t-2\tau} d\tau \\
&= e^{2t} \int_0^{t-t_0} \tau^{2\beta-2} e^{-2\tau} d\tau \\
&= \frac{2e^{2t}}{4^\beta} \int_0^{2t-2t_0} \xi^{2\beta-2} e^{-\xi} d\xi \\
&\le \frac{2e^{2t}}{4^\beta} \Gamma(2\beta - 1).
\end{aligned} \tag{11}$$

Since $\beta > 1/2$, $2\beta - 1 > 0$ and $\Gamma(2\beta - 1) \in \mathbf{R}_+$.

(2) Using a change of variables $\tau = t - s$ and $\xi = p\tau$ successively, we have the estimate

$$\begin{aligned}
\int_{t_0}^{t} (t-s)^{p(\beta-1)} e^{ps} ds &= \int_0^{t-t_0} \tau^{p(\beta-1)} e^{pt-p\tau} d\tau \\
&= e^{pt} \int_0^{t-t_0} \tau^{p(\beta-1)} e^{-p\tau} d\tau \\
&= \frac{e^{pt}}{p^{1+p(\beta-1)}} \int_0^{pt-pt_0} \xi^{p(\beta-1)} e^{-\xi} d\xi \\
&\le \frac{e^{pt}}{p^{1+p(\beta-1)}} \Gamma(1 + p(\beta - 1)).
\end{aligned} \tag{12}$$

Since $0 < \beta \le 1/2$, $p < 1/(1-\beta)$, $1 + p(\beta - 1) > 0$, and $\Gamma(1 + p(\beta - 1)) \in \mathbf{R}_+$. □

Theorem 4. *Suppose that a, b, c are nonnegative continuous functions on $[t_0, T)$, ϕ is a nonnegative continuous function on $[t_0 - r, t_0)$, $a(t_0) = \phi(t_0)$, and $t_0 \ge 0$, $r > 0$, $T > 0$ are constants. Suppose that the function w satisfies the following conditions:*

(1) *(q) condition, that is, w satisfies inequality* (7);

(2) *subadditivity, that is, for all $t, s \in \mathbf{R}_+$, $w(t+s) \le w(t) + w(s)$.*

If u satisfies (5), *then*

$$u(t) \le a(t) + W^{-1}\left[W\left(\int_{t_0}^{t} G(s)\,ds\right) + \int_{t_0}^{t} b(s)\,ds\right], \quad t \in [t_0, t_0 + r),$$

$$u(t) \le a(t) + W^{-1} \times \left\{W\left[W^{-1}\left(W\left(\int_{t_0}^{t_0+r} G(s)\,ds\right) + \int_{t_0}^{t_0+r} b(s)\,ds\right) + \int_{t_0+r}^{t} H(s)\,ds\right] + \int_{t_0+r}^{t} (b(s)+c(s))\,ds\right\}, \quad t \in [t_0 + r, T), \tag{13}$$

where

$$W(t) := \int_c^t \frac{ds}{w(s)}, \quad t \in (0,\infty),\ c > 0,\ W(\infty) = \infty, \tag{14}$$

$$G(t) := b(t)\,w(a(t)) + c(t)\,w(\phi(t-r)), \quad t \in [t_0, t_0 + r), \tag{15}$$

$$H(t) := b(t)\,w(a(t)) + c(t)\,w(a(t-r)), \quad t \in [t_0 + r, T). \tag{16}$$

Proof. Define a function $z(t)$ by the right side of (5), that is,

$$z(t) = \int_{t_0}^{t} [b(s)\,w(u(s)) + c(s)\,w(u(s-r))]\,ds, \quad t \in [t_0, T). \tag{17}$$

Then $z(t_0) = 0$, $u(t) \le a(t) + z(t)$, and $z(t)$ is a nonnegative, nondecreasing, and continuous function with $z'(t) = b(t)w(u(t)) + c(t)w(u(t-r))$, $t \in [t_0, T)$.

For $t \in [t_0, t_0 + r)$, by the subadditivity satisfied by w, we conclude

$$\begin{aligned} z'(t) &\le b(t)\,w(a(t) + z(t)) + c(t)\,w(\phi(t-r)) \\ &\le b(t)\,w(a(t)) + c(t)\,w(\phi(t-r)) + b(t)\,w(z(t)). \end{aligned} \tag{18}$$

Letting $s = t$ in (18) and integrating both sides of inequality (18) from t_0 to t, we obtain

$$\begin{aligned} z(t) &\le \int_{t_0}^{t} [b(s)\,w(a(s)) + c(s)\,w(\phi(s-r))]\,ds + \int_{t_0}^{t} b(s)\,w(z(s))\,ds \\ &\le \int_{t_0}^{\xi} [b(s)\,w(a(s)) + c(s)\,w(\phi(s-r))]\,ds + \int_{t_0}^{t} b(s)\,w(z(s))\,ds, \quad t_0 \le t \le \xi, \end{aligned} \tag{19}$$

where $t_0 \le \xi \le t_0 + r$ is chosen arbitrarily.

Define a function $z_1(t)$ by the right side of (19), that is,

$$z_1(t) = \int_{t_0}^{\xi} [b(s)\,w(a(s)) + c(s)\,w(\phi(s-r))]\,ds + \int_{t_0}^{t} b(s)\,w(z(s))\,ds, \quad t_0 \le t \le \xi. \tag{20}$$

Then, the function z_1 is a nonnegative, nondecreasing, and continuous function with

$$z_1(t_0) = \int_{t_0}^{\xi} [b(s)\,w(a(s)) + c(s)\,w(\phi(s-r))]\,ds, \quad z(t) \le z_1(t), \quad t_0 \le t \le \xi. \tag{21}$$

Differentiating z_1, we have

$$z_1'(t) \le b(t)\,w(z_1(t)), \quad t_0 \le t \le \xi. \tag{22}$$

From (22), we obtain

$$\frac{dz_1(t)}{w(z_1(t))} \le b(t)\,dt, \quad t_0 \le t \le \xi. \tag{23}$$

Using (21), from (23) we obtain

$$\begin{aligned} W(z_1(t)) &\le W(z_1(t_0)) + \int_{t_0}^{t} b(s)\,ds \\ &\le W\left(\int_{t_0}^{\xi} G(s)\,ds\right) + \int_{t_0}^{t} b(s)\,ds, \quad t_0 \le t \le \xi, \end{aligned} \tag{24}$$

where W, G are defined by (14) and (15), respectively. From (24), we observe

$$z_1(t) \le W^{-1}\left[W\left(\int_{t_0}^{\xi} G(s)\,ds\right) + \int_{t_0}^{t} b(s)\,ds\right], \quad t_0 \le t \le \xi. \tag{25}$$

Let $t = \xi$ in (25); we have

$$z_1(\xi) \le W^{-1}\left[W\left(\int_{t_0}^{\xi} G(s)\,ds\right) + \int_{t_0}^{\xi} b(s)\,ds\right]. \tag{26}$$

Since ξ is chosen arbitrarily, from (26), we have the estimation

$$\begin{aligned} z(t) &\le z_1(t) \\ &\le W^{-1}\left[W\left(\int_{t_0}^{t} G(s)\,ds\right) + \int_{t_0}^{t} b(s)\,ds\right], \quad t \in [t_0, t_0 + r). \end{aligned} \tag{27}$$

For $t \in [t_0 + r, T)$, using the subadditivity of w and monotony of w, z, from (17) we have

$$\begin{aligned} z'(t) &\le b(t)\,w(a(s) + z(t)) + c(t)\,w(a(t-r) + z(t-r)) \\ &\le b(t)\,w(a(t)) + c(t)\,w(a(t-r)) + b(t)\,w(z(t)) + c(t)\,w(z(t-r)) \\ &\le b(t)\,w(a(t)) + c(t)\,w(a(t-r)) + (b(t) + c(t)) \times w(z(t)). \end{aligned} \tag{28}$$

Letting $s = t$ in (28) and integrating both sides of inequality (28) from t_0 to t and using (27) we obtain

$$\begin{aligned} z(t) &\le z(t_0+r) + \int_{t_0+r}^{t} H(s)\,ds \\ &\quad + \int_{t_0+r}^{t} (b(s)+c(s))\,w(z(s))\,ds \\ &\le W^{-1}\left[W\left(\int_{t_0}^{t_0+r} G(s)\,ds\right) + \int_{t_0}^{t_0+r} b(s)\,ds\right] \\ &\quad + \int_{t_0+r}^{\xi} H(s)\,ds + \int_{t_0+r}^{t} (b(s)+c(s))\,w(z(s))\,ds, \\ &\qquad t \in [t_0+r,\xi], \end{aligned} \tag{29}$$

where $t_0 + r \le \xi \le T$, ξ is seen as a constant, and $H(s)$ is defined by (16).

Define a function z_2 by the right side of (29), that is,

$$\begin{aligned} z_2(t) &= W^{-1}\left[W\left(\int_{t_0}^{t_0+r} G(s)\,ds\right) + \int_{t_0}^{t_0+r} b(s)\,ds\right] \\ &\quad + \int_{t_0+r}^{\xi} H(s)\,ds + \int_{t_0+r}^{t} (b(s)+c(s))\,w(z(s))\,ds, \\ &\qquad t \in [t_0+r,\xi]. \end{aligned} \tag{30}$$

Obviously, z_2 is a nonnegative, nondecreasing, and continuous function with

$$\begin{aligned} z_2(t_0+r) &= W^{-1}\left[W\left(\int_{t_0}^{t_0+r} G(s)\,ds\right) + \int_{t_0}^{t_0+r} b(s)\,ds\right] \\ &\quad + \int_{t_0+r}^{\xi} [b(s)\,w(a(s)) + c(s)\,w(a(s-r))]\,ds, \end{aligned} \tag{31}$$

$$z(t) \le z_2(t), \quad t_0 + r \le t \le \xi. \tag{32}$$

Differentiating z_2, we have

$$z_2'(t) \le (b(t)+c(t))\,w(z_2(t)), \quad t_0 + r \le t \le \xi. \tag{33}$$

From (33), we have

$$\frac{dz_2(t)}{w(z_2(t))} \le (b(t)+c(t))\,dt, \quad t_0 + r \le t \le \xi. \tag{34}$$

Using (31), from (34), we have

$$\begin{aligned} W(z_2(t)) &\le W(z_2(t_0+r)) + \int_{t_0+r}^{t} (b(s)+c(s))\,ds \\ &\le W\left\{W^{-1}\left[W\left(\int_{t_0}^{t_0+r} G(s)\,ds\right) + \int_{t_0}^{t_0+r} b(s)\,ds\right]\right. \\ &\quad \left. + \int_{t_0+r}^{\xi} H(s)\,ds\right\} + \int_{t_0+r}^{t} (b(s)+c(s))\,ds, \\ &\qquad t \in [t_0+r,\xi]. \end{aligned} \tag{35}$$

It follows that

$$\begin{aligned} z_2(t) &\le W^{-1} \\ &\quad \times \left\{W\left\{W^{-1}\left[W\left(\int_{t_0}^{t_0+r} G(s)\,ds\right) + \int_{t_0}^{t_0+r} b(s)\,ds\right]\right.\right. \\ &\quad \left.\left. + \int_{t_0+r}^{\xi} H(s)\,ds\right\} + \int_{t_0+r}^{t} (b(s)+c(s))\,ds\right\}, \\ &\qquad t \in [t_0+r,\xi]. \end{aligned} \tag{36}$$

In (36), let $t = \xi$, and then we have

$$\begin{aligned} &z_2(\xi) \\ &\le W^{-1}\left\{W\left[W^{-1}\left(W\left(\int_{t_0}^{t_0+r} G(s)\,ds\right) + \int_{t_0}^{t_0+r} b(s)\,ds\right)\right.\right. \\ &\quad \left.\left. + \int_{t_0+r}^{\xi} H(s)\,ds\right] + \int_{t_0+r}^{\xi} (b(s)+c(s))\,ds\right\}. \end{aligned} \tag{37}$$

Since ξ is chosen arbitrarily, from (32) and (37), we obtain the estimation

$$\begin{aligned} z(t) &\le W^{-1}\left\{W\left[W^{-1}\left(W\left(\int_{t_0}^{t_0+r} G(s)\,ds\right) + \int_{t_0}^{t_0+r} b(s)\,ds\right)\right.\right. \\ &\quad \left.\left. + \int_{t_0+r}^{t} H(s)\,ds\right] + \int_{t_0+r}^{t} (b(s)+c(s))\,ds\right\}, \\ &\qquad t \in [t_0+r,T). \end{aligned} \tag{38}$$

Noting that $u(t) \le a(t) + z(t)$, from (27) and (38), we obtain our required estimations (13). □

Remark 5. When $w(u(t)) = u(t)$. The estimations (13) in Theorem 4 are reduced to the corresponding estimations in [20].

Theorem 6. *Suppose that a, b, c, u, w, ϕ, r satisfy the corresponding conditions in Theorem 4; β is a constant. If u satisfies (6), then the following assertions hold.*

(1) Suppose $\beta > 1/2$. Then

$$\begin{aligned} z(t) &\le A(t) + W^{-1}\left[W\left(\int_{t_0}^{t} I(s)\,ds\right) + \int_{t_0}^{t} B(s)\,ds\right], \\ &\qquad t \in [t_0, t_0+r), \\ z(t) &\le A(t) + W^{-1} \\ &\quad \times \left\{W\left[W^{-1}\left(W\left(\int_{t_0}^{t_0+r} I(s)\,ds\right) + \int_{t_0}^{t_0+r} B(s)\,ds\right)\right.\right. \\ &\quad \left.\left. + \int_{t_0+r}^{t} J(s)\,ds\right] + \int_{t_0+r}^{t} (B(s)+C(s))\,ds\right\}, \\ &\qquad t \in [t_0+r,T), \end{aligned} \tag{39}$$

where W is defined by (14) *in Theorem 4,*

$$I(s) := B(t)\, w(A(t)) + C(t)\, w(\Phi(t-r)), \quad t \in [t_0, t_0+r), \tag{40}$$

$$J(t) := B(t)\, w(A(t)) + C(t)\, w(A(t-r)), \quad t \in [t_0+r, T), \tag{41}$$

$$A(t) := \max\left\{3, e^{2r}\right\} \left[e^{t_0} a(t)\right]^2, \quad t \in [t_0, T), \tag{42}$$

$$B(t) := \frac{6}{4^\beta} \Gamma(2\beta - 1)\, b^2(t)\, R(t), \quad t \in [t_0, T), \tag{43}$$

$$C(t) := \frac{6}{4^\beta} \Gamma(2\beta - 1)\, e^{-2r} c^2(t)\, R(t-r), \quad t \in [t_0, T), \tag{44}$$

$$\Phi(t) := \max\left\{3, e^{2r}\right\} \left[e^{-t_0} \phi(t)\right]^2, \quad t \in [t_0, T), \tag{45}$$

and $R(t)$ is defined in (7) *in Definition 1.*

(2) Suppose that $\beta \in (0, 1/2]$, $p = 1+\beta$. Then

$$\begin{aligned} z(t) &\le D(t) + W^{-1}\left[W\left(\int_{t_0}^t K(s)\, ds\right) + \int_{t_0}^t E(s)\, ds\right], \\ &\qquad t \in [t_0, t_0+r), \\ z(t) &\le D(t) + W^{-1} \\ &\quad \left\{W\left[W^{-1}\left(W\left(\int_{t_0}^{t_0+r} K(s)\, ds\right) + \int_{t_0}^{t_0+r} E(s)\, ds\right)\right.\right. \\ &\quad \left.\left. + \int_{t_0+r}^t L(s)\, ds\right] + \int_{t_0+r}^t (E(s) + F(s))\, ds\right\}, \\ &\qquad t \in [t_0+r, T), \end{aligned} \tag{46}$$

where

$$K(s) := E(t)\, w(D(t)) + F(t)\, w(\Psi(t-r)), \quad t \in [t_0, t_0+r), \tag{47}$$

$$L(t) := E(t)\, w(D(t)) + F(t)\, w(D(t-r)), \quad t \in [t_0+r, T), \tag{48}$$

$$D(t) := \rho\left[e^{t_0} a(t)\right]^q, \quad t \in [t_0, T), \tag{49}$$

$$E(t) := 3\left[\frac{1}{p^{1+p(\beta-1)}} \Gamma(1 + p(\beta-1))\right]^{q/p} b^q(t)\, R(t), \quad t \in [t_0, T), \tag{50}$$

$$F(t) := 3\left[\frac{1}{p^{1+p(\beta-1)}} \Gamma(1 + p(\beta-1))\right]^{q/p} \times e^{-qr} c^q(t)\, R(t-r), \quad t \in [t_0, T), \tag{51}$$

$$\Psi(t) := \rho\left[e^{-t_0} \phi(t)\right]^q, \quad t \in [t_0, T), \tag{52}$$

and $q = 1 + 1/\beta$.

Proof. First we will prove assertion (1). Suppose that $\beta > 1/2$. Using Cauchy-Schwarz inequality, we obtain from (6) that

$$\begin{aligned} u(t) &\le a(t) + \int_{t_0}^t (t-s)^{\beta-1} e^s b(s)\, e^{-s} w(u(s))\, ds \\ &\quad + \int_{t_0}^t (t-s)^{\beta-1} e^{s-r} c(s)\, e^{-s+r} w(u(s-r))\, ds \\ &\le a(t) + \left[\int_{t_0}^t (t-s)^{2\beta-2} e^{2s} ds\right]^{1/2} \\ &\quad \times \left[\int_{t_0}^t b^2(s) e^{-2s} w^2(u(s)) ds\right]^{1/2} \\ &\quad + \left[\int_{t_0}^t (t-s)^{2\beta-2} e^{2s} ds\right]^{1/2} \\ &\quad \times \left[\int_{t_0}^t e^{-2r} c^2(s)\, e^{-2s+2r} w^2(u(s-r))\, ds\right]^{1/2}, \\ &\qquad t \in [t_0, T). \end{aligned} \tag{53}$$

Since w satisfies (q) condition, using (7) in Definition 1 and (9) in Lemma 3, from (53) we derive that

$$\begin{aligned} u(t) &\le a(t) + \left[\frac{2e^{2t}}{4^\beta} \Gamma(2\beta-1)\right]^{1/2} \\ &\quad \times \left[\int_{t_0}^t b^2(s)\, R(s)\, w\left(u^2(s)\, e^{-2s}\right) ds\right]^{1/2} \\ &\quad + \left[\frac{2e^{2t}}{4^\beta} \Gamma(2\beta-1)\right]^{1/2} \\ &\quad \times \left[\int_{t_0}^t e^{-2r} c^2(s)\, R(s-r)\, w\left(u^2(s-r)\, e^{-2s+2r}\right) ds\right]^{1/2}, \end{aligned} \tag{54}$$

for all $t \in [t_0, T)$. Using discrete Jensen inequality (8) with $n = 3$, $l = 2$, from (54) we obtain

$$\begin{aligned} u^2(t) &\le 3a^2(t) + 3\frac{2e^{2t}}{4^\beta} \Gamma(2\beta-1) \\ &\quad \times \left[\int_{t_0}^t b^2(s)\, R(s)\, w\left(u^2(s)\, e^{-2s}\right) ds\right. \\ &\quad \left. + \int_{t_0}^t e^{-2r} c^2(s)\, R(s-r)\, w\left(u^2(s-r)\, e^{-2s+2r}\right) ds\right], \\ &\qquad t \in [t_0, T). \end{aligned} \tag{55}$$

Let $v(t) = [e^{-t}u(t)]^2$ and $\lambda = \max\{3, e^{2r}\}$. From (55) we have

$$\begin{aligned} v(t) \le{}& \lambda\left[e^{t_0}a(t)\right]^2 + \frac{6}{4^{\beta}}\Gamma(2\beta-1) \\ &\times\left[\int_{t_0}^{t} b^2(s)R(s)w(v(s))\,ds\right. \\ &\left.+\int_{t_0}^{t} e^{-2r}c^2(s)R(s-r)w(v(s-r))\,ds\right], \\ & t\in[t_0,T). \end{aligned} \tag{56}$$

We observe that

$$\begin{aligned} v(t) &= \left[e^{-t}\phi(t)\right]^2 \le e^{2r}\left[e^{-t_0}\phi(t)\right]^2 \\ &\le \lambda\left[e^{-t_0}\phi(t)\right]^2 = \Phi(t), \quad t\in[t_0-r,t_0), \end{aligned} \tag{57}$$

$\Phi(t)$ is defined by (45). By the definitions of $A(t), B(t)$, and $C(t)$ in (42), (43), and (44), from (56) we see

$$\begin{aligned} v(t) \le{}& A(t) + \int_{t_0}^{t} B(s)w(v(s))\,ds \\ &+ \int_{t_0}^{t} C(s)w(v(s-r))\,ds, \quad t\in[t_0,T); \\ v(t) \le{}& \Phi(t), \quad t\in[t_0-r,t_0). \end{aligned} \tag{58}$$

We observe that (58) have the same form as (5) and $A(t), B(t), C(t)$ satisfy the corresponding conditions in Theorem 4. Applying Theorem 4 to (58), we obtain our required estimations (39).

(2) Now let us prove assertion (2). Suppose $\beta \in (0, 1/2]$, $p = 1+\beta$. Let $q = 1 + 1/\beta$; then $1/p + 1/q = 1$. Using Hölder inequality, from (6) we obtain

$$\begin{aligned} u(t) \le{}& a(t) + \int_{t_0}^{t}(t-s)^{\beta-1}e^{s}b(s)e^{-s}w(u(s))\,ds \\ &+ \int_{t_0}^{t}(t-s)^{\beta-1}e^{s-r}c(s)e^{-s+r}w(u(s-r))\,ds \\ \le{}& a(t) + \left[\int_{t_0}^{t}(t-s)^{p\beta-p}e^{ps}ds\right]^{1/p} \\ &\times\left[\int_{t_0}^{t} b^q(s)e^{-qs}w^q(u(s))ds\right]^{1/q} \\ &+\left[\int_{t_0}^{t}(t-s)^{p\beta-p}e^{ps}ds\right]^{1/p} \\ &\times\left[\int_{t_0}^{t} e^{-qr}c^q(s)e^{-qs+qr}w^q(u(s-r))\,ds\right]^{1/q}, \\ & t\in[t_0,T). \end{aligned} \tag{59}$$

Since w satisfies (q) condition, using (7) and (10), from (59) we derive

$$\begin{aligned} u(t) \le{}& a(t) + \left[\frac{e^{pt}}{p^{1+p(\beta-1)}}\Gamma(1+p(\beta-1))\right]^{1/p} \\ &\times\left[\int_{t_0}^{t} b^q(s)R(s)w(u^q(s)e^{-qs})\,ds\right]^{1/q} \\ &+\left[\frac{e^{pt}}{p^{1+p(\beta-1)}}\Gamma(1+p(\beta-1))\right]^{1/p} \\ &\times\left[\int_{t_0}^{t} e^{-qr}c^q(s)R(s-r)w(u^q(s-r)e^{-qs+qr})\,ds\right]^{1/q}, \end{aligned} \tag{60}$$

for all $t \in [t_0, T)$. Using Jensen inequality (8), from (60) we have

$$\begin{aligned} u^q(t) \le{}& 3a^q(t) + 3\left[\frac{e^{pt}}{p^{1+p(\beta-1)}}\Gamma(1+p(\beta-1))\right]^{q/p} \\ &\times\left[\int_{t_0}^{t} b^q(s)R(s)w(u^q(s)e^{-qs})\,ds\right. \\ &\left.+\int_{t_0}^{t} e^{-qr}c^q(s)R(s-r)w(u^q(s-r)e^{-qs+qr})\,ds\right], \\ & t\in[t_0,T). \end{aligned} \tag{61}$$

Let $v(t) = [e^{-t}u(t)]^q$ and $\rho = \max\{3, e^{qr}\}$. Then, we obtain from (61) that

$$\begin{aligned} v(t) \le{}& \rho\left[e^{t_0}a(t)\right]^q + 3\left[\frac{1}{p^{1+p(\beta-1)}}\Gamma(1+p(\beta-1))\right]^{q/p} \\ &\times\left[\int_{t_0}^{t} b^q(s)R(s)w(u^q(s)e^{-qs})\,ds\right. \\ &\left.+\int_{t_0}^{t} e^{-qr}c^q(s)R(s-r)w(u^q(s-r)e^{-qs+qr})\,ds\right], \\ & t\in[t_0,T). \end{aligned} \tag{62}$$

We observe that

$$\begin{aligned} v(t) &= \left[e^{-t}\phi(t)\right]^q \le e^{qr}\left[e^{-t_0}\phi(t)\right]^q \\ &\le \rho\left[e^{-t_0}\phi(t)\right]^q = \Psi(t), \quad t\in[t_0-r,t_0), \end{aligned} \tag{63}$$

where $\Psi(t)$ is defined by (52). Using definitions of $D(t), E(t)$ and $F(t)$ in (49), (50), and (51), from (62) we have

$$\begin{aligned} v(t) \le{}& D(t) + \int_{t_0}^{t} E(s)w(v(s))\,ds \\ &+ \int_{t_0}^{t} F(s)w(v(s-r))\,ds, \quad t\in[t_0,T), \\ v(t) \le{}& \Psi(t), \quad t\in[t_0-r,t_0). \end{aligned} \tag{64}$$

We observe that (64) have the same form as (5) and $D(t)$, $E(t)$, $F(t)$ satisfy the corresponding conditions in Theorem 4. Applying Theorem 4 to (64), we obtain our required estimations (46). $\square$

3. Application to Fractional Differential Equations (FDEs) with Delay

In this section, we apply our result to the following fractional differential equations (FDEs) with delay (see [20]):

$$\begin{aligned} D^{\beta}x(t) &= f(t, x(t), x(t-r)), \quad t \in [t_0, T), \\ x(t) &= \phi(t), \quad t \in [t_0 - r, t_0), \end{aligned} \tag{65}$$

where D^{β} represents the Caputo fractional derivative of order β $(\beta > 0)$, $f \in C([t_0, T) \times \mathbf{R} \times \mathbf{R}, \mathbf{R})$, and ϕ is as in Theorem 6.

Theorem 7. *Suppose that*

$$|f(t, x, y)| \le b(t)\, w(|x|) + c(t)\, w(|y|), \tag{66}$$

where $b(t), c(t), w$ are as in Theorem 6. Let $M = \max_{t \in [t_0 - r, t_0)} |\phi(t)|$. If $x(t)$ is any solution of IVP (65), then the following estimates hold.

(1) Suppose $1/2 < \beta \le 1$. Then

$$\begin{aligned} z(t) &\le A(t) + W^{-1}\left[W\left(\int_{t_0}^{t} \tilde{I}(s)\, ds\right) + \int_{t_0}^{t} \tilde{B}(s)\, ds\right], \\ &\qquad t \in [t_0, t_0 + r), \\ z(t) &\le A(t) + W^{-1}\left\{W\left[W^{-1}\left(W\left(\int_{t_0}^{t_0+r} \tilde{I}(s)\, ds\right) + \int_{t_0}^{t_0+r} \tilde{B}(s)\, ds\right) + \int_{t_0+r}^{t} \tilde{J}(s)\, ds\right] + \int_{t_0+r}^{t} \left(\tilde{B}(s) + \tilde{C}(s)\right) ds\right\}, \\ &\qquad t \in [t_0 + r, T), \end{aligned} \tag{67}$$

where W is defined by (14) *in Theorem 4,*

$$\begin{aligned} \tilde{I}(s) &:= \tilde{B}(t)\, w\left(\tilde{A}(t)\right) + \tilde{C}(t)\, w(\Phi(t-r)), \quad t \in [t_0, t_0 + r), \\ \tilde{J}(t) &:= \tilde{B}(t)\, w\left(\tilde{A}(t)\right) + \tilde{C}(t)\, w\left(\tilde{A}(t-r)\right), \quad t \in [t_0 + r, T), \\ \tilde{A}(t) &:= \max\left\{3, e^{2r}\right\}\left[e^{t_0} M\right]^2, \quad t \in [t_0, T), \\ \tilde{B}(t) &:= \frac{6\Gamma(2\beta - 1)\, b^2(t)\, R(t)}{\left(4^{\beta}\Gamma(\beta)\right)}, \quad t \in [t_0, T), \\ \tilde{C}(t) &:= \frac{6\Gamma(2\beta - 1)\, e^{-2r} c^2(t)\, R(t-r)}{\left(4^{\beta}\Gamma(\beta)\right)}, \quad t \in [t_0, T), \end{aligned} \tag{68}$$

and $R(t), \Phi$ are defined by (7) *and* (45)*, respectively.*

(2) Suppose that $\beta \in (0, 1/2]$, $p = 1 + \beta$. Then

$$\begin{aligned} z(t) &\le D(t) + W^{-1}\left[W\left(\int_{t_0}^{t} \tilde{K}(s)\, ds\right) + \int_{t_0}^{t} \tilde{E}(s)\, ds\right], \\ &\qquad t \in [t_0, t_0 + r), \\ z(t) &\le D(t) + W^{-1}\left\{W\left[W^{-1}\left(W\left(\int_{t_0}^{t_0+r} \tilde{K}(s)\, ds\right) + \int_{t_0}^{t_0+r} \tilde{E}(s)\, ds\right) + \int_{t_0+r}^{t} \tilde{L}(s)\, ds\right] + \int_{t_0+r}^{t} \left(\tilde{E}(s) + \tilde{F}(s)\right) ds\right\}, \\ &\qquad t \in [t_0 + r, T), \end{aligned} \tag{69}$$

where

$$\begin{aligned} \tilde{K}(s) &:= \tilde{E}(t)\, w\left(\tilde{D}(t)\right) + \tilde{F}(t)\, w(\Psi(t-r)), \quad t \in [t_0, t_0 + r), \\ \tilde{L}(t) &:= \tilde{E}(t)\, w\left(\tilde{D}(t)\right) + \tilde{F}(t)\, w\left(\tilde{D}(t-r)\right), \quad t \in [t_0 + r, T), \\ \tilde{D}(t) &:= \rho\left[e^{t_0} M\right]^q, \quad t \in [t_0, T), \\ \tilde{E}(t) &:= 3\left[\frac{1}{p^{1+p(\beta-1)}}\Gamma(1 + p(\beta - 1))\right]^{q/p} \times \left(\frac{b(t)}{\Gamma(\beta)}\right)^q R(t), \quad t \in [t_0, T), \\ \tilde{F}(t) &:= 3\left[\frac{1}{p^{1+p(\beta-1)}}\Gamma(1 + p(\beta - 1))\right]^{q/p} \times \left(\frac{c(t)}{\Gamma(\beta)}\right)^q e^{-qr} R(t-r), \quad t \in [t_0, T), \end{aligned} \tag{70}$$

and $q = 1 + 1/\beta$; $\Psi(t)$ is defined by (52).

Proof. The solution $x(t)$ of FDEs (65) can be written as (see [24])

$$\begin{aligned} x(t) &= \sum_{k=0}^{n-1} \frac{b_k}{k!}(t - t_0)^k + \frac{1}{\Gamma(\beta)} \times \int_{t_0}^{t} (t-s)^{\beta-1} f(s, x(s), x(s-r))\, ds, \quad t \in [t_0, T), \\ x(t) &= \phi(t), \quad t \in [t_0 - r, t_0). \end{aligned} \tag{71}$$

When $0 < \beta \le 1$, from (71) we obtain

$$\begin{aligned} &|x(t)| \\ &\le M + \frac{1}{\Gamma(\beta)} \\ &\quad \times \int_{t_0}^{t} (t-s)^{\beta-1} \left[b(s) w(|x(s)|) + c(s) w(|x(s-r)|) \right] ds, \\ &\qquad\qquad t \in [t_0, T), \\ &|x(t)| \le M, \quad t \in [t_0 - r, t_0). \end{aligned} \tag{72}$$

Applying Theorem 6 to (72), we obtain our required estimations (67) and (69). □

Remark 8. When $\beta > 1$. Let $\widetilde{D}(t) = M + \sum_{k=0}^{n-1}(b_k/k!)(t-t_0)^k$; we can obtain the estimations similar to (67) in Theorem 7.

Conflict of Interests

The authors declare that they have no competing interests.

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