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Research Article

On Perturbations of Generators of C_0-Semigroups

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We present a perturbation result for generators of C_0-semigroups which can be considered as an operator theoretic version of the Weiss-Staffans perturbation theorem for abstract linear systems. The results are illustrated by applications to the Desch-Schappacher and the Miyadera-Voigt perturbation theorems and to unbounded perturbations of the boundary conditions of a generator.

1. Introduction

In his classic [1] *"Perturbation Theory for Linear Operators"*, Kato addresses, among others, the following general problem.

Given (unbounded) operators A and P on a Banach space X, how should one define their "sum" A + P and which properties of A are preserved under the perturbation by P?

In the present paper we study this problem in the context of operator semigroups. Given the generator A of a C_0-semigroup on X, for which operators P is the (in a suitable way defined) sum $A + P$ again a generator?

Numerous results are known in this field (see, e.g., [2, Sections III.1–3 and related notes]), but no unifying and general theory is yet available.

Our aim is to go a step towards a more systematic perturbation theory for such generators. To this end we choose the following setting. For the generator A with domain $D(A) \subset X$ consider perturbations

$$P : D(P) \subset X \longrightarrow X_{-1}^{A}, \tag{1}$$

where X_{-1}^{A} is the extrapolated space associated with A (see [2, Section II.5.a]). The sum is then defined as $A_P := (A_{-1} + P)|_X$; that is,

$$\begin{aligned} &A_P x := A_{-1}x + Px \\ &\text{for } x \in D(A_P) := \{z \in D(P) : A_{-1}z + Pz \in X\}. \end{aligned} \tag{2}$$

For which P remains A_P a generator on X? The bounded perturbation theorem ([2, Section III.1]), the Desch-Schappacher ([2, Section III.3.a]), and the Miyadera-Voigt theorems ([2, Section III.3.c]) give some well-known answers in these cases.

It seems that the Weiss-Staffans theorem on the well-posedness of perturbed linear systems (cf. [3, Theorems 6.1 and 7.2] and [4, Sections 7.1 and 7.4]) is a general result in this direction. In the present paper we formulate and prove this result in a purely operator theoretic way avoiding, in particular, notions like abstract linear systems and Lebesgue- or Yosida-extensions.

More precisely (here we use the notation of Weiss, cf. [3]), the classical Weiss-Staffans theorem starts from an abstract linear system, that is, a quadruple $(\mathbb{T}, \Phi, \Psi, \mathbb{F})$ of operator families verifying a set of functional equations (for the precise definition see [3, Definition 5.1]). It then shows that for an admissible feedback operator K (cf. [3, Definition 3.5]) there exists a unique corresponding closed-loop system $(\mathbb{T}^K, \Phi^K, \Psi^K, \mathbb{F}^K)$. Moreover, it relates the generating operators (A, B, C, D) and (A^K, B^K, C^K, D^K) of these two systems. Since here $\mathbb{T}$ and $\mathbb{T}^K$ are C_0-semigroups with generators A and A^K, respectively, this result implicitly contains a perturbation theorem for generators of C_0-semigroups.

However, to apply this theorem to a perturbed operator A_P as appearing in (2) one first has to construct an abstract linear system with appropriate generating operators and a suitable admissible feedback operator incorporating the unperturbed generator A and the perturbation P. This makes it quite cumbersome to formulate and to apply the Weiss-Staffans theorem as a perturbation result for generators.

For this reason we start directly from a triple (A, B, C) of operators and then give conditions in terms of the semigroup generated by A and the operators B and C implying that A_P for $P = BC$ generates a C_0-semigroup. Even though in our approach it is not necessary, it is nevertheless helpful to interpret the perturbed generator as the state operator of a control system with feedback in order to give some motivation for the various definitions of "admissibility." For this reason in the sequel we use some common terminology from control theory.

More precisely, choose two Banach spaces X and U called *state-* and *observation-/control space*, respectively. (We assume that the observation and control spaces coincide. This is no restriction of generality and somewhat simplifies the presentation.) On these spaces consider the following operators:

(i) $A : D(A) \subset X \to X$, called the *state operator* (of the unperturbed system);

(ii) $B \in \mathscr{L}(U, X_{-1}^A)$, called the *control operator*;

(iii) $C \in \mathscr{L}(Z, U)$, called the *observation operator*,

where A is the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on X. Moreover, $D(C) = Z$ is a Banach space such that

$$X_1^A \overset{c}{\hookrightarrow} Z \overset{c}{\hookrightarrow} X, \tag{3}$$

where "$\overset{c}{\hookrightarrow}$" denotes a continuous linear injection and X_1^A is the domain $D(A)$ equipped with the graph norm. Then consider the linear control system

$$\Sigma(A, B, C) \quad \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & t \geq 0, \\ y(t) = Cx(t), & t \geq 0, \\ x(0) = x_0. \end{cases} \tag{4}$$

The solution of $\Sigma(A, B, C)$ is formally given by the variation of parameters formula

$$x(t) = T(t)x_0 + \int_0^t T_{-1}(t - s)Bu(s)\,ds. \tag{5}$$

Closing this system by putting $u(t) = y(t)$, one formally obtains the perturbed abstract Cauchy problem

$$\begin{aligned} \dot{x}(t) &= (A_{-1} + BC)x(t), \quad t \geq 0, \\ x(0) &= x_0, \end{aligned} \tag{6}$$

which is well-posed in X if and only if A_P for $P := BC \in \mathscr{L}(Z, X_{-1}^A)$ is a generator on X (cf. [2, Section II.6]).

Before elaborating this idea, we give a short summary of this paper.

Section 2 is dedicated to the notions of *admissibility* for control, observation, feedback, and pairs of operators. In Section 3 we state and prove the main results, that is, Theorems 10 and 14. In Section 4 we show how the Desch-Schappacher and Miyadera-Voigt theorems easily follow from Theorem 14 and give an application to the perturbation of the boundary condition of a generator in the spirit of Greiner [5].

2. Admissibility

Being only interested in the generator property of $A + P$ for some perturbation P, we can in the sequel assume without loss of generality that the growth bound $\omega_0(A) < 0$ and hence

$$0 \in \rho(A). \tag{7}$$

Taking $C = 0$ in the system $\Sigma(A, B, C)$ and considering the initial value $x_0 = 0$ it is natural to ask that for every control function $u \in L^p([0, t_0], U)$ one obtains a state $x(t_0) \in X$ for some/all $t_0 > 0$. Hence formula (5) is leading to the following definition (cf. [6, Definition 4.1], see also [7]).

Definition 1. The control operator $B \in \mathscr{L}(U, X_{-1}^A)$ is called *p-admissible* for some $1 \leq p < +\infty$ if there exists $t_0 > 0$ such that

$$\int_0^{t_0} T_{-1}(t_0 - s)Bu(s)\,ds \in X, \quad \forall u \in L^p([0, t_0], U). \tag{8}$$

Note that (8) becomes less restrictive for growing $p \in [1, +\infty)$.

Remark 2. The range condition (8) in the previous definition means that the operator $\mathscr{B}_{t_0} : L^p([0, t_0], U) \to X_{-1}^A$ given by

$$\mathscr{B}_{t_0}u := \int_0^{t_0} T_{-1}(t_0 - s)Bu(s)\,ds, \qquad u \in L^p([0, t_0], U) \tag{9}$$

has range $\mathrm{rg}(\mathscr{B}_{t_0}) \subseteq X$. Since obviously $\mathscr{B}_{t_0} \in \mathscr{L}(L^p([0, t_0], U), X_{-1}^A)$, the closed graph theorem implies that for admissible B the *controllability map* $\mathscr{B}_{t_0}$ belongs to $\mathscr{L}(L^p([0, t_0], U), X)$. On the other hand, using integration by parts, it follows that for every $u \in W^{1,p}([0, t_0], U)$

$$\begin{aligned} &\int_0^{t_0} T_{-1}(t_0 - s)Bu(s)\,ds \\ &\quad = A_{-1}^{-1}\Big(T_{-1}(t_0)Bu(0) - Bu(t_0) \\ &\qquad + \int_0^{t_0} T_{-1}(t_0 - s)Bu'(s)\,ds\Big) \in X. \end{aligned} \tag{10}$$

Since $W^{1,p}([0, t_0], U)$ is dense in $L^p([0, t_0], U)$, this shows that the range condition (8) is equivalent to the existence of some $M \geq 0$ such that

$$\begin{aligned} &\left\| \int_0^{t_0} T_{-1}(t_0 - s)Bu(s)\,ds \right\|_X \\ &\quad \leq M \cdot \|u\|_p, \quad \forall u \in W^{1,p}([0, t_0], U). \end{aligned} \tag{11}$$

Next, consider $\Sigma(A, B, C)$ with $B = 0$. Then it is reasonable to ask that every initial value $x_0 \in D(A)$ gives rise to an observation $y(\cdot) = CT(\cdot)x_0 \in L^p([0, t_0], U)$ for some/all $t_0 > 0$ which also depends continuously on x_0. This yields the following definition (cf. [8, Definition 6.1], see also [7]).

Definition 3. The observation operator $C \in \mathscr{L}(Z, U)$ is called *p-admissible* for some $1 \le p < +\infty$ if there exist $t_0 > 0$ and $M \ge 0$ such that

$$\int_0^{t_0} \|CT(s)x\|_U^p \, ds \le M \cdot \|x\|_X^p, \quad \forall x \in D(A). \tag{12}$$

Note that (12) becomes more restrictive for growing $p \in [1, +\infty)$.

Remark 4. The norm condition (12) in the previous definition combined with the denseness of $D(A) \subset X$ implies that there exists an *observability map* $\mathscr{C}_{t_0} \in \mathscr{L}(X, L^p([0, t_0], U))$ satisfying $\|\mathscr{C}_{t_0}\| \le M$ such that

$$(\mathscr{C}_{t_0} x)(s) = CT(s)x, \quad \forall x \in D(A), \; s \in [0, t_0]. \tag{13}$$

Finally, consider the system $\Sigma(A, B, C)$ with (possibly nonzero) p-admissible control and observation operators B and C. The following compatibility condition is needed to proceed (cf. [9, Section II.A]). For more information and several related conditions see [10, Theorem 5.8] and [4, Definition 5.1.1]. Recall that $Z = D(C)$.

Definition 5. The triple (A, B, C) (or the system $\Sigma(A, B, C)$) is called *compatible* if for some $\lambda \in \rho(A)$ we have

$$\operatorname{rg}(R(\lambda, A_{-1})B) \subset Z. \tag{14}$$

If the inclusion (14) holds for some $\lambda \in \rho(A)$, then it holds for all $\lambda \in \rho(A)$ by the resolvent identity. Moreover, the closed graph theorem implies that the operator

$$CR(\lambda, A_{-1})B \in \mathscr{L}(U), \quad \forall \lambda \in \rho(A). \tag{15}$$

Consider now a compatible control system $\Sigma(A, B, C)$ with initial value $x_0 = 0$. Then the input-output map of $\Sigma(A, B, C)$ which maps a control $u(\cdot)$ to the corresponding observation $y(\cdot)$ by (5) is formally given by

$$u(\cdot) \longmapsto y(\cdot) = C \int_0^{\cdot} T_{-1}(\cdot - s) Bu(s) \, ds. \tag{16}$$

Of course, the right-hand side does not in general make sense for arbitrary $u \in L^p([0, t_0], U)$ since the integral might give values $\notin Z = D(C)$. However, if

$$\begin{aligned} u &\in W_0^{2,p}([0, t_0], U) \\ &:= \{u \in W^{2,p}([0, t_0], U) : u(0) = u'(0) = 0\}, \end{aligned} \tag{17}$$

then integrating by parts twice and using (14) one obtains

$$\begin{aligned} &\int_0^r T_{-1}(r - s) Bu(s) \, ds \\ &= -A_{-1}^{-1} \Big(-Bu(r) + A_{-1}^{-1} Bu'(r) \\ &\qquad - \int_0^r T(r - s) A_{-1}^{-1} Bu''(s) \, ds \Big) \in Z. \end{aligned} \tag{18}$$

At this point it is reasonable to ask that the input-output map is continuous. This gives rise to the following definition.

Definition 6. The pair $(B, C) \in \mathscr{L}(U, X_{-1}^A) \times \mathscr{L}(Z, U)$ (or the system $\Sigma(A, B, C)$) is called *jointly p-admissible* for some $1 \le p < +\infty$ if B is a p-admissible control operator and C is a p-admissible observation operator and there exist $t_0 > 0$ and $M \ge 0$ such that

$$\begin{aligned} &\int_0^{t_0} \left\| C \int_0^r T_{-1}(r - s) Bu(s) \, ds \right\|_U^p dr \\ &\le M \cdot \|u\|_p^p, \quad \forall u \in W_0^{2,p}([0, t_0], U). \end{aligned} \tag{19}$$

Remark 7. If $\Sigma(A, B, C)$ is jointly p-admissible, then there exists a bounded *input-output map*

$$\begin{aligned} &\mathscr{F}_{t_0} \in \mathscr{L}(L^p([0, t_0], U)), \quad \text{such that} \\ &(\mathscr{F}_{t_0} u)(\cdot) = C \int_0^{\cdot} T_{-1}(\cdot - s) Bu(s) \, ds, \\ &\qquad \forall u \in W_0^{2,p}([0, t_0], U). \end{aligned} \tag{20}$$

We need a further definition.

Definition 8. An operator $F \in \mathscr{L}(U)$ is called a *p-admissible* feedback operator for some $1 \le p < +\infty$ if there exists $t_0 > 0$ such that $\mathrm{Id} - F\mathscr{F}_{t_0} \in \mathscr{L}(L^p([0, t_0], U))$ is invertible.

Note that $F = \mathrm{Id} \in \mathscr{L}(U)$ is admissible if $\|\mathscr{F}_{t_0}\| < 1$. For further reference we summarize some of the previous notions in a single notation.

Definition 9. Let A be the generator of a C_0-semigroup on a Banach space X, $B \in \mathscr{L}(U, X_{-1}^A)$ and $C \in \mathscr{L}(Z, U)$ for a Banach space Z satisfying $X_1^A \overset{c}{\hookrightarrow} Z \overset{c}{\hookrightarrow} X$. Then $P := BC \in \mathscr{L}(Z, X_{-1}^A)$ is called a *Weiss-Staffans perturbation for A* if for some $1 \le p < \infty$ the following hold:

(i) (A, B, C) is a compatible triple;

(ii) B is a p-admissible control operator;

(iii) C is a p-admissible observation operator;

(iv) (B, C) is a p-admissible pair;

(v) $\mathrm{Id} \in \mathscr{L}(U)$ is a p-admissible feedback operator.

3. The Weiss-Staffans Perturbation Theorem

In this section we state and prove the main results of this paper. These results can be considered as purely operator theoretic versions of perturbation theorems for abstract linear systems due to Weiss [3, Theorems 6.1 and 7.2 (1994)] in the Hilbert space case and Staffans [4, Theorems 7.1.2 and 7.4.5 (2005)] for Banach spaces. In particular, our approach avoids the use of the notions of abstract linear systems and Lebesgue extensions which are not needed if one is only interested in generators. For related results see also [11] and [12, Theorems 4.2 and 4.3].

Theorem 10. *Assume that $P = BC \in \mathscr{L}(Z, X_{-1}^{A})$ is a Weiss-Staffans perturbation for A. This means that there exist $1 \leq p < +\infty$, $t_0 > 0$ and $M \geq 0$ such that*

$$\begin{aligned}
&\text{(i)}\quad rg(R(\lambda, A_{-1})B) \subset Z, \quad \text{for some } \lambda \in \rho(A),\\
&\text{(ii)}\quad \int_0^{t_0} T_{-1}(t_0 - s)Bu(s)\,ds \in X, \quad \forall u \in L^p([0,t_0],U),\\
&\text{(iii)}\quad \int_0^{t_0} \|CT(s)x\|_U^p\,ds \leq M \cdot \|x\|_X^p, \quad \forall x \in D(A),\\
&\text{(iv)}\quad \int_0^{t_0} \left\| C\int_0^r T_{-1}(r-s)Bu(s)\,ds \right\|_U^p dr \leq M \cdot \|u\|_p^p \quad \forall u \in W_0^{2,p}([0,t_0],U),\\
&\text{(v)}\quad 1 \in \rho(\mathscr{F}_{t_0}), \quad \text{where } \mathscr{F}_{t_0} \in \mathscr{L}(L^p([0,t_0],U))
\end{aligned} \tag{21}$$

is given by (20). Then

$$\begin{aligned}
A_{BC} &:= (A_{-1} + BC)|_X,\\
D(A_{BC}) &:= \{x \in Z : (A_{-1} + BC)x \in X\},
\end{aligned} \tag{22}$$

generates a C_0-semigroup $(S(t))_{t\geq 0}$ on the Banach space X. Moreover, the perturbed semigroup $(S(t))_{t\geq 0}$ verifies the variation of parameters formula

$$S(t)x = T(t)x + \int_0^t T_{-1}(t-s)\cdot BC\cdot S(s)x\,ds \quad \forall t \geq 0,\ x \in D(A_{BC}). \tag{23}$$

For the proof we extend the controllability-, observability-, and input-output maps introduced in Remarks 2, 4, and 7 on $\mathbb{R}_+$ as follows. Recall that by assumption $\omega_0(A) < 0$.

Lemma 11. *Let (A, B, C) be compatible and (B, C) jointly p-admissible for some $1 \leq p < +\infty$. Then there exist*

(i) *a strongly continuous, uniformly bounded family $(\mathscr{B}_t)_{t\geq 0} \subset \mathscr{L}(L^p([0,+\infty),U),X)$;*

(ii) *a bounded operator $\mathscr{C}_\infty \in \mathscr{L}(X, L^p([0,+\infty),U))$;*

(iii) *a bounded operator $\mathscr{F}_\infty \in \mathscr{L}(L^p([0,+\infty),U))$,*

such that

$$\begin{aligned}
\mathscr{B}_t u &= \int_0^t T_{-1}(t_0 - s)Bu(s)\,ds, \quad \forall u \in L^p([0,+\infty),U),\\
(\mathscr{C}_\infty x)(s) &= CT(s)x, \quad \forall x \in D(A),\ s \in [0,+\infty),\\
(\mathscr{F}_\infty u)(\cdot) &= C\int_0^{\cdot} T_{-1}(\cdot - s)Bu(s)\,ds \quad \forall u \in W_0^{2,p}([0,+\infty),U).
\end{aligned} \tag{24}$$

Proof. The assertion for $(\mathscr{B}_t)_{t\geq 0}$ was proved in [13, Corollary 3.16]. The assertion for $\mathscr{C}_\infty$ was shown in [13, Lemma 3.9]. Finally, the assertion for $\mathscr{F}_\infty$ follows from [13, Remark 3.23]. □

For $\mu \geq 0$ we indicate in the sequel the controllability-, observability-, and input-output maps associated with the triple $(A - \mu, B, C)$ with the superscript "μ", for example,

$$(\mathscr{F}_\infty^\mu u)(\cdot) = C\int_0^{\cdot} e^{-\mu(\cdot - s)}T_{-1}(\cdot - s)Bu(s)\,ds, \quad \forall u \in W_0^{2,p}([0,+\infty),U). \tag{25}$$

Lemma 12 gives a condition such that the invertibility of $I - \mathscr{F}_{t_0}$ (see condition (v) of Theorem 10) implies the one of $I - \mathscr{F}_\infty^\mu$ for μ sufficiently large.

Lemma 12. *Let the assumptions of Theorem 10 be satisfied. If for $\mu \geq 0$*

$$\left\| T(t_0) + \mathscr{B}_{t_0}\left(1 - \mathscr{F}_{t_0}\right)^{-1}\mathscr{C}_{t_0} \right\| < e^{\mu t_0} \tag{26}$$

holds, then $1 \in \rho(\mathscr{F}_\infty^\mu)$.

Proof. Inspired by [14, (2.6)] and the proof of [15, Proposition 2.1] consider for $n \in \mathbb{N}$ the surjective isometry (denote by v^T the transposed vector of a vector v)

$$\begin{aligned}
J : L^p([0,nt_0],U) &\longrightarrow \prod_{k=1}^{n} L^p([0,t_0],U),\\
u &\longmapsto (u_1,\ldots,u_n)^T,
\end{aligned} \tag{27}$$

where $u_k : [0,t_0] \to U$, $u_k(s) := u((k-1)t_0 + s)$, and $\left\|(u_1,\ldots,u_n)^T\right\|_p^p := \sum_{k=1}^n \|u_k\|^p$.

Then $\mathscr{F}_{nt_0}$ is isometrically isomorphic to the matrix

$$J\mathscr{F}_{nt_0}J^{-1} = \begin{pmatrix}
\mathscr{F}_{t_0} & 0 & 0 & \cdots & \cdots & 0\\
\mathscr{C}_{t_0}T(t_0)^0\mathscr{B}_{t_0} & \mathscr{F}_{t_0} & 0 & \ddots & & \vdots\\
\mathscr{C}_{t_0}T(t_0)^1\mathscr{B}_{t_0} & \mathscr{C}_{t_0}\mathscr{B}_{t_0} & \ddots & \ddots & \ddots & \vdots\\
\vdots & \ddots & \ddots & \ddots & 0 & 0\\
\vdots & & \ddots & \mathscr{C}_{t_0}\mathscr{B}_{t_0} & \mathscr{F}_{t_0} & 0\\
\mathscr{C}_{t_0}T(t_0)^{n-2}\mathscr{B}_{t_0} & \cdots & \cdots & \mathscr{C}_{t_0}T(t_0)\mathscr{B}_{t_0} & \mathscr{C}_{t_0}\mathscr{B}_{t_0} & \mathscr{F}_{t_0}
\end{pmatrix}. \tag{28}$$

Since by assumption $1-\mathcal{F}_{t_0}$ is invertible, $1-\mathcal{F}_{nt_0}$ is invertible as well and

$$J\left(1-\mathcal{F}_{nt_0}\right)^{-1}J^{-1} = \begin{pmatrix} \mathcal{G} & 0 & 0 & \cdots & \cdots & 0 \\ \mathcal{G}\mathcal{C}_{t_0}\left(T(t_0)+\mathcal{B}_{t_0}\mathcal{G}\mathcal{C}_{t_0}\right)^0\mathcal{B}_{t_0}\mathcal{G} & \mathcal{G} & 0 & \ddots & & \vdots \\ \mathcal{G}\mathcal{C}_{t_0}\left(T(t_0)+\mathcal{B}_{t_0}\mathcal{G}\mathcal{C}_{t_0}\right)^1\mathcal{B}_{t_0}\mathcal{G} & \mathcal{G}\mathcal{C}_{t_0}\mathcal{B}_{t_0}\mathcal{G} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & 0 \\ \vdots & & \ddots & \mathcal{G}\mathcal{C}_{t_0}\mathcal{B}_{t_0}\mathcal{G} & \mathcal{G} & 0 \\ \mathcal{G}\mathcal{C}_{t_0}\left(T(t_0)+\mathcal{B}_{t_0}\mathcal{G}\mathcal{C}_{t_0}\right)^{n-2}\mathcal{B}_{t_0}\mathcal{G} & \cdots & \cdots & \mathcal{G}\mathcal{C}_{t_0}\left(T(t_0)+\mathcal{B}_{t_0}\mathcal{G}\mathcal{C}_{t_0}\right)\mathcal{B}_{t_0}\mathcal{G} & \mathcal{G}\mathcal{C}_{t_0}\mathcal{B}_{t_0}\mathcal{G} & \mathcal{G} \end{pmatrix}, \tag{29}$$

where we put $\mathcal{G} := (1-\mathcal{F}_{t_0})^{-1}$. By Lemma A.1 applied to $J(1-\mathcal{F}_{nt_0})^{-1}J^{-1}$ one obtains the estimate

$$\begin{aligned}\left\|\left(1-\mathcal{F}_{nt_0}\right)^{-1}\right\| \le \|\mathcal{G}\| + \left\|\mathcal{G}\mathcal{C}_{t_0}\right\| \cdot \left\|\mathcal{B}_{t_0}\mathcal{G}\right\| \\ \cdot \sum_{l=1}^{n-1}\left\|\left(T(t_0)+\mathcal{B}_{t_0}\mathcal{G}\mathcal{C}_{t_0}\right)\right\|^{l-1}.\end{aligned} \tag{30}$$

This shows that $\|(1-\mathcal{F}_{nt_0})^{-1}\|$ remains bounded as $n \to +\infty$ if (26) holds for $\mu = 0$.

If the estimate (26) only holds for some $\mu > 0$, consider the triple $(A-\mu, B, C)$. Let $M_{\varepsilon_\mu} \in \mathcal{L}(L^p([0,t_0],U))$ be the multiplication operator defined by

$$\left(M_{\varepsilon_\mu}u\right)(s) := e^{\mu s}\cdot u(s), \quad u \in L^p([0,t_0],U). \tag{31}$$

Then M_{ε_μ} is invertible with inverse $M_{\varepsilon_{-\mu}}$ and a simple computation shows that

$$\begin{aligned}\mathcal{B}_{t_0}^{\mu} &= e^{-\mu t_0}\mathcal{B}_{t_0}M_{\varepsilon_\mu}, \\ \mathcal{C}_{t_0}^{\mu} &= M_{\varepsilon_\mu}^{-1}\mathcal{C}_{t_0}, \\ \mathcal{F}_{t_0}^{\mu} &= M_{\varepsilon_\mu}^{-1}\mathcal{F}_{t_0}M_{\varepsilon_\mu}.\end{aligned} \tag{32}$$

By similarity this implies that $1 \in \rho(\mathcal{F}_{t_0}^{\mu})$. Hence, repeating the above reasoning for $(A-\mu, B, C)$ one obtains from (30) that $\|(1-\mathcal{F}_{nt_0}^{\mu})^{-1}\|$ remains bounded as $n \to +\infty$ if

$$\left\|e^{-\mu t_0}T(t_0) + \mathcal{B}_{t_0}^{\mu}\left(1-\mathcal{F}_{t_0}^{\mu}\right)^{-1}\mathcal{C}_{t_0}^{\mu}\right\| < 1. \tag{33}$$

Since by (32) one has

$$\begin{aligned}&e^{-\mu t_0}T(t_0) + \mathcal{B}_{t_0}^{\mu}\left(1-\mathcal{F}_{t_0}^{\mu}\right)^{-1}\mathcal{C}_{t_0}^{\mu} \\ &\quad = e^{-\mu t_0}\left(T(t_0) + \mathcal{B}_{t_0}\left(1-\mathcal{F}_{t_0}\right)^{-1}\mathcal{C}_{t_0}\right),\end{aligned} \tag{34}$$

the estimates (33) and (26) are equivalent. Summing up this shows that (26) implies that

$$K := \sup_{n\in\mathbb{N}}\left\|\left(1-\mathcal{F}_{nt_0}^{\mu}\right)^{-1}\right\| < +\infty. \tag{35}$$

Using this fact we finally show that $1 \in \rho(\mathcal{F}_{\infty}^{\mu})$. Observe first that $(1-\mathcal{F}_{\infty}^{\mu})u = 0$ for some $u \in L^p([0,+\infty),U)$ implies that $(1-\mathcal{F}_{nt_0}^{\mu})(u|_{[0,nt_0]}) = 0$ for every $n \in \mathbb{N}$. Since $(1-\mathcal{F}_{nt_0}^{\mu})$ is injective for every $n \in \mathbb{N}$, this gives that $u = 0$; that is, $1-\mathcal{F}_{\infty}^{\mu}$ is injective.

To show surjectivity fix some $v \in L^p([0,+\infty),U)$ and define for $n \in \mathbb{N}$

$$u_n := \left(1-\mathcal{F}_{nt_0}^{\mu}\right)^{-1}\left(v|_{[0,nt_0]}\right) \in L^p([0,nt_0],U); \tag{36}$$

that is, u_n is the unique solution in $L^p([0,nt_0],U)$ of the equation

$$\left(1-\mathcal{F}_{nt_0}^{\mu}\right)u = v|_{[0,nt_0]}. \tag{37}$$

However, for $m \ge n$ one has $(\mathcal{F}_{mt_0}^{\mu}u_m)|_{[0,nt_0]} = \mathcal{F}_{nt_0}^{\mu}(u_m|_{[0,nt_0]})$; hence also $u_m|_{[0,nt_0]} \in L^p([0,nt_0],U)$ solves (37). This implies that

$$u_m|_{[0,nt_0]} = u_n. \tag{38}$$

Thus one can define

$$u(s) := \lim_{n\to+\infty}u_n(s), \quad s \in [0,+\infty). \tag{39}$$

Since by (35) it follows that $\|u_n\| \le K\cdot\|v\|$ for all $n \in \mathbb{N}$, Fatou's lemma implies that $u \in L^p([0,+\infty),U)$. Moreover, by construction

$$\left(\left(1-\mathcal{F}_{\infty}^{\mu}\right)u\right)|_{[0,nt_0]} = \left(1-\mathcal{F}_{nt_0}^{\mu}\right)u_n = v|_{[0,nt_0]} \quad \forall n \in \mathbb{N}, \tag{40}$$

which implies $(1-\mathcal{F}_{\infty}^{\mu})u = v$. Since $v \in L^p([0,t_0],U)$ was arbitrary, this shows that $1-\mathcal{F}_{\infty}^{\mu}$ is surjective. Hence $1-\mathcal{F}_{\infty}^{\mu}$ is bijective and therefore $1 \in \rho(\mathcal{F}_{\infty}^{\mu})$ as claimed. □

Next we show that the invertibility of $\mathrm{Id}-\mathcal{F}_{\infty}^{\mu}$ implies for sufficiently large λ the invertibility of the *"transfer function"* $\mathrm{Id} - CR(\lambda, A_{-1})B$ of the system $\Sigma(A,B,C)$ with feedback $u(t) = y(t)$. In the following the Laplace transform of a function u is denoted by

$$(\mathcal{L}u)(\lambda) := \hat{u}(\lambda) := \int_0^{+\infty}e^{-\lambda r}u(r)\,dr. \tag{41}$$

Lemma 13. *Assume that* $1 \in \rho(\mathcal{F}_\infty^\mu)$ *for some* $\mu \geq 0$. *Then* $1 \in \rho(CR(\lambda, A_{-1})B)$ *for all* $\lambda \in \mathbb{C}$ *satisfying* $\operatorname{Re}\lambda > \mu$ *and*

$$\mathcal{L}\left((Id - \mathcal{F}_\infty^\mu)^{-1}u\right)(\lambda) = \left(Id - CR\left(\lambda, A_{-1}\right)B\right)^{-1} \cdot \hat{u}(\lambda) \qquad \forall u \in L^p([0,+\infty), U). \tag{42}$$

Proof. Assume first that $\mu = 0$. Then it is well known that $\mathcal{F}_\infty = \mathcal{F}_\infty^\mu$ is shift invariant (cf. [16]); that is, $\mathcal{F}_\infty$ commutes with the right shift. Then also $\mathcal{G} := \mathrm{Id} - \mathcal{F}_\infty \in \mathcal{L}(L^p([0,+\infty),U))$ is shift invariant and by [16, Theorem 2.3] and [13, Lemma 3.19] one obtains for $u \in L^p([0,+\infty),U)$

$$\widehat{(\mathcal{G}u)}(\lambda) = \left(\mathrm{Id} - CR\left(\lambda, A_{-1}\right)B\right) \cdot \hat{u}(\lambda), \quad \operatorname{Re}\lambda > 0. \tag{43}$$

Let $\mathcal{R} := \mathcal{G}^{-1} \in \mathcal{L}(L^p([0,+\infty),U))$. Then clearly the right shift also commutes with $\mathcal{R}$; that is, this operator is shift invariant as well. Hence again by [16, Theorem 2.3] there exists $R(\lambda) \in \mathcal{L}(U)$ such that

$$\widehat{(\mathcal{R}u)}(\lambda) = R(\lambda) \cdot \hat{u}(\lambda), \quad \operatorname{Re}\lambda > 0, \qquad u \in L^p([0,+\infty), U). \tag{44}$$

Summing up one obtains for all $u \in L^p([0,+\infty),U)$

$$\begin{aligned}\hat{u}(\lambda) &= \widehat{(\mathcal{R}\mathcal{G}u)}(\lambda) = R(\lambda) \cdot \widehat{(\mathcal{G}u)}(\lambda) \\ &= R(\lambda) \cdot \left(\mathrm{Id} - CR\left(\lambda, A_{-1}\right)B\right) \\ &\quad \cdot \hat{u}(\lambda) = \widehat{(\mathcal{G}\mathcal{R}u)}(\lambda) \\ &= \left(\mathrm{Id} - CR\left(\lambda, A_{-1}\right)B\right) \cdot \widehat{(\mathcal{R}u)}(\lambda) \\ &= \left(\mathrm{Id} - CR\left(\lambda, A_{-1}\right)B\right) \cdot R(\lambda) \cdot \hat{u}(\lambda).\end{aligned} \tag{45}$$

Taking $u(s) = e^{-s}v$ for some $v \in U$, this implies

$$\begin{aligned}\frac{1}{1+\lambda} \cdot v &= R(\lambda) \cdot \left(\mathrm{Id} - CR\left(\lambda, A_{-1}\right)B\right) \cdot \frac{1}{1+\lambda} \cdot v \\ &= \left(\mathrm{Id} - CR\left(\lambda, A_{-1}\right)B\right) \cdot R(\lambda) \cdot \frac{1}{1+\lambda} \cdot v, \\ &\qquad \operatorname{Re}\lambda > 0.\end{aligned} \tag{46}$$

Hence $R(\lambda) = (\mathrm{Id} - CR(\lambda, A_{-1})B)^{-1}$.

If $\mu > 0$, then by the same reasoning applied to $\mathcal{F}_\infty^\mu$ one obtains that

$$\begin{aligned}1 &\in \rho\left(CR\left(\lambda, A_{-1} - \mu\right)B\right) \\ &= \rho\left(CR\left(\lambda + \mu, A_{-1}\right)B\right), \quad \forall \operatorname{Re}\lambda > 0.\end{aligned} \tag{47}$$

Clearly this implies our claim in case $\mu > 0$ and the proof is complete. □

We are now well prepared to prove the main result of this section.

Proof of Theorem 10. The idea of the proof is to define an operator family $(S(t))_{t\geq 0} \subset \mathcal{L}(X)$ and then to verify that it is a C_0-semigroup with generator A_{BC}.

To this end, assume that the condition (26) in Lemma 12 holds for $\mu = 0$. Then $\mathrm{Id} - \mathcal{F}_\infty$ is invertible and one can define for $t \geq 0$

$$S(t) := T(t) + \mathcal{B}_t\left(\mathrm{Id} - \mathcal{F}_\infty\right)^{-1}\mathcal{C}_\infty \in \mathcal{L}(X). \tag{48}$$

Since $(T(t))_{t\geq 0}$ and $(\mathcal{B}_t)_{t\geq 0}$ are both strongly continuous and uniformly bounded, the same holds for $(S(t))_{t\geq 0}$. We proceed and compute the Laplace transform of $S(\cdot)x : [0,+\infty) \to X$ for $x \in X$. Since

$$S(\cdot)x = T(\cdot)x + T_{-1}(\cdot)B * \left(1 - \mathcal{F}_\infty\right)^{-1}\mathcal{C}_\infty x, \tag{49}$$

the convolution theorem for the Laplace transform (or [13, Lemma 3.12]) and Lemma 13 imply for every $x \in X$ and $\operatorname{Re}\lambda > 0$

$$\begin{aligned}\mathcal{L}(S(\cdot)x)(\lambda) &= R(\lambda, A)x + R\left(\lambda, A_{-1}\right)B \\ &\quad \cdot \mathcal{L}\left(\left(1 - \mathcal{F}_\infty\right)^{-1}\mathcal{C}_\infty x\right)(\lambda) \\ &= R(\lambda, A)x + R\left(\lambda, A_{-1}\right)B \\ &\quad \cdot \left(\mathrm{Id} - CR\left(\lambda, A_{-1}\right)B\right)^{-1} \\ &\quad \cdot CR(\lambda, A)x =: Q(\lambda)x.\end{aligned} \tag{50}$$

We now show that $Q(\lambda) = R(\lambda, A_{BC})$. First note that by the compatibility condition (14) one has

$$\operatorname{rg}(Q(\lambda)) \subset D(A) + Z = Z = D(C). \tag{51}$$

Moreover,

$$\begin{aligned}&\left(\lambda - A_{-1} - BC\right) \cdot Q(\lambda) \\ &\quad = \mathrm{Id} - BCR(\lambda, A) + B \cdot \mathrm{Id} \\ &\qquad \cdot \left(\mathrm{Id} - CR\left(\lambda, A_{-1}\right)B\right)^{-1} \\ &\qquad \cdot CR(\lambda, A) - B \cdot CR\left(\lambda, A_{-1}\right)B \\ &\qquad \cdot \left(\mathrm{Id} - CR\left(\lambda, A_{-1}\right)B\right)^{-1}CR(\lambda, A) = \mathrm{Id}.\end{aligned} \tag{52}$$

This implies that $Q(\lambda)$ is a right inverse and $\operatorname{rg}(Q(\lambda)) \subset D(A_{BC})$. To show that it is also a left inverse take $x \in D(A_{BC}) \subset Z = D(C)$. Then we obtain

$$\begin{aligned}&Q(\lambda) \cdot \left(\lambda - A_{-1} - BC\right)x \\ &\quad = x - R\left(\lambda, A_{-1}\right)BCx + R\left(\lambda, A_{-1}\right) \\ &\qquad \cdot B\left(\mathrm{Id} - CR\left(\lambda, A_{-1}\right)B\right)^{-1} \cdot \mathrm{Id} \\ &\qquad \cdot Cx - R\left(\lambda, A_{-1}\right) \cdot B\left(\mathrm{Id} - CR\left(\lambda, A_{-1}\right)B\right)^{-1} \\ &\qquad \cdot CR\left(\lambda, A_{-1}\right)B \cdot Cx = x,\end{aligned} \tag{53}$$

and hence it follows that $Q(\lambda) = R(\lambda, A_{BC})$ as claimed. Summing up we showed that $(S(t))_{t\geq 0} \subset \mathscr{L}(X)$ is a strongly continuous family with Laplace transform $R(\lambda, A_{BC})$. By [17, Theorem 3.1.7] this implies that $(S(t))_{t\geq 0}$ is a C_0-semigroup with generator A_{BC}.

To verify the variation of parameters formula (23) one first notes that by Lemma 13 and the explicit representation of $R(\lambda, A_{BC})$ in (50) one has for all $x \in D(A_{BC})$ and $\operatorname{Re}\lambda > \mu = 0$ that

$$\mathscr{L}\left(\left(1-\mathscr{F}_{\infty}\right)^{-1}\mathscr{C}_{\infty}(\cdot)\,x\right)(\lambda) = \mathscr{L}\left(CS(\cdot)\,x\right)(\lambda). \tag{54}$$

By the uniqueness of the Laplace transform this implies that

$$\left(1-\mathscr{F}_{\infty}\right)^{-1}\mathscr{C}_{\infty}(\cdot)\,x = CS(\cdot)\,x, \tag{55}$$

and the assertion follows from the definition of $(S(t))_{t\geq 0}$ in (49).

Now assume that (26) only holds for some $\mu > 0$. Then repeating the same reasoning for the triple $(A-\mu, B, C)$ one concludes as before that $(A-\mu)_{BC} = ((A-\mu)_{-1} + BC)|_X = A_{BC} - \mu$ is a generator. Clearly this implies that A_{BC} generates a strongly continuous semigroup $(S(t))_{t\geq 0}$. Moreover, one obtains that the pair of rescaled semigroups $(e^{-\mu t}T(t))_{t\geq 0}$ and $(e^{-\mu t}S(t))_{t\geq 0}$ verify the variation of parameters formula (23) which implies that this formula holds for the pair $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ as well. □

As already remarked in the introduction, with increasing $p \in [1, +\infty)$ the p-admissibility of the control and observation operator becomes weaker and stronger, respectively.

Assuming that the input-output map maps L^{α} to L^{β} for some (the main cases we have in mind are $\alpha < p = \beta < +\infty$ and $1 < \alpha = p < \beta$) $1 \leq \alpha \leq p \leq \beta < +\infty$ satisfying $\alpha < \beta$, one can drop the invertibility condition $1 \in \rho(\mathscr{F}_{t_0})$ in Theorem 10 (and sometimes even the compatibility condition (14), cf. Remark 15).

Theorem 14. *Assume that conditions (i)–(iii) in Theorem 10 are satisfied. Moreover, suppose there exist* $1 \leq \alpha \leq p \leq \beta < \infty$ *with* $\alpha < \beta$, $p > 1$, *and* $M \geq 0$ *such that*

$$\begin{aligned}\left(\text{iv}'\right) \quad \int_0^{t_0}\left\|C\int_0^r T_{-1}(r-s)\,Bu(s)\,ds\right\|_U^{\beta} dr \leq M\cdot\|u\|_{\alpha}^{\beta} \\ \forall u \in W_0^{2,\alpha}\left(\left[0,t_0\right],U\right).\end{aligned} \tag{56}$$

Then A_{BC} given by (22) generates a C_0-semigroup $(S(t))_{t\geq 0}$ on the Banach space X verifying the variation of parameters formula (23).

Proof. By Theorem 10 it suffices to show that $1 \in \rho(\mathscr{F}_{t_1})$ for some $t_1 > 0$. By assumption the operator

$$\begin{aligned}&\widetilde{F}_t : W_0^{2,\alpha}\left([0,t],U\right) \subset L^{\alpha}\left([0,t],U\right) \longrightarrow L^{\beta}\left([0,t],U\right), \\ &\widetilde{F}_t u := C\int_0^{\cdot} T_{-1}(\cdot - s)\,Bu(s)\,ds\end{aligned} \tag{57}$$

has a bounded extension $F_t \in \mathscr{L}(L^{\alpha}([0,t],U), L^{\beta}([0,t],U))$ for every $t \in (0, t_0]$. We distinguish 2 cases and use in both of them Jensen's inequality as follows:

$$\|u\|_r \leq t^{(1/r)-(1/s)}\cdot\|u\|_s, \tag{58}$$

for $1 \leq r < s \leq +\infty$ and $u \in L^s([0,t],U) \subset L^r([0,t],U)$.

Case $\alpha < p$. Then F_t belongs to $\mathscr{L}(L^{\alpha}([0,t],U), L^{p}([0,t],U))$ with norm (denote the norm of a bounded linear operator $F : L^r \to L^s$ by $\|F\|_{rs}$) $\|F_t\|_{\alpha p} \leq \|F_{t_0}\|_{\alpha p}$. This implies, by (58) for $r = \alpha$ and $s = p$, that

$$\left\|F_t u\right\|_p \leq \left\|F_t\right\|_{\alpha p}\cdot\|u\|_{\alpha} \leq t^{(1/\alpha)-(1/p)}\cdot\left\|F_{t_0}\right\|_{\alpha p}\cdot\|u\|_p. \tag{59}$$

Case $p < \beta$. In this case, $F_t \in \mathscr{L}(L^{p}([0,t],U), L^{\beta}([0,t],U))$ with norm $\|F_t\|_{p\beta} \leq \|F_{t_0}\|_{p\beta}$. This implies, by (58) for $r = p$ and $s = \beta$, that

$$\begin{aligned}\left\|F_t u\right\|_p &\leq t^{(1/p)-(1/\beta)}\cdot\left\|F_t u\right\|_{\beta} \\ &\leq t^{(1/p)-(1/\beta)}\cdot\left\|F_{t_0}\right\|_{p\beta}\cdot\|u\|_p.\end{aligned} \tag{60}$$

Hence in both cases, considering $\mathscr{F}_t := F_t|_{L^p([0,t],U)} \in \mathscr{L}(L^p([0,t],U))$, one concludes that there exists $t_1 > 0$ such that $\|\mathscr{F}_{t_1}\| < 1$ which implies $1 \in \rho(\mathscr{F}_{t_1})$. □

Remark 15. Assume as in Theorem 14 that $1 \leq \alpha \leq p \leq \beta \leq +\infty$ with $\alpha < \beta$ and $1 < p < +\infty$. If there exist $t_0 > 0$ and a dense subspace $D \subset L^{\alpha}([0,t_0],U)$ such that for every $u \in D$

(i) $\int_0^r T_{-1}(r-s)Bu(s)\,ds \in Z$ for almost all $0 < r \leq t_0$;

(ii) the map $[0,t_0] \ni r \mapsto C\int_0^r T_{-1}(r-s)Bu(s)\,ds$ is in $L^{\beta}([0,t_0],U)$;

(iii) there exists $M \geq 0$ such that

$$\begin{aligned}\int_0^{t_0}\left\|C\int_0^r T_{-1}(r-s)\,Bu(s)\,ds\right\|_U^{\beta} dr \\ \leq M\cdot\|u\|_{\alpha}^{\beta}, \quad \forall u \in D,\end{aligned} \tag{61}$$

then also the compatibility condition (14) is satisfied.

To verify this assertion define $\widetilde{F}_t : D \subset L^{\alpha}([0,t],U) \to L^{\beta}([0,t],U)$ as in (57) with $W_0^{2,\alpha}([0,t],U)$ replaced by the space D. By assumption, $\widetilde{F}_t$ has a unique bounded extension $F_t : L^{\alpha}([0,t],U) \to L^{\beta}([0,t],U)$. As above take $\mathscr{F}_t := F_t|_{L^p([0,t],U)} \in \mathscr{L}(L^p([0,t],U))$. Then, by Hölder's inequality (or (58) for $r = 1$ and $s = \beta$), one obtains for every $v \in U$ (define $(f \otimes x)(s) := f(s)\cdot x$ for all $s \in [0,t_0]$ where

$f : [0,t_0] \to \mathbb{C}$ is a scalar function; moreover, by $\mathbb{1}$ denote the constant one function on the interval $[0,t_0]$)

$$\begin{aligned}
\left\| \frac{1}{t} \int_0^t (\mathcal{F}_t \mathbb{1} \otimes v)(s)\, ds \right\| &\le \frac{1}{t} \int_0^t \left\| (\mathcal{F}_t \mathbb{1} \otimes v)(s) \right\| ds = \frac{1}{t} \cdot \left\| F_t \mathbb{1} \otimes v \right\|_1 \\
&\le \frac{1}{t} \cdot t^{1-(1/\beta)} \cdot \left\| F_t \mathbb{1} \otimes v \right\|_\beta \\
&\le t^{-1/\beta} \cdot \left\| F_t \right\|_{\alpha\beta} \cdot \left\| \mathbb{1} \otimes v \right\|_\alpha \\
&\le t^{(1/\alpha)-(1/\beta)} \cdot \left\| F_{t_0} \right\|_{\alpha\beta} \cdot \|v\|_U \longrightarrow 0, \quad \text{as } t \longrightarrow 0^+.
\end{aligned} \tag{62}$$

By [10, Theorem 5.8] in the Hilbert space case or [4, Theorems 5.6.4 and 5.6.5] in the general case this convergence implies the compatibility condition (14).

4. Applications

We now give some applications of our abstract results. First we show that Theorem 14 can be considered as a simultaneous generalization of the Desch-Schappacher and the Miyadera-Voigt perturbation theorems. Moreover, we generalize a result of Greiner concerning the perturbation of the boundary conditions of a generator.

4.1. The Desch-Schappacher Perturbation Theorem. The following result was proved in [18, Theorem 5, Proposition 8]; see also [2, Corollary III.3.4] and [19, Corollary 5.5.1].

Theorem 16 (see [18]). *Assume that for $B \in \mathcal{L}(X, X_{-1}^A)$ there exist $1 \le p < +\infty$, $t_0 > 0$, and $M \ge 0$ such that*

$$\int_0^{t_0} T_{-1}(t_0 - s) Bu(s)\, ds \in X, \quad \forall u \in L^p([0,t_0], X). \tag{63}$$

Then $(A_B, D(A_B))$ given by

$$\begin{aligned}
A_B x &:= (A_{-1} + B) x, \\
D(A_B) &:= \{x \in X : (A_{-1} + B) x \in X\},
\end{aligned} \tag{64}$$

is the generator of a C_0-semigroup $(S(t))_{t\ge0}$ on X.

Remark that one could consider the condition (63) also for $p = \infty$ or $u \in C([0,t_0], U)$. However, in this case one needs an additional norm estimate to ensure that condition (v) in Theorem 10 is satisfied (cf. [2, Corollary III.3.3.]). Moreover, note that in a certain sense the Desch-Schappacher theorem depends only on the range but not on the "size" of the perturbation B. In particular, if B satisfies the assumption of Theorem 16, then also BF satisfies it for every $F \in \mathcal{L}(X)$.

Proof of Theorem 16. Let $U = Z = X$ and $C = \mathrm{Id}$. Then by assumption $B \in \mathcal{L}(X, X_{-1}^A)$ is a p-admissible control operator and conditions (i)–(iii) in Theorem 10 are clearly satisfied. We will prove that (ii) implies condition (iv$'$) from Theorem 14. To this end we first verify that the function

$$[0,t_0] \ni r \longmapsto v(r) := \int_0^r T_{-1}(r-s) Bu(s)\, ds \in X \tag{65}$$

is continuous for every $u \in L^p([0,t_0], X)$. For such u define $u_t : [0,t_0] \to U$ by

$$u_t(s) := \begin{cases} 0 & \text{if } 0 \le s \le t_0 - t \\ u(s - t_0 + t) & \text{if } t_0 - t < s \le t_0; \end{cases} \tag{66}$$

that is, u_t is just the right translation of u by $t_0 - t$. Then $u_t \in L^p([0,t_0], X)$ and using Remark 2 one obtains from $v(r) = \mathcal{B}_{t_0} u_r$ that for $r_0, r_1 \in [0,t_0]$

$$\begin{aligned}
\|v(r_0) - v(r_1)\| &= \left\| \mathcal{B}_{t_0} (u_{r_0} - u_{r_1}) \right\| \\
&\le \left\| \mathcal{B}_{t_0} \right\| \cdot \left\| u_{r_0} - u_{r_1} \right\|_p \\
&\longrightarrow 0, \quad \text{as } r_1 \longrightarrow r_0,
\end{aligned} \tag{67}$$

where the last step follows from the strong continuity of the nilpotent right translation semigroup on $L^p([0,t_0], X)$. Next define the operator

$$\begin{aligned}
F_{t_0} &: L^p([0,t_0], X) \longrightarrow C([0,t_0], X), \\
(F_{t_0} u)(r) &:= \int_0^r T_{-1}(r-s) Bu(s)\, ds, \\
& r \in [0,t_0].
\end{aligned} \tag{68}$$

The operator F_{t_0} is well-defined. Moreover, the estimate

$$\begin{aligned}
\left\| (F_{t_0} u)(r) \right\| &\le \left\| \mathcal{B}_{t_0} \right\| \cdot \|u_r\|_p \le \left\| \mathcal{B}_{t_0} \right\| \cdot \|u\|_p \\
& \forall u \in L^p([0,t_0], X), \quad r \in [0,t_0],
\end{aligned} \tag{69}$$

shows that $F_{t_0} \in \mathcal{L}(L^p([0,t_0], X), C([0,t_0], X)) \subset \mathcal{L}(L^p([0,t_0], X), L^\beta([0,t_0], X))$ for all $\beta \ge 1$. Choosing $\beta > p$ this implies condition (iv$'$) and hence the proof is complete. □

Remark 17. The proofs of Theorems 14 and 16 imply the following: if $B \in \mathcal{L}(U, X_{-1}^A)$ is a p-admissible control operator for some $1 \le p < +\infty$ then for every bounded $C \in \mathcal{L}(X, U)$ the triple (A, B, C) is compatible and jointly p-admissible. Moreover, in this case every $F \in \mathcal{L}(U)$ is a p-admissible feedback operator for the system $\Sigma(A, B, C)$.

4.2. The Miyadera-Voigt Perturbation Theorem. As another application we consider the following version of the Miyadera-Voigt perturbation theorem (cf. [20, 21], see also [2, Corollary III.3.16] and [19, Theorem 5.4.2]).

Theorem 18 (see [20, 21]). *Assume that for $C \in \mathcal{L}(X_1^A, X)$ there exist $1 < p < +\infty$, $t_0 > 0$, and $M \ge 0$ such that*

$$\int_0^{t_0} \|CT(s)x\|_X^p\, ds \le M \cdot \|x\|_X^p, \quad \forall x \in D(A). \tag{70}$$

Then $(A_C, D(A_C))$ *given by*

$$\begin{aligned} A_C x &:= (A + C)\, x, \\ D(A_C) &:= D(A), \end{aligned} \tag{71}$$

is the generator of a C_0*-semigroup on* X.

Observe that one could consider condition (70) also for $p = 1$. However, in this case one needs $M < 1$ to ensure that condition (v) in Theorem 10 is satisfied (cf. [2, Corollary III.3.16]). Moreover, note that in a certain sense the Miyadera-Voigt Theorem 18 (for $p > 1$) depends only on the domain but not on the "size" of the perturbation C. In particular, if C satisfies the assumption of Theorem 18, then also FC satisfies it for every $F \in \mathscr{L}(X)$.

Proof of Theorem 18. Let $U = X$, $Z = X_1^A$, and $B = \mathrm{Id}$. Then, by assumption, $C \in \mathscr{L}(Z, X)$ is a p-admissible observation operator and conditions (i)–(iii) in Theorem 10 are clearly satisfied. We will show that condition (iii) implies condition (iv′) from Theorem 14. To this end fix $0 \le \gamma < \delta \le t_0$ and $x \in D(A)$. Then for $u = \mathbb{1}_{[\gamma,\delta]} \otimes x$ one obtains

$$\begin{aligned} C \int_0^r T(r-s)\, u(s)\, ds &= CA^{-1} \int_\gamma^r \mathbb{1}_{[\gamma,\delta]}(s) \cdot T(r-s)\, Ax\, ds \\ &= \int_\gamma^r \mathbb{1}_{[\gamma,\delta]}(s) \cdot CT(r-s)\, x\, ds. \end{aligned} \tag{72}$$

Using (72), condition (iii), the triangle and Hölder's inequality for $f \in L^1(a,b)$ and $p \ge 1$

$$\begin{aligned} &\int_0^{t_0} \left\| C \int_0^r T(r-s)\, u(s)\, ds \right\|_X^p dr \\ &\le \int_\gamma^{t_0} \left(\int_\gamma^r \mathbb{1}_{[\gamma,\delta]}(s) \cdot \| CT(r-s)\, x \|_X\, ds \right)^p dr \\ &= \int_\gamma^\delta \left(\int_\gamma^r \| CT(r-s)\, x \|_X\, ds \right)^p dr \\ &\quad + \int_\delta^{t_0} \left(\int_\gamma^\delta \| CT(r-s)\, x \|_X\, ds \right)^p dr \\ &\le \int_\gamma^\delta (r-\gamma)^{p-1} \int_\gamma^r \| CT(r-s)\, x \|_X^p\, ds\, dr \\ &\quad + \int_\delta^{t_0} \int_\gamma^\delta (\delta-\gamma)^{p-1} \| CT(r-s)\, x \|_X^p\, ds\, dr \\ &\le \int_\gamma^\delta (r-\gamma)^{p-1} M \cdot \|x\|_X^p\, dr \\ &\quad + \int_\gamma^\delta (\delta-\gamma)^{p-1} \int_\delta^{t_0} \| CT(r-s)\, x \|_X^p\, dr\, ds \\ &\le \frac{M}{p} \cdot (\delta-\gamma)^p \cdot \|x\|_X^p \\ &\quad + \int_\gamma^\delta (\delta-\gamma)^{p-1} M \cdot \|x\|_X^p\, ds \\ &= M \left(1 + \frac{1}{p} \right) \cdot (\delta-\gamma)^p \cdot \|x\|_X^p \\ &=: K^p \cdot (\delta-\gamma)^p \cdot \|x\|_X^p. \end{aligned} \tag{73}$$

Let now $u = \sum_{k=1}^n \mathbb{1}_{[\gamma_k,\delta_k]} \otimes x_k \in L^1([0,t_0], X)$ be a step function where the intervals $[\gamma_k, \delta_k] \subset [0, t_0]$ are pairwise disjoint and $x_k \in D(A)$ for $k = 1, \dots, n$. Then from (73) one obtains

$$\begin{aligned} &\left(\int_0^{t_0} \left\| C \int_0^r T(r-s)\, u(s)\, ds \right\|_X^p dr \right)^{1/p} \\ &\quad \le K \cdot \sum_{k=1}^n (\delta_k - \gamma_k) \cdot \|x_k\|_X = K \cdot \|u\|_1. \end{aligned} \tag{74}$$

Since the step functions having values in $D(A)$ are dense in $L^1([0,t_0], X)$, this implies condition (iv′) for $\alpha = 1$ and $\beta = p$. This completes the proof. □

Remark 19. The proofs of Theorems 14 and 18 imply the following: if $C \in \mathscr{L}(Z, U)$ is a p-admissible observation operator for some $1 \le p < +\infty$ then for every bounded $B \in \mathscr{L}(U, X)$ the triple (A, B, C) is compatible and jointly p-admissible. Moreover, if $p > 1$ then in addition every $F \in \mathscr{L}(U)$ is a p-admissible feedback operator for the system $\Sigma(A, B, C)$.

4.3. Perturbing the Boundary Conditions of a Generator.

In this section we show how Theorem 10 can be used to generalize significantly the approach by Greiner in [5] to perturbations of boundary conditions of a generator. To explain the general setup we consider the following:

(i) two Banach spaces (in this section denote the elements of X by f instead of x) X and ∂X, the latter called "boundary space";

(ii) a closed, densely defined "maximal" operator ("maximal" concerns the size of the domain, e.g., a differential operator without boundary conditions) $A_m : D(A_m) \subseteq X \to X$;

(iii) the Banach space $[D(A_m)] := (D(A_m), \|\cdot\|_{A_m})$ where $\|f\|_{A_m} := \|f\| + \|A_m f\|$ is the graph norm;

(iv) two "boundary" operators $L, \Phi \in \mathscr{L}([D(A_m)], \partial X)$.

Then define two restrictions $A, A^\Phi \subset A_m$ by

$$\begin{aligned} D(A) &:= \{ f \in D(A_m) : Lf = 0 \} = \ker L, \\ D(A^\Phi) &:= \{ f \in D(A_m) : Lf = \Phi f \}. \end{aligned} \tag{75}$$

In many applications X, ∂X, and $D(A_m)$ are function spaces and L is a "trace-type" operator which restricts a function in $D(A_m)$ to (a part of) the boundary of its domain. Hence one can consider A^Φ with boundary condition $Lf = \Phi f$ as a perturbation of the operator A with abstract "Dirichlet type" boundary condition $Lf = 0$.

In order to treat this setup within our framework we make the following assumptions:

(i) the operator A generates a strongly continuous semigroup $(T(t))_{t\geq 0}$ on X;

(ii) the boundary operator $L : D(A_m) \to \partial X$ is surjective.

Under these assumptions the following lemma, shown by Greiner [5, Lemma 1.2], is the key to write A^Φ as a Weiss-Staffans perturbation of A.

Lemma 20. *Let the above assumptions (i) and (ii) be satisfied. Then for each $\lambda \in \rho(A)$ the operator $L|_{\ker(\lambda - A_m)}$ is invertible and $D_\lambda := (L|_{\ker(\lambda-A_m)})^{-1} : \partial X \to \ker(\lambda - A_m) \subseteq X$ is bounded.*

Using this so-called *Dirichlet operator* D_λ one obtains the following representation of A^Φ where for simplicity we assume that A is invertible.

Lemma 21. *If $0 \in \rho(A)$, then*

$$A^\Phi = \left(A_{-1} - A_{-1}D_0 \cdot \Phi\right)\big|_X; \qquad (76)$$

that is, $A^\Phi = A_{BC}$ for $U := \partial X$, $Z := [D(A_m)]$, and

$$\begin{aligned} B &:= -A_{-1}D_0 \in \mathscr{L}\left(U, X_{-1}^A\right), \\ C &:= \Phi \in \mathscr{L}(Z, U). \end{aligned} \qquad (77)$$

Proof. Denote the operator on the right-hand side of (76) by $\widetilde{A}^\Phi$. Then

$$\begin{aligned} f \in D\left(\widetilde{A}^\Phi\right) &\Longleftrightarrow f - D_0\Phi f \in D(A) \\ &\Longleftrightarrow Lf = LD_0\Phi f = \Phi f \\ &\Longleftrightarrow f \in D\left(A^\Phi\right). \end{aligned} \qquad (78)$$

Moreover, for $f \in D(A^\Phi)$ we have

$$\begin{aligned} \widetilde{A}^\Phi f &= A\left(f - D_0\Phi f\right) = A_m\left(f - D_0\Phi f\right) = A_m f \\ &= A^\Phi f, \end{aligned} \qquad (79)$$

as claimed. □

We mention that in [5, Theorem 2.1] the operator $\Phi \in \mathscr{L}(X, U)$ is bounded and the assumptions imply that $A_{-1}D_0$ is a 1-admissible control operator. Hence in this case A^Φ is a generator by the Desch-Schappacher theorem.

By using Theorem 10 one can now deal also with unbounded Φ.

Corollary 22. *Assume that for some $1 \leq p < +\infty$ the pair $(A_{-1}D_0, \Phi)$ is jointly p-admissible and that $Id \in \mathscr{L}(\partial X)$ is a p-admissible feedback operator for A. Then A^Φ is the generator of a C_0-semigroup on X.*

Proof. One only has to show the compatibility condition (14). This, however, immediately follows from

$$\begin{aligned} \operatorname{rg}\left(R\left(\lambda, A_{-1}\right)B\right) &= \operatorname{rg}\left(\left(\mathrm{Id} - \lambda R(\lambda, A)\right)D_0\right) \\ &\subset \ker\left(A_m\right) + D(A) \subseteq D\left(A_m\right) \\ &= Z. \end{aligned} \qquad (80)$$

□

Remark 23. We note that in [22, Theorem 4.1] the authors study a similar problem in the context of regular linear systems.

As a simple but typical example for the previous corollary consider the space $X := L^p[0,1]$ and the first derivative $A_m := d/ds$ with domain $D(A_m) := W^{1,p}[0,1]$ (c.f. [5, Example 1.1.(c)]). As boundary space choose $\partial X = \mathbb{C}$, as boundary operator the point evaluation $L = \delta_1$ and as perturbation some $\Phi \in \left(W^{1,p}[0,1]\right)'$. This gives rise to the differential operators $A, A^\Phi \subset d/ds$ with domains

$$\begin{aligned} D(A) &:= \left\{f \in W^{1,p}[0,1] : f(1) = 0\right\}, \\ D\left(A^\Phi\right) &:= \left\{f \in W^{1,p}[0,1] : f(1) = \Phi f\right\}. \end{aligned} \qquad (81)$$

Then clearly the assumptions (i) and (ii) made above are satisfied; in particular A generates the nilpotent left-shift semigroup given by

$$\left(T(t)f\right)(s) = \begin{cases} f(s+t) & \text{if } s+t \leq 1, \\ 0 & \text{else.} \end{cases} \qquad (82)$$

However, A^Φ is not always a generator. For example if $\Phi = \delta_1$, then $A^\Phi = A_m$ and $\sigma(A^\Phi) = \mathbb{C}$; hence A^Φ is not a generator. Thus one needs an additional assumption on Φ.

Definition 24. A bounded linear functional $\Phi : C[0,1] \to \mathbb{C}$ has *little mass* in $r = 1$ if there exist $q < 1$ and $\delta > 0$ such that

$$|\Phi f| \leq q \cdot \|f\|_\infty, \qquad (83)$$

for every $f \in C[0,1]$ satisfying $\operatorname{supp} f \subset [1-\delta, 1]$.

Note that $W^{1,p}[0,1] \overset{c}{\hookrightarrow} C[0,1]$ and hence $(C[0,1])' \subset [D(A_m)]'$. Now the following holds.

Corollary 25. *If $\Phi \in (C[0,1])'$ has little mass in $r = 1$, then for all $1 \leq p < +\infty$ the operator A^Φ is the generator of a strongly continuous semigroup on $L^p[0,1]$.*

Proof. By Corollary 22 it suffices to show that for the triple $(A, A_{-1}D_0, \Phi)$ the conditions (ii)–(v) of Theorem 10 are satisfied. To this end, note that $0 \in \rho(A)$ and that the Dirichlet operator $D_0 : \mathbb{C} \to L^p[0,1]$ is given by $D_0\alpha = \alpha \cdot \mathbb{1}$ where $\mathbb{1}(s) = 1$ for all $s \in [0,1]$.

(ii) By Remark 2 it suffices to verify estimate (11) where we may assume that $u \in W_0^{1,p}[0,t_0]$ for some $0 < t_0 \le 1$. Using integration by parts and [23, Theorem 4.2] we conclude that (for a function g defined on an interval denote in the sequel by $\tilde{g}$ its extension to $\mathbb{R}$ by the value 0)

$$\begin{aligned}\int_0^{t_0} T_{-1}(t_0 - s) Bu(s)\,ds &= -\int_0^{t_0} T_{-1}(t_0 - s) A_{-1}D_0 u(s)\,ds \\ &= D_0 u(t_0) - \int_0^{t_0} T(t_0 - s) D_0 u'(s)\,ds \\ &= u(t_0)\cdot\mathbb{1} - \int_0^{t_0} (T(t_0 - s)\mathbb{1})\cdot u'(s)\,ds \\ &= u(t_0)\cdot\mathbb{1} - \int_{\max\{0,\cdot+t_0-1\}}^{t_0} u'(s)\,ds \\ &= u(\max\{0,\cdot+t_0-1\}) = \tilde{u}(\cdot + t_0 - 1).\end{aligned} \tag{84}$$

This implies $\left\|\mathcal{B}_{t_0}u\right\|_X = \left\|\mathcal{B}_{t_0}u\right\|_p \le \|u\|_p$ for all $u \in W_0^{1,p}[0,t_0]$ which shows (ii).

(iii) By the Riesz-Markov representation theorem there exists a regular complex Borel measure μ on $[0,1]$ such that

$$\Phi f = \int_0^1 f(r)\,d\mu(r), \quad \forall f \in C[0,1]. \tag{85}$$

Using Fubini's theorem and Hölder's inequality one obtains for $0 < t_0 \le 1$ and $f \in D(A)$

$$\begin{aligned}\int_0^{t_0} |CT(s)f|^p ds &= \int_0^{t_0} \left|\Phi\tilde{f}(\cdot + s)\right|^p ds \\ &\le \int_0^{t_0} \left(\int_0^1 \left|\tilde{f}(r+s)\right| d|\mu|(r)\right)^p ds \\ &\le \int_0^{t_0} (|\mu|[0,1])^{p-1} \cdot \int_0^1 \left|\tilde{f}(r+s)\right|^p d|\mu|(r)\,ds \\ &= \|\mu\|^{p-1}\cdot\int_0^1\int_0^{t_0}\left|\tilde{f}(r+s)\right|^p ds\,d|\mu|(r) \\ &\le \|\mu\|^p \cdot \|f\|_p^p,\end{aligned} \tag{86}$$

where $\|\mu\| := |\mu|[0,1]$ (which coincides with $\|\Phi\|_\infty$). This proves (iii).

(iv) From (84) one obtains for $0 < t_0 \le 1$ and $u \in W_0^{1,p}[0,t_0]$ by similar arguments as in (iii)

$$\begin{aligned}\int_0^{t_0} \left|C\int_0^r T_{-1}(r-s) Bu(s)\,ds\right|^p dr &= \int_0^{t_0} |\Phi\tilde{u}(\cdot + r - 1)|^p dr \\ &= \int_0^{t_0} \left|\int_{1-r}^1 u(s+r-1)\,d\mu(s)\right|^p dr \\ &\le \int_0^{t_0} (|\mu|[1-r,1])^{p-1} \\ &\quad \cdot \int_{1-r}^1 |u(s+r-1)|^p d|\mu|(s)\,dr \\ &\le (|\mu|[1-t_0,1])^{p-1} \\ &\quad \cdot \int_{1-t_0}^1\int_{1-s}^1 |u(s+r-1)|^p dr\,d|\mu|(s) \\ &\le (|\mu|[1-t_0,1])^p\cdot\|u\|_p^p.\end{aligned} \tag{87}$$

This shows (iv).

(v) Since by assumption Φ has little mass in $r = 1$, it follows that $|\mu|[1-t_0,1] < 1$ for sufficiently small $t_0 > 0$. Hence from estimate (87) and the denseness of $W_0^{1,p}[0,t_0]$ in $L^p[0,t_0]$ it follows that $\left\|\mathcal{F}_{t_0}\right\| \le |\mu|[1-t_0,1] < 1$ for $0 < t_0 \le 1$ sufficiently small. This implies $1 \in \rho(\mathcal{F}_{t_0})$ as claimed. □

Remarks. (i) Corollary 25 could be easily generalized (with essentially the same proof) to the first derivative on $L^p([0,1],\mathbb{C}^n)$. One could even go further and prove a similar result on $L^p([0,1],E)$ for a (possibly infinite dimensional) Banach space E provided the boundary operator Φ has a representation as a Riemann-Stieltjes integral as in (85). See also [22, Example 5.1].

(ii) In most cases the admissibility of the identity as a feedback operator is verified by showing that $\left\|\mathcal{F}_{t_0}\right\| < 1$ for sufficiently small $t_0 > 0$. Choosing $\Phi = \alpha\delta_1$, by (84), one obtains $\mathcal{F}_{t_0} = \alpha\mathrm{Id}$ for all $t_0 > 0$; hence $1 \in \rho(\mathcal{F}_{t_0})$ if and only if $\alpha \ne 1$. This provides an example where our perturbation theorem is applicable even if $\left\|\mathcal{F}_{t_0}\right\| > 1$ for all $t_0 > 0$. Note that for $\alpha = 1$ one obtains $A^\Phi = A_m$; hence in this case A^Φ cannot be a generator.

Appendix

Estimating the p-Norm of a Triangular Toeplitz Matrix

For the proof of Lemma 12 one needed the following result.

Lemma A.1. *For a Banach space X endow $\mathcal{X} := X^n$, $n \in \mathbb{N}$, with the p-norm*

$$\left\|(x_1,\ldots,x_n)^T\right\|_p := \left(\sum_{k=1}^n \|x_k\|^p\right)^{1/p}, \tag{A.1}$$

for some $1 \le p \le +\infty$. *Moreover, let* $T_0, \dots, T_{n-1} \in \mathscr{L}(X)$. *Then the norm of the Toeplitz operator matrix*

$$\mathscr{T} := \left(T_{j-i}\right)_{i,j=1}^{n} = \begin{pmatrix} T_0 & 0 & 0 & \cdots & \cdots & 0 \\ T_1 & T_0 & 0 & \ddots & & \vdots \\ T_2 & T_1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & 0 \\ \vdots & & \ddots & T_1 & T_0 & 0 \\ T_{n-1} & \cdots & \cdots & T_2 & T_1 & T_0 \end{pmatrix}_{n\times n} \in \mathscr{L}(\mathscr{X}), \quad \text{(A.2)}$$

can be estimated as

$$\|\mathscr{T}\| \le \sum_{j=0}^{n-1} \|T_j\|. \quad \text{(A.3)}$$

Proof. Let $x=(x_1,\dots,x_n)^T \in \mathscr{X}$. Then one can estimate

$$\begin{aligned} \|\mathscr{T} x\|_p &= \left(\sum_{j=1}^{n} \left\| \sum_{i=1}^{j} T_{j-i} x_i \right\|^p \right)^{1/p} \\ &\le \left(\sum_{j=1}^{n} \left(\sum_{i=1}^{j} \|T_{j-i}\| \cdot \|x_i\| \right)^p \right)^{1/p} \\ &= \Bigg(\sum_{j=1}^{n} (((\|T_0\|, \|T_1\|, \dots, \|T_{n-1}\|) \\ &\qquad * (\|x_1\|, \|x_2\|, \dots, \|x_n\|))(j))^p \Bigg)^{1/p} \quad \text{(A.4)} \\ &= \|(\|T_0\|, \|T_1\|, \dots, \|T_{n-1}\|) \\ &\qquad * (\|x_1\|, \|x_2\|, \dots, \|x_n\|)\|_p \\ &\le \|(\|T_0\|, \|T_1\|, \dots, \|T_{n-1}\|)\|_1 \\ &\qquad \cdot \|(\|x_1\|, \|x_2\|, \dots, \|x_n\|)\|_p \\ &= \sum_{j=0}^{n-1} \|T_j\| \cdot \|x\|_p, \end{aligned}$$

where the second last step follows from Young's inequality applied to the convolution of sequences. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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