Research Article

Maps Preserving Peripheral Spectrum of Generalized Jordan Products of Self-Adjoint Operators

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Let $\mathscr{A}_1$ and $\mathscr{A}_2$ be standard real Jordan algebras of self-adjoint operators on complex Hilbert spaces H_1 and H_2, respectively. For $k \geq 2$, let $(i_1, \ldots, i_m)$ be a fixed sequence with $i_1, \ldots, i_m \in \{1, \ldots, k\}$ and assume that at least one of the terms in $(i_1, \ldots, i_m)$ appears exactly once. Define the generalized Jordan product $T_1 \circ T_2 \circ \cdots \circ T_k = T_{i_1} T_{i_2} \cdots T_{i_m} + T_{i_m} \cdots T_{i_2} T_{i_1}$ on elements in $\mathscr{A}_i$. Let $\Phi : \mathscr{A}_1 \rightarrow \mathscr{A}_2$ be a map with the range containing all rank-one projections and trace zero-rank two self-adjoint operators. We show that Φ satisfies that $\sigma_\pi(\Phi(A_1) \circ \cdots \circ \Phi(A_k)) = \sigma_\pi(A_1 \circ \cdots \circ A_k)$ for all $A_1, \ldots, A_k$, where $\sigma_\pi(A)$ stands for the peripheral spectrum of A, if and only if there exist a scalar $c \in \{-1, 1\}$ and a unitary operator $U : H_1 \rightarrow H_2$ such that $\Phi(A) = cUAU^*$ for all $A \in \mathscr{A}_1$, or $\Phi(A) = cUA^tU^*$ for all $A \in \mathscr{A}_1$, where A^t is the transpose of A for an arbitrarily fixed orthonormal basis of H_1. Moreover, $c = 1$ whenever m is odd.

1. Introduction

Recently, the study of maps preserving spectrum of products of operators attracted attentions of researchers. In [1], Molnár characterized surjective maps Φ on bounded linear operators acting on a Hilbert space preserving the spectrum of the product of operators; that is, AB and $\Phi(A)\Phi(B)$ always have the same spectrum. This similar question was studied by Huang and Hou in [2] by replacing the spectrum by several spectral functions such as the left spectrum and spectral boundary. Hou et al. [3, 4] studied, respectively, further the maps Φ between certain operator algebras preserving the spectrum of a generalized product $T_1 * T_2 * \cdots * T_k$ and a generalized Jordan product $T_1 \circ T_2 \circ \cdots \circ T_k$ of low rank operators. Note that the linear maps between Banach algebras which preserve the spectrum are extensively studied in connection with a longstanding open problem due to Kaplansky on invertibility preserving linear maps ([5–10] and the references therein).

Moreover, there has been considerable interest in studying peripheral spectrum preserving maps on operator

algebras. Recall that the peripheral spectrum of an element T in a complex Banach algebra $\mathscr{A}$ is defined by

$$\sigma_\pi(T) = \{z \in \sigma(T) : |z| = r(T)\}, \tag{1}$$

where $\sigma(T)$ and $r(T)$ stand for the spectrum and the spectral radius of T, respectively. Recall also that a set-valued map $\Lambda : \mathscr{A} \rightarrow 2^{\mathbb{C}}$ is said to be a spectral function if $\emptyset \neq \Lambda(T) \subseteq \sigma(T)$ for every $T \in \mathscr{A}$. Since $\sigma(T)$ is compact, $\sigma_\pi(T)$ is a well-defined nonempty set and is an important spectral function. Observe that it is always true that $\sigma_\pi(TS) = \sigma_\pi(ST)$. In [11], Tonev and Luttman studied maps preserving peripheral spectrum of the usual operator products on standard operator algebras. Recall that a standard operator algebra is a subalgebra of $\mathscr{B}(X)$ that contains the identity I and all finite rank operators, where $\mathscr{B}(X)$ stands for as usual the Banach algebra of all bounded linear operators on Banach space X. They studied also the corresponding problems in uniform algebras (see [12, 13]). Miura and Honma [14] generalized the result in [13] and characterized surjective maps ϕ and ψ satisfying $\sigma_\pi(\phi(T)\psi(S)) = \sigma_\pi(TS)$ on standard operator

algebras. Cui and Li studied in [15] the maps preserving peripheral spectrum of Jordan products $AB + BA$ of operators on standard operator algebras. In [16] the maps preserving peripheral spectrum of Jordan semitriple products BAB of operators were characterized. The authors studied in [17, 18], respectively, further the maps between certain operator algebras which preserve peripheral spectrum of a generalized product $T_1 * T_2 * \cdots * T_k$ and a generalized Jordan product $T_1 \circ T_2 \circ \cdots \circ T_k$ as defined below.

Definition 1. Fix a positive integer $k \geq 2$ and a finite sequence $(i_1, i_2, \ldots, i_m)$ such that $\{i_1, i_2, \ldots, i_m\} = \{1, 2, \ldots, k\}$ and there is an i_p not equal to i_q for all other q; that is, i_p appears just one time in the sequence. For operators $T_1, \ldots, T_k$, the operators,

$$T_1 * T_2 * \cdots * T_k = T_{i_1} T_{i_2} \cdots T_{i_m}, \qquad (2)$$

$$T_1 \circ T_2 \circ \cdots \circ T_k = T_{i_1} T_{i_2} \cdots T_{i_m} + T_{i_m} \cdots T_{i_2} T_{i_1} \qquad (3)$$

are, respectively, called generalized product and generalized Jordan product of $T_1, \ldots, T_k$, while m is called the width of the products.

Evidently, the generalized Jordan product $T_1 \circ \cdots \circ T_k$ (the generalized product $T_1 * \cdots * T_k$) covers the Jordan product $T_1 T_2 + T_2 T_1$ and the Jordan triple product $T_1 T_2 T_3 + T_3 T_2 T_1$ (the usual product $T_1 T_2$ and the Jordan semitriple product $T_1 T_2 T_1$), and so forth. We also remark that the notations $T_1 * T_2 * \cdots * T_k$ and $T_1 \circ T_2 \circ \cdots \circ T_k$ are not unique for $T_1, T_2, \ldots, T_k$ because they depend on the choice of the integers $k \geq 2$, $m \geq 2$, and the sequence $(i_1, i_2, \ldots, i_m)$. In this paper, we presume that k, m, and the sequence $(i_1, i_2, \ldots i_m)$ are arbitrary but fixed throughout the paper.

Let us consider the case of Hilbert spaces. Denote by $\mathscr{B}(H)$ the set of all bounded linear operators on a complex Hilbert space H and T^* the adjoint of $T \in \mathscr{B}(H)$. If $T = T^*$, T is self-adjoint. Denote by $\mathscr{B}_s(H)$ the real Jordan algebra of all self-adjoint operators in $\mathscr{B}(H)$. A real Jordan subalgebra of $\mathscr{B}_s(H)$ is said to be standard if it contains the identity I and all finite rank self-adjoint operators. In [14] Miura and Honma characterized the surjective maps between standard operator algebras on Hilbert spaces that preserve the peripheral spectrum of skew products $T^* S$ of operators. Cui and Li studied in [15] the maps preserving peripheral spectrum of skew Jordan products $AB^* + B^* A$ of operators on standard operator algebras on complex Hilbert spaces. A characterization of maps preserving peripheral spectrum of Jordan products of self-adjoint operators $AB + BA$ on standard real Jordan subalgebras of $\mathscr{B}_s(H)$ was also given in [15]. In [16] the maps preserving peripheral spectrum of Jordan skew semitriple products $BA^* B$ of operators were characterized, and then, the maps preserving peripheral spectrum of the skew generalized products of operators on Hilbert space H were characterized in [17].

Products of self-adjoint operators in Hilbert space play a role in several different areas of pure and applied mathematics. In this paper, we characterize the maps preserving the peripheral spectrum of generalized Jordan products of self-adjoint operators between the standard real Jordan algebras of self-adjoint operators on complex Hilbert spaces. Let $\mathscr{A}_i$

be a standard real Jordan algebra in $\mathscr{B}_s(H_i)$, $i = 1, 2$, and $\Phi : \mathscr{A}_1 \rightarrow \mathscr{A}_2$ a map with range containing all rank-one projections and all rank-two self-adjoint operators with zero trace. We show that Φ satisfies that $\sigma_\pi(\Phi(A_1) \circ \cdots \circ \Phi(A_k)) = \sigma_\pi(A_1 \circ \cdots \circ A_k)$ for all $A_1, \ldots, A_k$ in $\mathscr{A}_1$ if and only if there exist a scalar $c \in \{-1, 1\}$ and a unitary operator $U : H_1 \rightarrow H_2$ such that $\Phi(A) = cUAU^*$ for all $A \in \mathscr{A}_1$, or $\Phi(A) = cUA^tU^*$ for all $A \in \mathscr{A}_1$, where A^t is the transpose of A with respect to an arbitrary but fixed orthonormal basis of H_1. Moreover, $c = 1$ whenever m is odd. We also characterize the maps from $\mathscr{A}_1$ into $\mathscr{A}_2$ that preserves the peripheral spectrum of generalized product on $\mathscr{A}_i$.

2. Generalized Jordan Products of Self-Adjoint Operators

Let H_1 and H_2 be two complex Hilbert spaces and $\mathscr{B}_s(H_1)$ and $\mathscr{B}_s(H_2)$ the real linear spaces of all self-adjoint operators in $\mathscr{B}(H_1)$ and $\mathscr{B}(H_2)$, respectively. Then $\mathscr{B}_s(H_1)$ and $\mathscr{B}_s(H_2)$ are real Jordan algebras. Recall that a standard real Jordan algebra on H_i is a Jordan subalgebra of $\mathscr{B}_s(H_i)$ which contains all finite rank self-adjoint operators and the identity operator. In this section, we will characterize maps preserving peripheral spectrum of generalized Jordan products of self-adjoint operators.

Theorem 2. *Let $\mathscr{A}_1$ and $\mathscr{A}_2$ be standard real Jordan algebras of self-adjoint operators on complex Hilbert spaces H_1 and H_2, respectively. Consider the product $T_1 \circ \cdots \circ T_k$ defined in (3) of Definition 1 with the width m. Assume that $\Phi : \mathscr{A}_1 \rightarrow \mathscr{A}_2$ is a map the range of which contains all rank-one projections and all rank-two self-adjoint operators with zero trace. Then Φ satisfies*

$$\sigma_\pi\left(\Phi\left(A_1\right) \circ \cdots \circ \Phi\left(A_k\right)\right) = \sigma_\pi\left(A_1 \circ \cdots \circ A_k\right) \qquad (4)$$

for all $A_1, A_2, \ldots, A_k \in \mathscr{A}_1$ if and only if there exist a unitary operator $U \in \mathscr{B}(H_1, H_2)$ and a scalar $c \in \{-1, 1\}$ such that either

(1) *$\Phi(A) = cUAU^*$ for every $A \in \mathscr{A}_1$, or*

(2) *$\Phi(A) = cUA^tU^*$ for every $A \in \mathscr{A}_1$. Here A^t is the transpose of A with respect to an arbitrary but fixed orthonormal basis of H_1.*

Moreover, $c = 1$ whenever m is odd.

To prove Theorem 2, we consider the special case that $A_{i_p} = A$ and $A_{i_q} = B$ for all $q \neq p$. Thus there exist nonnegative integers r, s with $r + s = m - 1 \geq 1$ such that $A_1 \circ A_2 \circ \cdots \circ A_k = B^r AB^s + B^s AB^r$. For this special case we have.

Theorem 3. *Let $\mathscr{A}_1$ and $\mathscr{A}_2$ be standard real Jordan algebras of self-adjoint operators on complex Hilbert spaces H_1 and H_2, respectively. Assume that $\Phi : \mathscr{A}_1 \rightarrow \mathscr{A}_2$ is a map the range of which contains all rank-one projections and all rank-two*

self-adjoint operators with zero trace, and r, s are nonnegative integers with $r + s \geq 1$. Then Φ satisfies

$$
\sigma_\pi \left(B^r A B^s + B^s A B^r \right)
$$
$$
= \sigma_\pi \left(\Phi(B)^r \Phi(A) \Phi(B)^s + \Phi(B)^s \Phi(A) \Phi(B)^r \right) \tag{5}
$$

for all $A, B \in \mathscr{A}_1$ if and only if there exist a unitary operator $U \in \mathscr{B}(H_1, H_2)$ and a scalar $c \in \{-1, 1\}$ such that $\Phi(A) = cUAU^*$ for every $A \in \mathscr{A}_1$, or $\Phi(A) = cUA^tU^*$ for every $A \in \mathscr{A}_1$. Moreover, $c = 1$ whenever $r + s$ is even. Here A^t is the transpose of A with respect to an arbitrary but fixed orthonormal basis of H_1.

If Φ meets (4), then it also meets (5) for some r, s with $r + s = m - 1$ by taking $A_{i_p} = A$ and $A_{i_q} = B$ for $q \neq p$. Hence it is obvious that the truth of Theorem 3 will imply the truth of Theorem 2.

Thus we focus our attention to prove Theorem 3. We will do it by decomposing the proof in a number of steps and use of technical lemmas.

Note that, if $s = r > 0$, then the question is reduced to the generalized product $B^r A B^r$ of self-adjoint operators, which will be discussed in the next section. So, unless specified otherwise, we always assume in this section that $s > r \geq 0$.

Lemma 4. *For any unit vector $x \in H$ and nonzero $B \in \mathscr{B}_s(H)$, we have*

$$
\sigma_\pi \left(B^r x \otimes xB^s + B^s x \otimes xB^r \right)
$$
$$
= \begin{cases} \left\{ \langle B^{r+s}x, x \rangle + \|B^r x\| \|B^s x\| \right\} & \text{if } \langle B^{r+s}x, x \rangle > 0; \\ \left\{ \langle B^{r+s}x, x \rangle - \|B^r x\| \|B^s x\| \right\} & \text{if } \langle B^{r+s}x, x \rangle < 0; \\ \left\{ \pm \|B^r x\| \|B^s x\| \right\} & \text{if } \langle B^{r+s}x, x \rangle = 0. \end{cases} \tag{6}
$$

Proof. In fact, if there exist nonzero $\alpha, \beta \in \mathbb{R}$ such that $B^r x = \alpha x$, $B^s x = \beta x$, clearly (6) holds. Now assume that $B^r x$ and x or $B^s x$ and x are linearly independent. Then there exist nonzero $\gamma \in \mathbb{R}$ and $z \in H$ such that $(B^r x \otimes xB^s + B^s x \otimes xB^r)z = \gamma z$; that is,

$$
\langle B^s z, x \rangle B^r x + \langle B^r z, x \rangle B^s x = \gamma z. \tag{7}
$$

It follows that

$$
\langle B^s z, x \rangle \langle B^r x, x \rangle + \langle B^r z, x \rangle \langle B^s x, x \rangle = \gamma \langle z, x \rangle, \tag{8}
$$

$$
\langle B^s z, x \rangle \langle B^{r+s} x, x \rangle + \langle B^r z, x \rangle \langle B^{2s} x, x \rangle = \gamma \langle B^s z, x \rangle, \tag{9}
$$

$$
\langle B^r z, x \rangle \langle B^{r+s} x, x \rangle + \langle B^s z, x \rangle \langle B^{2r} x, x \rangle = \gamma \langle B^r z, x \rangle. \tag{10}
$$

We consider the following two cases.

Case 1 ($\langle B^{r+s}x, x \rangle = 0$). If $\langle B^r z, x \rangle \neq 0$, it follows from (10) that $\langle B^s z, x \rangle \neq 0$. Then (9) and (10) imply that $\gamma = \pm \|B^r x\| \|B^s x\|$. If $\langle B^r z, x \rangle = 0$, it follows from (9) that

$\langle B^s z, x \rangle = 0$, but this contradicts (7). So $\sigma_\pi(B^r x \otimes xB^s + B^s x \otimes xB^r) = \{\pm \|B^r x\| \|B^s x\|\}$.

Case 2 ($\langle B^{r+s}x, x \rangle \neq 0$). In this case, there must be $\langle B^r z, x \rangle \neq 0$ and $\langle B^s z, x \rangle \neq 0$. Then it follows from (9) and (10) that

$$
\left(\gamma - \langle B^{r+s}x, x \rangle \right)^2 = \|B^r x\|^2 \|B^s x\|^2, \tag{11}
$$

which implies that $\gamma = \langle B^{r+s}x, x \rangle \pm \|B^r x\| \|B^s x\|$. So

$$
\sigma \left(B^r x \otimes xB^s + B^s x \otimes xB^r \right) = \left\{ 0, \langle B^{r+s}x, x \rangle \pm \|B^r x\| \|B^s x\| \right\}. \tag{12}
$$

Now the result follows immediately. $\square$

In Lemmas 5 and 6, we always assume that $\Phi : \mathscr{A}_1 \to \mathscr{A}_2$ is a map satisfying (5) with range containing all rank-one projections and all rank-two self-adjoint operators of zero trace, and assume that r, s are nonnegative integers with $r+s \geq 1$. Recall that a self-adjoint operator A is said to be positive, denote by $A \geq 0$, if $\langle Ax, x \rangle \geq 0$ for all $x \in H$; while $A \geq B$ means that $A - B \geq 0$.

Lemma 5. *$\Phi(I) = I$ or $-I$. $\Phi(I) = -I$ may occur only if $r + s$ is odd.*

Proof. For any $A, B \in \mathscr{A}_1$, since

$$
\sigma_\pi \left(B^r A B^s + B^s A B^r \right)
$$
$$
= \sigma_\pi \left(\Phi(B)^r \Phi(A) \Phi(B)^s + \Phi(B)^s \Phi(A) \Phi(B)^r \right), \tag{13}
$$

it follows that $r(A) = r(\Phi(A))$ holds for every $A \in \mathscr{A}_1$. Let $\Phi(I) = B$. By the assumption on the range of Φ, for any unit vector $y \in H_2$, there exists $A \in \mathscr{A}_1$ such that $\Phi(A) = y \otimes y$. We consider the following two cases.

Case 1 ($s > r = 0$). It follows from (5) that

$$
\sigma_\pi \left(2A^s \right) = \sigma_\pi \left(\Phi(A)^s \Phi(I) + \Phi(I) \Phi(A)^s \right)
$$
$$
= \sigma_\pi \left(y \otimes yB + By \otimes y \right), \tag{14}
$$

which implies that $\|By \otimes y + y \otimes yB\| = 2$ for all unit vectors $y \in H_2$. Then by Lemma 4, we have

$$
2 = \|By \otimes y + y \otimes yB\| = |\langle By, y \rangle| + \|By\| \leq 2\|By\|, \tag{15}
$$

and hence $\|By\| \geq 1$ for all unit vectors $y \in H_2$, and $\|B\| \geq 1$. On the other hand, for any unit vector $y \in H_2$, we have $2|\langle By, y \rangle| \leq |\langle By, y \rangle| + \|By\| = 2$. Hence $|\langle By, y \rangle| \leq 1$ holds for all unit vectors $y \in H_2$ and consequently, $\|B\| \leq 1$. So we must have $\|B\| = 1$ and $\|By\| = 1$ for all unit vectors $y \in H_2$. Now it follows from (15) that $B = \varepsilon I$ with $\varepsilon \in \{-1, 1\}$. In particular, if s is even, as $A^s \geq 0$, (14) and (15) imply that $\langle By, y \rangle = 1$ for all unit vectors $y \in H_2$ and hence $B = I$.

Case 2 ($s > r > 0$). By (5) we have

$$
\sigma_\pi \left(2A^{r+s} \right) = \sigma_\pi \left(\Phi(A)^s \Phi(I) \Phi(A)^r + \Phi(A)^r \Phi(I) \Phi(A)^s \right)
$$
$$
= \sigma_\pi \left(2y \otimes yBy \otimes y \right), \tag{16}
$$

which implies that $\|y \otimes yBy \otimes y\| = \|A^{r+s}\| = 1$ for all unit vectors $y \in H_2$. Then $|\langle By, y \rangle| = 1$ holds for each unit vector y, and so $B = \varepsilon I$ with $\varepsilon \in \{-1, 1\}$. Particularly, if $r + s$ is even, then $A^{r+s} \geq 0$ and it follows from (16) that $\langle By, y \rangle = 1$ holds for every unit vector y. Hence $B = I$. $\qquad \square$

If $\Phi(I) = -I$, then $-\Phi$ satisfies the conditions in Theorem 3, so we may as well assume $\Phi(I) = I$ in the sequel, and thus $\sigma_\pi(A) = \sigma_\pi(\Phi(A))$ holds for every $A \in \mathscr{A}_1$.

Lemma 6. Φ *preserves rank-one projections in both directions.*

Proof. We consider the following two cases.

Case 1 ($s > r = 0$). Consider the following.

Case 1.1 (s is even). For any unit vector $x \in H_1$, let $\Phi(x \otimes x) = B$ and $\Phi(I - x \otimes x) = T$. It follows from $\{0\} = \sigma_\pi((x \otimes x)^s(I - x \otimes x) + (I - x \otimes x)(x \otimes x)^s) = \sigma_\pi(B^sT + TB^s)$ that $B^sT + TB^s = 0$.

Note that if $A \geq 0$, and $AS + SA = 0$, then $AS = SA = 0$. If fact $AS + SA = 0$ implies that $A^2S = SA^2$. Since $A \geq 0$, we must have $AS = SA$ and $2AS = AS + SA = 0$, which forces $AS = SA = 0$.

Now, as $B^s \geq 0$ and $B^sT + TB^s = 0$, we see that $B^sT = TB^s = 0$. It follows from $\sigma_\pi(B) = \{1\}$ that $B^s \neq 0$, which implies that $\{0\} \neq \operatorname{ran} B^s \subseteq \ker T$, where $\operatorname{ran} T$ stands for the range of T. For any unit vector $y \in \ker T$, pick $A \in \mathscr{A}_1$ such that $\Phi(A) = y \otimes y$. It follows from $\sigma_\pi((I - x \otimes x)A^s + A^s(I - x \otimes x)) = \sigma_\pi(Ty \otimes y + y \otimes yT) = \{0\}$ that $(I - x \otimes x)A^s + A^s(I - x \otimes x) = 0$, which, together with $A^s \geq 0$, implies that $(I - x \otimes x)A^s = A^s(I - x \otimes x) = 0$. So we have $A = x \otimes x$ and $\Phi(x \otimes x) = y \otimes y$ is rank-one.

Case 1.2 (s is odd). For any unit vector $x \in H_1$, let $A = x \otimes x$ and $\Phi(A) = B$. We will prove that B is a rank-one projection.

Claim 1.2.1 ($\dim \ker(B - I) = 1$). Note that $\sigma_\pi(B) = \sigma_\pi(A) = \{1\}$. Then $1 \in \sigma(B) \subseteq (-1, 1]$. It follows that either (i) $\dim \ker(B - I) \geq 1$ or (ii) $B - I$ is injective but not surjective.

Assume that (ii) occurs. Since $1 \in \sigma_\pi(B)$, we have $\|B\| = 1$ and $B \leq I$. So, according to some suitable space decomposition of H_2, B has an operator matrix representation of the form

$$\begin{pmatrix} a & 0 & b & 0 & 0 \\ 0 & a & 0 & c & 0 \\ b & 0 & * & * & * \\ 0 & c & * & * & * \\ 0 & 0 & * & * & * \end{pmatrix}, \qquad (17)$$

where $a > 1/2$ and $b, c \geq 0$. To see this, one can first choose three orthonormal vectors x_1, x_2, x_3 such that $1 - d < \langle Bx_j, x_j \rangle < 1$ for some sufficiently small $d \in (0, 1/4)$. Suppose the compression $\widehat{B}$ of B onto the span of $\{x_1, x_2, x_3\}$ has eigenvalues $\mu_1 \geq \mu_2 \geq \mu_3$. Then

$$\mu_2 \geq \frac{(\mu_2 + \mu_3)}{2} \geq \frac{[(3 - 3d) - 1]}{2} > \frac{1}{2}. \qquad (18)$$

Let $\mu_2 = a$. Then $\widehat{B}$ is similar to

$$\begin{pmatrix} a & 0 & * \\ 0 & a & * \\ * & * & * \end{pmatrix}. \qquad (19)$$

Thus, there exists a space decomposition such that B has an operator matrix of the form

$$\begin{pmatrix} aI_2 & B_{12} \\ B_{12}^* & * \end{pmatrix}. \qquad (20)$$

Clearly, there are unitary U, V such that $UB_{12}V^*$ has operator matrix of the form

$$\begin{pmatrix} b & 0 & 0 \\ 0 & c & 0 \end{pmatrix}, \qquad (21)$$

where $b, c \geq 0$. So B has the desired operator matrix form. Under the same decomposition, take $S = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \oplus 0$; then $\sigma_\pi(BS^s + S^sB)$ has two different points with $r(BS^s + S^sB) \geq 2a > 1$ and there exists $R \in \mathscr{A}_1$ such that $\Phi(R) = S$. It follows that $\sigma_\pi(R) = \{-1, 1\}$. So $\|R\| = 1$ and $\|R^su\| \leq 1$ for all unit vectors $u \in H_1$. But $\sigma_\pi(AR^s + R^sA) = \sigma_\pi(x \otimes xR^s + R^sx \otimes x)$ is either a singleton or $\{\pm\|R^sx\|\}$ with $\|R^sx\| \leq 1$. This contradicts the fact $r(AR^s + R^sA) = r(BS^s + S^sB) \geq 2a > 1$.

So $\dim \ker(B - I) \geq 1$. Assume that $\dim \ker(B - I) = n \geq 2$. According to the space decomposition $H_2 = \ker(B - I) \oplus \ker(B - I)^\perp$, B has an operator matrix $I_n \oplus N$. Under the same space decomposition, take $M = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \oplus 0$. Similar to the previous discussion, one gets a contradiction again. So $\dim \ker(B - I) = 1$.

Claim 1.2.2. There exists a unit vector $y \in H_2$ such that $B = y \otimes y$.

If it is not true, then, by Claim 1.2.1, there exist a unit vector $y \in \ker(B - I)$ and a nonzero $B_2 \in \mathscr{A}_2$ with $B_2y = 0$ such that $B = y \otimes y + B_2$. So there exists a unit vector $z \in [y]^\perp$ such that $B_2z \neq 0$. Let $C_1 = y \otimes y$ and $C_2 = z \otimes z$. Then $\sigma_\pi(BC_1^s + C_1^sB) = \sigma_\pi(BC_1 + C_1B) = \{2\}$, $C_1C_2^s + C_2^sC_1 = 0$, and $BC_2^s + C_2^sB \neq 0$. Since the range of Φ contains all rank-one projections, there exist D_1 and D_2 in $\mathscr{A}_1$ such that $\Phi(D_1) = C_1$ and $\Phi(D_2) = C_2$. Then $\sigma_\pi(D_1) = \sigma_\pi(D_2) = \{1\}$, $\sigma_\pi(AD_1^s + D_1^sA) = \sigma_\pi(BC_1^s + C_1^sB) = \{2\}$, $D_1D_2^s + D_2^sD_1 = 0$, and $AD_2^s + D_2^sA \neq 0$.

Since $\{2\} = \sigma_\pi(AD_1^s + D_1^sA) = \sigma_\pi(x \otimes xD_1^s + D_1^sx \otimes x)$, it follows from (6) that $|\langle D_1^sx, x \rangle| + \|D_1^sx\| = 2$, which, together with $\|D_1\| = 1$, implies that $D_1^sx = x$. So, according to the space decomposition $H_1 = [x] \oplus [x]^\perp$, $D_1^s = [1] \oplus Z$ with $\sigma(Z) \subseteq (-1, 1]$. If D_2 has an operator matrix $\left(\begin{smallmatrix} v_{11} & V_{12} \\ V_{12}^* & V_{22} \end{smallmatrix}\right)$ accordingly, then

$$0 = D_1^sD_2 + D_2D_1^s = \begin{pmatrix} 2v_{11} & V_{12} + V_{12}Z \\ ZV_{12}^* + V_{12}^* & ZV_{22} + V_{22}Z \end{pmatrix}. \qquad (22)$$

Since $I + Z$ is invertible, we see that $V_{12} = 0$. Clearly, $v_{11} = 0$. So, $D_2 = 0 \oplus V_{22}$. But then it contradicts the fact that $AD_2^s + D_2^sA \neq 0$. So Claim 1.2.2 holds and Φ preserves rank-one projections.

Conversely, assume that $\Phi(A)$ is a rank-one projection; then a similar discussion shows that A is a rank-one projection, too.

Case 2 ($s > r > 0$). Consider the following.

Case 2.1 ($r + s$ is even). For any unit vector $x \in H_1$, let $\Phi(x \otimes x) = B$ and $\Phi(I - x \otimes x) = T$. It follows from $\{0\} = \sigma_\pi((I - x \otimes x)^r (x \otimes x)(I - x \otimes x)^s + (I - x \otimes x)^s (x \otimes x)(I - x \otimes x)^r) = \sigma_\pi(T^r BT^s + T^s BT^r)$ that $T^r BT^s + T^s BT^r = 0$. Since $T^{s-r} \geq 0$ and $T^r BT^r T^{s-r} + T^{s-r} T^r BT^r = T^r BT^s + T^s BT^r = 0$, we see that $T^s BT^r = T^r BT^s = 0$. If $\ker T = \{0\}$, then $\ker T^r = \{0\}$ and $\ker T^s = \{0\}$, which, together with $T^r BT^s = 0$, imply that $BT^s = 0$, and thus $B = 0$, a contradiction.

So, $\ker T \neq \{0\}$. Take a unit vector $y \in \ker T$ and $A \in \mathscr{A}_1$ such that $\Phi(A) = y \otimes y$. It follows from $\sigma_\pi(A^r (I - x \otimes x)A^s + A^s (I - x \otimes x)A^r) = \sigma_\pi(2y \otimes yTy \otimes y) = \{0\}$ that $A^r (I - x \otimes x)A^s + A^s (I - x \otimes x)A^r = 0$, which, together with $A^{s-r} \geq 0$, implies that $A^r (I - x \otimes x)A^s = A^s (I - x \otimes x)A^r = 0$. Hence we have $A = x \otimes x$ and $\Phi(x \otimes x) = y \otimes y$.

Case 2.2 ($r + s$ is odd). For any unit vectors $x \in H_1$, let $A = x \otimes x$ and $\Phi(A) = B$. We will prove that B is a rank-one projection.

Claim 2.2.1 ($\dim \ker(B - I) = 1$). Note that $\sigma_\pi(B) = \sigma_\pi(A) = \{1\}$. Then $1 \in \sigma(B) \subseteq (-1, 1]$. It follows that either (i) $\dim \ker(B - I) \geq 1$ or (ii) $B - I$ is injective but not surjective.

Assume that (ii) occurs. Since $1 \in \sigma_\pi(B)$, we have $\|B\| = 1$ and $B \leq I$. So, like shown in Case 1.2.1, with respect to some suitable space decomposition of H_2, B has an operator matrix representation of the form

$$\begin{pmatrix} a & 0 & b & 0 & 0 \\ 0 & a & 0 & c & 0 \\ b & 0 & * & * & * \\ 0 & c & * & * & * \\ 0 & 0 & * & * & * \end{pmatrix}, \tag{23}$$

where $a > 1/2$ and $b, c \geq 0$. Under the same decomposition, take $S = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \oplus 0$, and then $\{\pm 2a\} = \sigma_\pi(S^r BS^s + S^s BS^r)$. As S has rank-two and zero trace, there exists $R \in \mathscr{A}_1$ such that $\Phi(R) = S$. It follows that $\sigma_\pi(R) = \{-1, 1\}$. So $\|R\| = 1$ and $\|R^s u\| \|R^r u\| \leq 1$ for all unit vectors $u \in H_1$. But $\sigma_\pi(R^r AR^s + R^s AR^r) = \sigma_\pi(R^r x \otimes xR^s + R^s x \otimes xR^r) = \{\pm \|R^r x\| \|R^s x\|\}$. This contradicts the fact $r(AR^s + R^s A) = r(BS^s + S^s B) \geq 2a > 1$.

So $\dim \ker(B - I) \geq 1$. Assume that $\dim \ker(B - I) = n \geq 2$. According to the space decomposition $H_2 = \ker(B - I) \oplus \ker(B - I)^\perp$, B has an operator matrix $I_n \oplus N$. Under the same space decomposition, take $M = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \oplus 0$. Similar to the previous discussion, one gets a contradiction again. So $\dim \ker(B - I) = 1$.

Claim 2.2.2. There exists a unit vector $y \in H_2$ such that $B = y \otimes y$.

If it is not true, then, by Claim 2.2.1, there exist a unit vector $y \in \ker(B - I)$ and a nonzero $B_2 \in \mathscr{A}_2$ with $B_2 y = 0$ such that $B = y \otimes y + B_2$. So there exists a unit vector $z \in [y]^\perp$ such that $B_2^r z \neq 0$ and $B_2^s z \neq 0$. Let $C_1 = y \otimes y$ and $C_2 = z \otimes z$. Since the range of Φ contains all rank-one projections,

there exist D_1 and D_2 in $\mathscr{A}_1$ such that $\Phi(D_1) = C_1$ and $\Phi(D_2) = C_2$. Then $\sigma_\pi(D_1^r AD_1^s + D_1^s AD_1^r) = \sigma_\pi(C_1^r BC_1^s + C_1^s BC_1^r) = \sigma_\pi(2y \otimes yBy \otimes y) = \{2\}$, which, together with (6), implies that $\langle D_1^{r+s} x, x \rangle + \|D_1^r x\| \|D_1^s x\| = 2$. It follows from $\langle D_1^{r+s} x, x \rangle \leq \|D_1^r x\| \|D_1^s x\| \leq 1$ that $\langle D_1^{r+s} x, x \rangle = 1$. So $D_1^{r+s} x = x$, and according to the space decomposition $H_1 = [x] \oplus [x]^\perp$, $D_1^{r+s} = [1] \oplus Z$ with $\sigma(Z) \subseteq (-1, 1]$. Thus under the same space decomposition we have $D_1 = [1] \oplus Y$ with $\sigma(Y) \subseteq (-1, 1]$. Write D_2 in the operator matrix $\left(\begin{smallmatrix} v_{11} & V_{12} \\ V_{12}^* & V_{22} \end{smallmatrix}\right)$ accordingly; then

$$\begin{aligned} 0 &= D_1^s D_2 D_1^r + D_1^r D_2 D_1^s \\ &= \begin{pmatrix} 2v_{11} & V_{12}(Y^s + Y^r) \\ (Y^s + Y^r)V_{12}^* & Y^r V_{22}Y^s + Y^s V_{22}Y^r \end{pmatrix}. \end{aligned} \tag{24}$$

Clearly, $v_{11} = 0$. So, $A^r D_2 A^s + A^s D_2 A^r = 0$. But then this contradicts the fact that $\sigma_\pi(A^r D_2 A^s + A^s D_2 A^r) = \sigma_\pi(B^r C_2 B^s + B^s C_2 B^r) \neq \{0\}$. So Claim 2.2.2 holds and Φ preserves rank-one projections.

Conversely, assume that $\Phi(A)$ is a rank-one orthogonal projection; then, a similar discussion implies that A is a rank-one projection. $\qquad\square$

The following lemma was proved in [19].

Lemma 7. *Let H be a complex Hilbert space and $A, B \in \mathscr{B}(H)$ self-adjoint operators. If $|\langle Ax, x \rangle| + \|Ax\| \|x\| = |\langle Bx, x \rangle| + \|Bx\| \|x\|$ holds for all $x \in H$, then $A = \pm B$.*

Now we are in a position to present our proof of Theorem 3, except the case $r = s$.

Proof of Theorem 3. The "if" part is obvious. Let us check the "only if" part.

By Lemma 6, Φ preserves rank-one projections in both directions. It follows that there exists a bijective map $T : H_1 \to H_2$ such that

$$\Phi(x \otimes x) = Tx \otimes Tx \tag{25}$$

for all unit vectors $x \in H_1$, where $\|Tx\| = \|x\|$ and $T(\lambda x) = \lambda Tx$ for any $x \in H_1$ and $\lambda \in \mathbb{C}$.

We consider the following two cases.

Case 1 ($s > r = 0$). For any unit vectors $x, y \in H_1$, we have $\sigma_\pi((y \otimes y)(x \otimes x)^s + (x \otimes x)^s(y \otimes y)) = \sigma_\pi((Ty \otimes Ty)(Tx \otimes Tx)^s + (Tx \otimes Tx)^s(Ty \otimes Ty))$. By (6), $\langle x, y \rangle = 0$ if and only if $\langle Tx, Ty \rangle = 0$, and when $\langle x, y \rangle \neq 0$,

$$|\langle Tx, Ty \rangle|^2 + |\langle Tx, Ty \rangle| = |\langle x, y \rangle|^2 + |\langle x, y \rangle|. \tag{26}$$

It follows that

$$|\langle Tx, Ty \rangle| = |\langle x, y \rangle| \tag{27}$$

holds for all $x, y \in H_1$.

Wigner's theorem [20] states that every bijective map T between Hilbert spaces H_1, H_2 satisfying (27) must have the form $Tx = \phi(x)Ux$ for any $x \in H_1$, where U is a unitary or a conjugate unitary (i.e., antiunitary) operator and ϕ is

a so-called phase-function which means that its values are of modulus one. Thus, by Wigner's theorem, there exists a unitary or conjugate unitary operator $U : H_1 \rightarrow H_2$ such that $\Phi(x \otimes x) = Ux \otimes Ux$ for every unit vector $x \in H_1$.

Assume first that U is unitary. Let $A \in \mathcal{A}_1$ be arbitrary. For any unit vector $x \in H_1$,

$$\begin{aligned}
\sigma_\pi &\left(A(x \otimes x)^s + (x \otimes x)^s A\right) \\
&= \sigma_\pi \left(\Phi(A)(Ux \otimes Ux)^s + (Ux \otimes Ux)^s \Phi(A)\right).
\end{aligned} \tag{28}$$

Applying (6), for any unit vector $x \in H_1$, one has

$$|\langle Ax, x \rangle| + \|Ax\| = \left|\langle U^* \Phi(A) Ux, x \rangle\right| + \left\|U^* \Phi(A) Ux\right\|, \tag{29}$$

and hence Lemma 7 implies that $U^* \Phi(A) U = \pm A$. Hence, $\Phi(A) = \pm UAU^*$ for every $A \in \mathcal{A}_1$. We claim that $\Phi(A) = UAU^*$ for every $A \in \mathcal{A}_1$. Otherwise, there exists some nonzero B_0 such that $\Phi(B_0) = -UB_0U^*$. Let $\mathcal{M}_+ = \{A \in \mathcal{A}_1 : \Phi(A) = UAU^*\}$ and $\mathcal{M}_- = \{B \in \mathcal{A}_1 : B \neq 0, \Phi(B) = -UBU^*\}$. Then $\mathcal{M}_+ \cap \mathcal{M}_- = \emptyset$, $\mathcal{M}_+ \cup \mathcal{M}_- = \mathcal{A}_1$, and $B_0 \in \mathcal{M}_-$. Note that, as $\Phi(I) = I$, $I \in \mathcal{M}_+$. It follows that, for any $B \in \mathcal{M}_-$, we have $\sigma_\pi(2B) = \sigma_\pi(IB + BI) = \sigma_\pi(\Phi(I)\Phi(B) + \Phi(B)\Phi(I)) = -\sigma_\pi(2B)$. Therefore, $\sigma_\pi(B) = \{-\|B\|, \|B\|\}$ holds for all $B \in \mathcal{M}_-$. Let $B = \int_{-\|B\|}^{\|B\|} \lambda dE_\lambda$ be the spectral resolution of B. Then the spectral project $E = \int_{(1/2)\|B\|}^{\|B\|} dE_\lambda \neq 0$. Though we do not know if $E \in \mathcal{A}_1$, we can take unit vector $x \in EH_1$ so that $\langle Bx, x \rangle \geq (1/2)\|B\| > 0$. Thus, $\langle Bx, x \rangle + \|B\| \geq \|B\| > 0$. By Lemma 4, we have $\sigma_\pi(x \otimes xB + Bx \otimes x) = \{\langle Bx, x \rangle + \|B\|\}$. Since $x \otimes x \in \mathcal{A}_1$ and $\sigma_\pi(x \otimes x) = \{1\}$, $x \otimes x \in \mathcal{M}_+$. But then,

$$\begin{aligned}
\{\langle Bx, x \rangle + \|B\|\} &= \sigma_\pi(x \otimes xB + Bx \otimes x) \\
&= \sigma_\pi(\Phi(x \otimes x)\Phi(B) + \Phi(B)\Phi(x \otimes x)) \\
&= -\sigma_\pi(x \otimes xB + Bx \otimes x) \\
&= \{-\langle Bx, x \rangle - \|B\|\},
\end{aligned} \tag{30}$$

a contradiction. So, $\Phi(A) = UAU^*$ holds for every $A \in \mathcal{A}_1$.

Now assume that U is conjugate unitary. Take arbitrarily an orthonormal basis $\{e_i\}_{i \in \Lambda}$ of H and define J by $J(\sum_{i \in \Lambda} \xi_i e_i) = \sum_{i \in \Lambda} \overline{\xi_i} e_i$. Then $J : H_1 \rightarrow H_1$ is conjugate unitary and $J^2 = I$. Let $V = UJ$. Then V is unitary and a similar discussion as above implies that $\Phi(A) = VA^t V^*$ for all $A \in \mathcal{A}_1$ and A^t is the transpose of A for the orthonormal basis $\{e_i\}_{i \in \Lambda}$ of H_1.

Case 2 ($s > r > 0$). For any unit vectors $x, y \in H_1$, we have

$$\begin{aligned}
\left\{2|\langle x, y \rangle|^2\right\} &= \sigma_\pi\left((x \otimes x)^r (y \otimes y)(x \otimes x)^s \right. \\
&\quad \left. + (x \otimes x)^s (y \otimes y)(x \otimes x)^r\right) \\
&= \sigma_\pi\left((Tx \otimes Tx)^r (Ty \otimes Ty)(Tx \otimes Tx)^s \right. \\
&\quad \left. + (Tx \otimes Tx)^s (Ty \otimes Ty)(Tx \otimes Tx)^r\right) \\
&= \left\{2|\langle Tx, Ty \rangle|^2\right\}.
\end{aligned} \tag{31}$$

Hence

$$|\langle Tx, Ty \rangle| = |\langle x, y \rangle| \tag{32}$$

holds for all $x, y \in H_1$. Thus, by Wigner's theorem, there exists a unitary or conjugate unitary operator $U : H_1 \rightarrow H_2$ such that $\Phi(x \otimes x) = Ux \otimes Ux$ for every unit vector $x \in H_1$.

Now assume that U is unitary. Let $A \in \mathcal{A}_1$ be arbitrary. For any unit vector $x \in H_1$, since

$$\begin{aligned}
\{2\langle Ax, x \rangle\} &= \sigma_\pi\left((x \otimes x)^r A(x \otimes x)^s + (x \otimes x)^s A(x \otimes x)^r\right) \\
&= \sigma_\pi\left((Ux \otimes Ux)^r \Phi(A)(Ux \otimes Ux)^s \right. \\
&\quad \left. + (Ux \otimes Ux)^s \Phi(A)(Ux \otimes Ux)^r\right) \\
&= \{2\langle \Phi(A) Ux, Ux \rangle\},
\end{aligned} \tag{33}$$

we have

$$\langle Ax, x \rangle = \langle \Phi(A) Ux, Ux \rangle \quad \forall \text{unit vectors } x \in H_1. \tag{34}$$

So we get $\Phi(A) = UAU^*$ for every $A \in \mathcal{A}_1$.

Similar to the case $s > r = 0$, if U is conjugate unitary, then there exists a unitary operator V such that $\Phi(A) = VA^t V^*$ for all $A \in \mathcal{A}_1$.

Hence we have shown that, in the case $\Phi(I) = I$, there exists a unitary U such that either $\Phi(A) = UAU^*$ for every $A \in \mathcal{A}_1$; or $\Phi(A) = UA^t U^*$ for every $A \in \mathcal{A}_1$, where A^t is the transpose of A with respect to an arbitrarily given orthonormal basis of H_1.

If $\Phi(I) = -I$, considering $\Psi = -\Phi$ gives $\Phi(A) = -UAU^*$ for every $A \in \mathcal{A}_1$ or $\Phi(A) = -UA^t U^*$ for every $A \in \mathcal{A}_1$. It is clear that this case does not occur if $r + s$ is even. □

3. Generalized Products of Self-Adjoint Operators on Hilbert Spaces

In this section, we will characterize maps preserving peripheral spectrum of generalized products of self-adjoint operators. Its special case, Theorem 10, makes up for the gap for the case $s = r$ in the proof of Theorem 3.

Let $\mathcal{A}$ be a real Jordan algebra in $\mathcal{B}_s(H)$. If a generalized product $T_1 * T_2 * \cdots * T_k$ defined in (2) satisfies that $T_1 * T_2 * \cdots * T_k \in \mathcal{A}$ for any $T_1, T_2, \ldots, T_k \in \mathcal{A}$, that is, the general product is closed in $\mathcal{A}$, we say that $T_1 * T_2 * \cdots * T_k$ is a generalized product on $\mathcal{A}$. The following lemma was proved in [3].

Lemma 8. *Let $T_1 * T_2 * \cdots * T_k = T_{i_1} \cdots T_{i_p} \cdots T_{i_m}$ be a generalized product on a standard real Jordan algebra $\mathcal{A} \subseteq \mathcal{B}_s(H)$ defined as in (2) of Definition 1. Then there exists a positive integer n with $m = 2n - 1$ such that $i_p = n$, and $i_j = i_{2n-j}$ for all $j = 1, \ldots, n$.*

The following is the main result in this section. Observe that we do not need the assumption that the range of the map contains all rank-two self-adjoint operators with zero trace.

Theorem 9. *Let $\mathscr{A}_1$ and $\mathscr{A}_2$ be standard real Jordan algebras of self-adjoint operators on complex Hilbert spaces H_1 and H_2, respectively. Consider the generalized product $T_1 * \cdots * T_k$ on $\mathscr{B}_s(H_i)$ as in Lemma 8 with width m. Assume that $\Phi : \mathscr{A}_1 \rightarrow \mathscr{A}_2$ is a map the range of which contains all rank-one projections. Then Φ satisfies*

$$\sigma_\pi \left(\Phi\left(A_1\right) * \cdots * \Phi\left(A_k\right) \right) = \sigma_\pi \left(A_1 * \cdots * A_k \right) \quad (35)$$

for all $A_1, A_2, \ldots, A_k \in \mathscr{A}_1$ if and only if one of the following conditions holds.

(1) *There exists a unitary operator $U : H_1 \rightarrow H_2$ such that $\Phi(A) = UAU^*$ for all $A \in \mathscr{A}_1$.*

(2) *There exists a unitary operator $U : H_1 \rightarrow H_2$ such that $\Phi(A) = UA^tU^*$ for all $A \in \mathscr{A}_1$,*

where A^t is the transpose of A for an arbitrarily but fixed orthonormal basis of H_1.

To prove Theorem 9, we consider the special case by taking $A_{i_p} = A$ and $A_{i_q} = B$ if $q \neq p$. By Lemma 8, there exists positive integer $r(= n)$ with $2r = m - 1 \geq 2$ such that $A_1 * A_2 * \cdots * A_k = B^r A B^r$. It is clear that Theorem 9 is an immediate consequence of the following result.

Theorem 10. *Let $\mathscr{A}_1$ and $\mathscr{A}_2$ be standard real Jordan algebras of delf-adjoint operators on complex Hilbert spaces H_1 and H_2, respectively. Assume that $\Phi : \mathscr{A}_1 \rightarrow \mathscr{A}_2$ is a map the range of which contains all rank-one projections and r is nonnegative integer with $r \geq 1$. Then Φ satisfies*

$$\sigma_\pi \left(B^r A B^r \right) = \sigma_\pi \left(\Phi(B)^r \Phi(A) \Phi(B)^r \right) \quad \forall A, B \in \mathscr{A}_1 \quad (36)$$

if and only if one of the following conditions holds.

(1) *There exists a unitary operator $U : H_1 \rightarrow H_2$ such that $\Phi(A) = UAU^*$ for all $A \in \mathscr{A}_1$.*

(2) *There exists a unitary operator $U : H_1 \rightarrow H_2$ such that $\Phi(A) = UA^tU^*$ for all $A \in \mathscr{A}_1$,*

where A^t is the transpose of A for an arbitrarily but fixed orthonormal basis of H_1.

To prove Theorem 10, it suffices to check the "only if" part. Assume in the following that $\Phi : \mathscr{A}_1 \rightarrow \mathscr{A}_2$ is a map satisfying (36) with range containing all rank-one projections.

Lemma 11. $\Phi(I) = I$.

Proof. It follows from (36) that $r(A) = r(\Phi(A))$ holds for every $A \in \mathscr{A}_1$. Let $\Phi(I) = B$. For any unit vector $y \in H_2$, there exists $A \in \mathscr{A}_1$ such that $\Phi(A) = y \otimes y$. Then

$$\sigma_\pi \left(A^{2r} \right) = \sigma_\pi \left(\Phi(A)^r \Phi(I) \Phi(A)^r \right) = \sigma_\pi \left(y \otimes yBy \otimes y \right), \quad (37)$$

which, together with $A^{2r} \geq 0$, implies that $\langle By, y \rangle = 1$ for all unit vectors $y \in H_2$. So $B = I$. $\square$

Lemma 12. Φ *preserves rank-one projections in both directions.*

Proof. For any unit vector $x \in H_1$, let $\Phi(x \otimes x) = B$ and $\Phi(I - x \otimes x) = T$. It follows from $\{0\} = \sigma_\pi((x \otimes x)^r (I - x \otimes x)(x \otimes x)^r) = \sigma_\pi(B^r T B^r)$ that $B^r T B^r = 0$. Similarly, we have $T^r B T^r = 0$. If $\ker T = \{0\}$, then $\ker T^r = \{0\}$, which, together with $T^r B T^r = 0$, implies that $BT^r = 0$, and thus $B = 0$, a contradiction.

So, there exist a unit vector $y \in \ker T$. Take $A \in \mathscr{A}_1$ such that $\Phi(A) = y \otimes y$. It follows from $\sigma_\pi(A^r(I - x \otimes x)A^r) = \sigma_\pi(y \otimes yTy \otimes y) = \{0\}$ that $A^r(I - x \otimes x)A^r = 0$, which implies that $A = x \otimes x$. Thus $\Phi(x \otimes x) = y \otimes y$ and, therefore, Φ preserves rank-one projections.

Similarly one can show that $\Phi(A)$ is a rank-one projection will imply that A is a rank-one projection. $\square$

Proof of Theorem 10. By Lemma 12, Φ preserves rank-one projections in both directions. It follows that there exists a bijective map $T : H_1 \rightarrow H_2$ such that

$$\Phi(x \otimes x) = Tx \otimes Tx \quad (38)$$

for all unit vectors $x \in H_1$, where $\|Tx\| = \|x\|$ and $T(\lambda x) = \lambda Tx$ for any $x \in H_1$ and $\lambda \in \mathbb{C}$.

For any unit vectors $x, y \in H_1$, we have

$$\left\{ \left| \langle x, y \rangle \right|^2 \right\} = \sigma_\pi \left((x \otimes x)^r (y \otimes y)(x \otimes x)^r \right)$$

$$= \sigma_\pi \left((Tx \otimes Tx)^r (Ty \otimes Ty)(Tx \otimes Tx)^r \right) \quad (39)$$

$$= \left\{ \left| \langle Tx, Ty \rangle \right|^2 \right\}.$$

Hence

$$\left| \langle Tx, Ty \rangle \right| = \left| \langle x, y \rangle \right| \quad (40)$$

holds for all $x, y \in H_1$. Thus, by Wigner's theorem again, there exists a unitary or conjugate unitary operator $U : H_1 \rightarrow H_2$ such that $\Phi(x \otimes x) = Ux \otimes Ux$ for every unit vector $x \in H_1$.

Now assume that U is unitary. Let $A \in \mathscr{A}_1$ be arbitrary. For any unit vector $x \in H_1$, since

$$\{\langle Ax, x \rangle\} = \sigma_\pi \left((x \otimes x)^r A(x \otimes x)^r \right)$$

$$= \sigma_\pi \left((Ux \otimes Ux)^r \Phi(A)(Ux \otimes Ux)^r \right) \quad (41)$$

$$= \{\langle \Phi(A)Ux, Ux \rangle\},$$

we have

$$\langle Ax, x \rangle = \langle \Phi(A)Ux, Ux \rangle \quad \forall \text{unit vectors } x \in H_1. \quad (42)$$

Hence we get $\Phi(A) = UAU^*$ for every $A \in \mathscr{A}_1$.

Similarly, U is conjugate unitary implies that there exists a unitary operator such that $\Phi(A) = VA^tV^*$ for all $A \in \mathscr{A}_1$. $\square$

Remark 13. Finally, we remark that if we do not require that the generalized product is closed in the involved standard real Jordan algebras $\mathscr{A}_i$, $i = 1, 2$, we can still obtain a characterization of the maps Φ from $\mathscr{A}_1$ into $\mathscr{A}_2$ with range containing all rank-one projections which preserves the peripheral spectrum of an arbitrarily given generalized product. In fact, such maps have the same form stated in Theorem 2.

Conflict of Interests

Acknowledgments

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