Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 183197, 4 pages
http://dx.doi.org/10.1155/2014/183197

Research Article

L^0-Linear Modulus of a Random Linear Operator

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Received 5 November 2013; Accepted 23 February 2014; Published 27 March 2014

Academic Editor: Shawn X. Wang

We prove that there exists a unique L^0-linear modulus for an a.s. bounded random linear operator on a specifical random normed module, which generalizes the classical case.

1. Introduction

In 1964, Chacon and Krengel began to study linear modulus of a linear operator and proved that there exists a unique linear modulus for a bounded linear operator [1], which plays an important role in the work of mean ergodicity for linear operators and linear operators semigroups [2–5]. Recently, the mean ergodicity for random linear operators has been investigated in [6–8], and its further developments should naturally include the study of L^0-linear modulus of a random linear operator on a random normed module. The purpose of this paper is to investigate the existence of the L^0-linear modulus for an a.s. bounded random linear operator on a specifical random normed module.

The notion of random normed modules (briefly, RN modules), which was first introduced in [9] and subsequently elaborated in [10], is a random generalization of ordinary normed spaces. In the last ten years the theory of RN modules together with their random conjugate spaces has obtained systematic and deep developments [11–17]; in particular, the recently developed L^0-convex analysis, which has been a powerful tool for the study of conditional risk measures, is just based on the theory of RN modules together with their random conjugate spaces [12, 17–20]. One of the key points in the process of applying the theory of RN modules to random analysis and the theory of conditional risk measures is to properly construct the two classes of RN modules $L^0(\mathscr{E}, X)$ and $L^p_{\mathscr{F}}(\mathscr{E})$, where $L^0(\mathscr{E}, X)$ is the RN module of equivalence classes of X-valued random variables defined on a probability space $(\Omega, \mathscr{E}, P)$ and $L^p_{\mathscr{F}}(\mathscr{E})$ is the $L^0(\mathscr{F}, R)$-module generated by $L^p(\mathscr{E})$; see [17], for the construction of $L^0(\mathscr{E}, X)$. In particular, $L^p_{\mathscr{F}}(\mathscr{E})$ constructed in [18] will be used in this paper and thus we give the details of its construction as follows.

Let $(\Omega, \mathscr{E}, P)$ be a probability space, $\mathscr{F}$ a sub σ-algebra of $\mathscr{E}$, and $\overline{L}^0(\mathscr{E})$ (or $L^0(\mathscr{E})$) the set of equivalence classes of $\mathscr{E}$-measurable extended real-valued (real-valued) random variables on Ω. Let $\overline{L}^0_+(\mathscr{E}) = \{\xi \in \overline{L}^0(\mathscr{E}) \mid \xi \geq 0\}$ and $L^0_+(\mathscr{E}) = \{\xi \in L^0(\mathscr{E}) \mid \xi \geq 0\}$. Similarly, one can understand such notations as $\overline{L}^0(\mathscr{F})$, $L^0(\mathscr{F})$, $\overline{L}^0_+(\mathscr{F})$, and $L^0_+(\mathscr{F})$. Define the mapping $|||\cdot|||_p : \overline{L}^0(\mathscr{E}) \to \overline{L}^0_+(\mathscr{F})$ by

$$|||x|||_p = \left[E\left(|x|^p \mid \mathscr{F}\right)\right]^{1/p} \tag{1}$$

for any $x \in \overline{L}^0(\mathscr{E})$ and $1 \leq p < \infty$, where $E(|x|^p \mid \mathscr{F}) = \lim_{n\to\infty} E(|x|^p \wedge n \mid \mathscr{F})$ denotes the extended conditional expectation and let

$$L^p_{\mathscr{F}}(\mathscr{E}) = \left\{x \in L^0(\mathscr{E}) \mid |||x|||_p \in L^0_+(\mathscr{F})\right\}. \tag{2}$$

Then, $(L^p_{\mathscr{F}}(\mathscr{E}), |||\cdot|||_p)$ is an RN module. In fact, $L^p_{\mathscr{F}}(\mathscr{E})$ is exactly the $L^0(\mathscr{F})$-module generated by $L^p(\mathscr{E})$, namely, $L^p_{\mathscr{F}}(\mathscr{E}) = L^0(\mathscr{F}) \cdot L^p(\mathscr{E}) := \{\xi x \mid \xi \in L^0(\mathscr{F}) \text{ and } x \in L^p(\mathscr{E})\}$, where $L^p(\mathscr{E}) = \{x \in L^0(\mathscr{E}) \mid E[|x|^p] < \infty\}$.

The remainder of this paper is organized as follows: in Section 2 we briefly recall some necessary notions and facts and in Section 3 we present and prove our main results.

2. Preliminaries

In the sequel of this paper, $(\Omega, \mathcal{F}, P)$ denotes a given probability space, K the scalar field R of real numbers or C of complex numbers, N the set of positive integers, and $L^0(\mathcal{F}, K)$ the algebra over K of equivalence classes of K-valued $\mathcal{F}$-measurable random variables on Ω under the ordinary scalar multiplication, addition, and multiplication operations on equivalence classes.

Proposition 1 (see [21]). *$\overline{L}^0(\mathcal{F}, R)$ is a complete lattice under the ordering $\leq$: $\xi \leq \eta$ if and only if $\xi^0(\omega) \leq \eta^0(\omega)$, for P-almost all ω in Ω (briefly, a.s.), where ξ^0 and η^0 are arbitrarily chosen representatives of ξ and η, respectively, and has the following nice properties.*

(1) Every subset A of $\overline{L}^0(\mathcal{F}, R)$ has a supremum (denoted by $\bigvee A$) and an infimum (denoted by $\bigwedge A$) and there exist two sequences $\{a_n, n \in N\}$ and $\{b_n, n \in N\}$ in A such that $\bigvee_{n\geq 1} a_n = \bigvee A$ and $\bigwedge_{n\geq 1} b_n = \bigwedge A$.

(2) If A is directed (dually directed), namely, for any two elements c_1 and c_2 in A, there exists some c_3 in A such that $c_1 \bigvee c_2 \leq c_3$ ($c_1 \bigvee c_2 \geq c_3$); then the above $\{a_n, n \in N\}$ ($\{b_n, n \in N\}$) can be chosen as nondecreasing (nonincreasing).

(3) $L^0(\mathcal{F}, R)$, as a sublattice of $\overline{L}^0(\mathcal{F}, R)$, is complete in the sense that every subset with an upper bound (a lower bound) has a supremum (an infimum).

Let ξ and η be two elements in $L^0(\mathcal{F}, R)$; then $\xi < \eta$ is understood as usual, namely, $\xi \leq \eta$ and $\xi \neq \eta$. For $A \in \mathcal{F}$, $\xi > \eta$ on A means $\xi^0(\omega) > \eta^0(\omega)$ P-a.s. on A, where ξ^0 and η^0 are arbitrarily chosen representatives of ξ and η, respectively. Specially, we denote $L^0_+(\mathcal{F}) = \{\xi \in L^0(\mathcal{F}, R) \mid \xi \geq 0\}$ and $L^0_{++}(\mathcal{F}) = \{\xi \in L^0(\mathcal{F}, R) \mid \xi > 0 \text{ on } \Omega\}$.

Definition 2 (see [10, 17]). An ordered pair $(S, \|\cdot\|)$ is called a random normed module (briefly, an RN module) over K with base $(\Omega, \mathcal{F}, P)$ if S is a left module over the algebra $L^0(\mathcal{F}, K)$ and $\|\cdot\|$ is a mapping from S to $L^0_+(\mathcal{F})$ such that the following three axioms are satisfied:

(1) $\|x\| = 0$ if and only if $x = \theta$ (the null vector of S);

(2) $\|\xi x\| = |\xi|\|x\|$, for all $\xi \in L^0(\mathcal{F}, K)$ and $x \in S$;

(3) $\|x + y\| \leq \|x\| + \|y\|$, for all $x, y \in S$.

Cleary, $(L^0(\mathcal{F}, K), |\cdot|)$ is an RN module over K with base $(\Omega, \mathcal{F}, P)$.

Let $(S, \|\cdot\|)$ be an RN module over K with base $(\Omega, \mathcal{F}, P)$. For any $\varepsilon > 0$, $0 < \lambda < 1$, denote $N_\theta(\varepsilon, \lambda) = \{x \in S \mid P\{\omega \in \Omega \mid \|x\|(\omega) < \varepsilon\} > 1 - \lambda\}$; then $\mathcal{U}_\theta = \{N_\theta(\varepsilon, \lambda) \mid \varepsilon > 0, 0 < \lambda < 1\}$ is a local base at θ of some Hausdorff linear topology, called the (ε, λ)-topology induced by $\|\cdot\|$. In this paper, given an RN module $(S, \|\cdot\|)$ over K with base $(\Omega, \mathcal{F}, P)$, it is always assumed that $(S, \|\cdot\|)$ is endowed with the (ε, λ)-topology. In this paper, it suffices to notice that the (ε, λ)-topology for an RN module $(S, \|\cdot\|)$ is a metrizable linear topology and a sequence $\{x_n, n \in N\}$ in S converges in the (ε, λ)-topology to some $x \in S$ if and only if $\{\|x_n - x\|, n \in N\}$ converges in probability P to 0. It should be pointed out that the (ε, λ)-topology for $(L^0(\mathcal{F}, K), |\cdot|)$ is exactly the topology of convergence in probability.

Example 3. Let X be a normed space over K and $L^0(\mathcal{F}, X)$ the linear space of equivalence classes of X-valued $\mathcal{F}$-random variables on Ω. The module multiplication operation $\cdot$: $L^0(\mathcal{F}, K) \times L^0(\mathcal{F}, X) \to L^0(\mathcal{F}, X)$ is defined by $\xi \cdot x$ = the equivalence class of $\xi^0 x^0$, where ξ^0 and x^0 are the respective arbitrarily chosen representatives of $\xi \in L^0(\mathcal{F}, K)$ and $x \in L^0(\mathcal{F}, X)$, and $(\xi^0 x^0)(\omega) = \xi^0(\omega) x^0(\omega)$, for all $\omega \in \Omega$. Furthermore, the mapping $\|\cdot\| : L^0(\mathcal{F}, X) \to L^0_+(\mathcal{F})$ by $\|x\|$ = the equivalence class of $\|x^0\|$, for all $x \in L^0(\mathcal{F}, X)$, where x^0 is as above. Then it is easy to see that $(L^0(\mathcal{F}, X), \|\cdot\|)$ is an RN module over K with base $(\Omega, \mathcal{F}, P)$.

Definition 4 (see [22]). Let $(S^1, \|\cdot\|_1)$ and $(S^2, \|\cdot\|_2)$ be two RN modules over K with base $(\Omega, \mathcal{F}, P)$. A linear operator T from S^1 to S^2 is called a random linear operator; further, the random linear operator T is called a.s. bounded if there exists some $\xi \in L^0_+(\mathcal{F})$ such that $\|Tx\|_2 \leq \xi \cdot \|x\|_1$ for any $x \in S^1$. Denote by $B(S^1, S^2)$ the linear space of a.s. bounded random linear operators from S^1 to S^2; define $\|\cdot\| : B(S^1, S^2) \to L^0_+(\mathcal{F})$ by $\|T\| := \bigwedge\{\xi \in L^0_+(\mathcal{F}) \mid \|Tx\|_2 \leq \xi \cdot \|x\|_1$, for all $x \in S^1\}$ for any $T \in B(S^1, S^2)$; then it is easy to see that $(B(S^1, S^2), \|\cdot\|)$ is an RN module over K with base $(\Omega, \mathcal{F}, P)$.

Proposition 5 (see [22]). *Let $(S^1, \|\cdot\|_1)$ and $(S^2, \|\cdot\|_2)$ be two RN modules over K with base $(\Omega, \mathcal{F}, P)$. Then, we have the following statements:*

(1) *$T \in B(S^1, S^2)$ if and only if T is a continuous module homomorphism;*

(2) *if $T \in B(S^1, S^2)$, then $\|T\| = \bigvee\{\|Tx\|_2 \mid x \in S^1$ and $\|x\|_1 \leq 1\}$, where denotes the identity element in $L^0(\mathcal{F})$.*

3. Main Results and Proofs

The main result of this paper is Theorem 7, which will be derived from Lemma 6.

Let S_ε be the class of simple $\mathcal{E}$-measurable functions and $L^1_{\mathcal{F}}(\mathcal{E})_+ = \{\xi \in L^1_{\mathcal{F}}(\mathcal{E}) \mid \xi \geq 0\}$; then Lemma 6 holds.

Lemma 6. *$L^0(\mathcal{F})(S_\varepsilon \cap L^1_+(\mathcal{E}))$ is dense in $L^1_{\mathcal{F}}(\mathcal{E})_+$.*

Proof. For any $f \in L^1_{\mathcal{F}}(\mathcal{E})_+$, there exist $\xi \in L^0(\mathcal{F})$ and $g \in L^1(\mathcal{E})$ such that

$$f = \xi \cdot g = I_{[\xi>0]\cap[g>0]} \cdot \xi \cdot g + I_{[\xi<0]\cap[g<0]} \cdot \xi \cdot g. \tag{3}$$

Clearly, $I_{[g>0]} \cdot g \in L^1_+(\mathcal{E})$; thus there exists a sequence $\{g_n, n \in N\} \subset S_\varepsilon \cap L^1_+(\mathcal{E})$ such that

$$g_n \nearrow I_{[g>0]} \cdot g \quad \text{a.s. on } \Omega \tag{4}$$

as $n \to \infty$; that is, $I_{[g>0]} \cdot g - g_n \searrow 0$ a.s. on Ω as $n \to \infty$. Since $I_{[\xi>0]} \cdot \xi \in L^0(\mathscr{F})$, it follows that

$$E\left[\left|I_{[\xi>0]\cap[g>0]} \cdot \xi \cdot g - I_{[\xi>0]} \cdot \xi \cdot g_n\right| \mid \mathscr{F}\right] = I_{[\xi>0]} \cdot \xi \cdot E\left[\left|I_{[g>0]} \cdot g - g_n\right| \mid \mathscr{F}\right]. \tag{5}$$

Furthermore, since $E[I_{[g>0]} \cdot g - g_1] < \infty$, it follows that $E[|I_{[g>0]} \cdot g - g_n| \mid \mathscr{F}]$ converges to 0 a.s. on Ω as $n \to \infty$. Hence, $E[|I_{[g>0]} \cdot g - g_n| \mid \mathscr{F}]$ converges to 0 in probability P as $n \to \infty$. Consequently, $|||I_{[\xi>0]\cap[g>0]} \cdot \xi \cdot g - I_{[\xi>0]} \cdot \xi \cdot g_n|||_1$ converges to 0 in the (ε, λ)-topology as $n \to \infty$.

Next, observe that $I_{[\xi<0]\cap[g<0]} \cdot \xi \cdot g = I_{[-\xi>0]\cap[-g>0]} \cdot (-\xi) \cdot (-g)$; it follows from the above discussion that there exists a sequence $\{h_n, n \in N\} \subset S_\varepsilon \cap L^1_+(\mathscr{E})$ such that $I_{[-\xi>0]} \cdot (-\xi) \cdot h_n$ converges to $I_{[-\xi>0]\cap[-g>0]} \cdot (-\xi) \cdot (-g)$ in the (ε, λ)-topology induced by $||| \cdot |||_1$ as $n \to \infty$.

Let

$$f_n = I_{[\xi>0]} \cdot \xi \cdot g_n + I_{[-\xi>0]} \cdot (-\xi) \cdot h_n. \tag{6}$$

Then,

$$f_n = \left[I_{[\xi>0]} \cdot \xi + I_{[-\xi>0]} \cdot (-\xi)\right] \times \left(I_{[\xi>0]} \cdot g_n + I_{[-\xi>0]} \cdot h_n\right) \in L^0(\mathscr{F})\left(S_\varepsilon \cap L^1_+(\mathscr{E})\right) \tag{7}$$

and f_n converges to f in the (ε, λ)-topology induced by $||| \cdot |||_1$ as $n \to \infty$, which shows that $L^0(\mathscr{F})(S_\varepsilon \cap L^1_+(\mathscr{E}))$ is dense in $L^1_{\mathscr{F}}(\mathscr{E})_+$. □

Now we can present and prove the main result below.

Theorem 7. *Let T be an a.s. bounded random linear operator on $L^1_{\mathscr{F}}(\mathscr{E})$. Then there exists a unique positive a.s. bounded random linear operator $\mathscr{T}$ on $L^1_{\mathscr{F}}(\mathscr{E})$, called the L^0-linear modulus of T, such that*

(1) $|||\mathscr{T}|||_1 \le |||T|||_1$,

(2) $|Tf| \le \mathscr{T}|f|$ *for any* $f \in L^1_{\mathscr{F}}(\mathscr{E})$,

(3) $\mathscr{T}f = \bigvee\{|Tg| \mid g \in L^1_{\mathscr{F}}(\mathscr{E})$ *and* $|g| \le f\}$ *for any* $f \in L^1_{\mathscr{F}}(\mathscr{E})_+$.

Proof. Let $\mathscr{P}$ denote the family of all finite measurable partitions of Ω to $\mathscr{E}$; that is, for any $D \in \mathscr{P}$, there exists $D_i \in \mathscr{E}$ $(i = 1, 2, \ldots, k(D))$ such that $\sum_{i=1}^{k(D)} D_i = \Omega$, where $k(D)$ is a finite number with respect to D. It is known that $\mathscr{P}$ is partially ordered in the usual way: $D \le D'$ in $\mathscr{P}$ means that D' is a refinement of D; that is, the sets D_i are unions of sets of D'. For any $f \in L^1_{\mathscr{F}}(\mathscr{E})_+$, define

$$Q(D, T, f) = \sum_{i=1}^{k(D)} \left|T\left(\tilde{I}_{D_i} \cdot f\right)\right|. \tag{8}$$

Then, for any fixed f, $Q(D, T, f)$ is monotone increasing on $\mathscr{P}$. Furthermore,

$$\begin{aligned} E&\left[|Q(D, T, f)| \bigwedge n \mid \mathscr{F}\right] \\ &= E\left[\left(\sum_{i=1}^{k(D)} \left|T\left(\tilde{I}_{D_i} \cdot f\right)\right|\right) \bigwedge n \mid \mathscr{F}\right] \\ &= E\left[\sum_{i=1}^{k(D)} \left(\left|T\left(\tilde{I}_{D_i} \cdot f\right)\right| \bigwedge n\right) \mid \mathscr{F}\right] \\ &= \sum_{i=1}^{k(D)} E\left[\left|T\left(\tilde{I}_{D_i} \cdot f\right)\right| \bigwedge n \mid \mathscr{F}\right], \end{aligned} \tag{9}$$

letting $n \to \infty$ in (9), yields that

$$|||Q(D, T, f)|||_1 = \sum_{i=1}^{k(D)} \left|\left|\left|T(\tilde{I}_{D_i} \cdot f)\right|\right|\right|_1. \tag{10}$$

Observe that

$$\begin{aligned} |||f|||_1 &= \lim_{n\to\infty} E\left[|f| \bigwedge n \mid \mathscr{F}\right] \\ &= \lim_{n\to\infty} E\left[\left|\sum_{i=1}^{k(D)} \left(\tilde{I}_{D_i} \cdot f\right)\right| \bigwedge n \mid \mathscr{F}\right] \\ &= \sum_{i=1}^{k(D)} \lim_{n\to\infty} E\left[\left|\tilde{I}_{D_i} \cdot f\right| \bigwedge n \mid \mathscr{F}\right] \\ &= \sum_{i=1}^{k(D)} \left|\left|\left|\tilde{I}_{D_i} \cdot f\right|\right|\right|_1. \end{aligned} \tag{11}$$

Combining (10) and (11), we have

$$\begin{aligned} |||Q(D, T, f)|||_1 &\le \sum_{i=1}^{k(D)} |||T|||_1 \cdot \left|\left|\left|\tilde{I}_{D_i} \cdot f\right|\right|\right|_1 \\ &= |||T|||_1 \cdot |||f|||_1, \end{aligned} \tag{12}$$

which shows that the net $\{Q(D, T, f),\ D \in \mathscr{P}\}$ is not only monotone increasing on $\mathscr{P}$ but also L^0-norm bounded with respect to $||| \cdot |||_1$. Thus, we can define $\mathscr{T}$ by

$$\mathscr{T}f = \bigvee \{Q(D, T, f),\ D \in \mathscr{P}\} \in L^1_{\mathscr{F}}(\mathscr{E})_+. \tag{13}$$

Then, $\mathscr{T}$ is a positive a.s bounded random linear operator according to Lemma 6 and from inequality (12) we get $|||\mathscr{T}|||_1 \le |||T|||_1$.

For any $f \in L^1_{\mathscr{F}}(\mathscr{E})_+$, let

$$Lf = \bigvee \left\{|Tg| \mid g \in L^1_{\mathscr{F}}(\mathscr{E}),\ |g| \le f\right\}. \tag{14}$$

For any $g \in L^0(\mathscr{F})(S_\varepsilon \cap L^1(\mathscr{E}))$ and $|g| \le f$, it is clear that $|Tg| \le \mathscr{T}|g|$, and further observe that $\mathscr{T}|g| \le \mathscr{T}f$ since $\mathscr{T}$ is positive. Consequently,

$$|Tg| \le \mathscr{T}\,|g| \le \mathscr{T}f, \tag{15}$$

which shows that

$$Lf \leq \mathcal{T} f \tag{16}$$

holds. If the converse inequality of (16) does not hold, then there exists an $f \in L^1_{\mathcal{F}}(\mathcal{E})_+$, a $D \in \mathcal{P}$, an $A_0 \in \mathcal{E}$ with $P(A_0) > 0$, and an $\varepsilon > 0$ such that

$$Q(D, T, f) \geq Lf + \varepsilon \quad \text{on } A_0. \tag{17}$$

Now there exists a set $A_1 \subset A_0$ with $P(A_1) > 0$ and a $\xi_1 \in L^0(\mathcal{E}, C)$ with $|\xi_1| = \tilde{I}_{A_1}$ such that

$$\left| T\left(\xi_1 \cdot \tilde{I}_{D_1} \cdot f\right) - \left| T\left(\tilde{I}_{D_1} \cdot f\right)\right|\right| < \frac{\varepsilon}{2k(D)} \tag{18}$$

on A_1. Continuing in this way we find $A_0 \supset A_1 \supset A_2 \supset \cdots \supset A_{k(D)}$ with $P(A_{k(D)}) > 0$ and $\xi_i \in L^0(\mathcal{E}, C)$ with $|\xi_i| = \tilde{I}_{A_i}$ $(i = 1, 2, \ldots, k(D))$ such that

$$\left| T\left(\xi_i \cdot \tilde{I}_{D_i} \cdot f\right) - \left| T\left(\tilde{I}_{D_i} \cdot f\right)\right|\right| < \frac{\varepsilon}{2k(D)} \tag{19}$$

on A_i. Setting $g = \sum_{i=1}^{k(D)} \xi_i \cdot \tilde{I}_{D_i} \cdot f$ we have $|g| \leq |f|$ and this leads to a contradiction with inequality (17) on $A_{k(D)}$.

This completes the proof. □

If we put $\mathcal{F} = \{\Omega, \Phi\}$, then the following classical result holds.

Corollary 8 (see [1]). *Let T be a bounded linear operator on $L^1(\mathcal{E})$. Then, there exists a unique positive bounded linear operator $\mathcal{T}$ on $L^1(\mathcal{E})$, called the linear modulus of T, such that*

(1) $\|\mathcal{T}\|_1 \leq \|T\|_1$,

(2) $|Tf| \leq \mathcal{T}|f|$ *for any* $f \in L^1(\mathcal{E})$,

(3) $\mathcal{T} f = \bigvee\{|Tg| \mid g \in L^1(\mathcal{E})$ *and* $|g| \leq f\}$ *for any* $f \in L^1_+(\mathcal{E})$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors would like to express their sincere gratitude to Professor Guo Tiexin for his invaluable suggestions. This research is supported by the National Natural Science Foundation of China (Grant no. 11301380) and the Higher School Science and Technology Development Fund Project in Tianjin (Grant no. 20031003).

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