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Research Article

Hierarchical Fixed Point Problems in Uniformly Smooth Banach Spaces

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We propose some relaxed implicit and explicit viscosity approximation methods for hierarchical fixed point problems for a countable family of nonexpansive mappings in uniformly smooth Banach spaces. These relaxed viscosity approximation methods are based on the well-known viscosity approximation method and hybrid steepest-descent method. We obtain some strong convergence theorems under mild conditions.

1. Introduction

Let X be a real Banach space and U the unit sphere of X; that is, $U = \{x \in X : \|x\| = 1\}$. Recall that X is said to be smooth if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{1}$$

exists for all $x, y \in U$; in this case, X is also said to have a Gâteaux differentiable norm. X is said to have a uniformly Gâteaux differentiable norm if for each $y \in U$, the limit is attained uniformly for $x \in U$. Moreover, it is said to be uniformly smooth if this limit is attained uniformly for $x, y \in U$. The norm of X is said to be the Fréchet differential if for each $x \in U$, this limit is attained uniformly for $y \in U$. In addition, we define a function $\rho : [0, \infty) \to [0, \infty)$ called the modulus of smoothness of X as follows:

$$\rho(\tau) = \sup \left\{ \frac{1}{2} \left(\|x + y\| + \|x - y\| \right) - 1 : x, y \in X, \; \|x\| = 1, \|y\| = \tau \right\}. \tag{2}$$

It is known that X is uniformly smooth if and only if $\lim_{\tau \to 0} \rho(\tau)/\tau = 0$.

Let X be a real Banach space and let J denote the normalized duality mapping from X to 2^{X^*} given by

$$J(x) = \left\{ x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2 \right\}, \quad \forall x \in X, \tag{3}$$

where X^* denotes the dual space of X and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. We use $\mathrm{Fix}(T)$ to denote the set of fixed points of the mapping T. It is well known that if X is smooth, then J is single-valued and norm-to-weak* continuous, whereas if X is a Banach space with a uniformly Gâteaux differentiable norm, then J is single-valued and norm-to-weak* uniformly continuous on bounded subsets of X. Further, if X is a uniformly smooth Banach space, then J is single-valued and norm-to-norm uniformly continuous on bounded subsets of X. In what follows, we still denote by J the single-valued normalized duality mapping.

Let C be a nonempty closed convex subset of X. Recall that a mapping $T : C \to C$ is said to be L-Lipschitzian if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \le L \|x - y\|, \quad \forall x, y \in C. \tag{4}$$

In particular, if $L = 1$, then T is said to be nonexpansive; that is,

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{5}$$

We use the notation $\rightharpoonup$ to indicate the weak convergence and the one $\rightarrow$ to indicate the strong convergence.

Definition 1. Let $A : C \rightarrow X$ be a mapping of C into X. Then A is said to be

(i) accretive if for each $x, y \in C$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y)\rangle \geq 0, \tag{6}$$

where J is the normalized duality mapping;

(ii) α-strongly accretive if for each $x, y \in C$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y)\rangle \geq \alpha\|x - y\|^2, \tag{7}$$

for some $\alpha \in (0, 1)$;

(iii) pseudocontractive if for each $x, y \in C$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y)\rangle \leq \|x - y\|^2; \tag{8}$$

(iv) β-strongly pseudocontractive if for each $x, y \in C$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y)\rangle \leq \beta\|x - y\|^2, \tag{9}$$

for some $\beta \in (0, 1)$;

(v) λ-strictly pseudocontractive if for each $x, y \in C$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y)\rangle \leq \|x - y\|^2 - \lambda\|x - y - (Ax - Ay)\|^2, \tag{10}$$

for some $\lambda \in (0, 1)$.

In a real smooth Banach space X we say that an operator A is strongly positive [1] if there exists a constant $\overline{\gamma} > 0$ with the property

$$\begin{gathered} \langle Ax, J(x)\rangle \geq \overline{\gamma}\|x\|^2, \\ \|aI - bA\| = \sup_{\|x\| \leq 1} |\langle (aI - bA)x, J(x)\rangle|, \\ a \in [0, 1], \quad b \in [-1, 1], \end{gathered} \tag{11}$$

where I is the identity mapping.

Recently, the problem of convergence of implicit iterative algorithms to a common fixed point for a family of nonexpansive mappings and its extensions to Hilbert spaces or Banach spaces have been considered by many authors; see [1–9] and the references therein.

Yao et al. [10] introduced the following Halpern-type implicit iterative algorithm,

$$x_n = \alpha_n u + \beta_n x_{n-1} + \gamma_n T x_n, \quad \forall n \geq 1, \tag{12}$$

and proved a strong convergence theorem under suitable conditions.

On the other hand, let C be a nonempty closed convex subset of a real Hilbert space H and let $A : C \rightarrow H$ be a nonlinear mapping. The classical variational inequality problem (VIP) is to find $x^* \in C$ such that

$$\langle Ax^*, x - x^*\rangle \geq 0, \quad \forall x \in C. \tag{13}$$

If we assume that C is the fixed point set of a nonexpansive mapping T and S is another nonexpansive mapping (not necessarily with fixed points), the problem (13) becomes the VIP of finding $x^* \in \mathrm{Fix}(T)$ such that

$$\langle (I - S)x^*, x - x^*\rangle \geq 0, \quad \forall x \in \mathrm{Fix}(T), \tag{14}$$

introduced first by Moudafi and Maingé in [11], which is called hierarchical fixed point problem.

In particular, whenever $\mathrm{Fix}(S) \neq \emptyset$, all elements of $\mathrm{Fix}(S)$ are solutions of VIP (14). If S is a ρ-contraction (i.e., $\|Sx - Sy\| \leq \rho\|x - y\|$ for some $\rho \in (0, 1)$), the set of solutions of VIP (14) is a singleton and it is well known as a viscosity problem, which was first introduced by Moudafi [12] and then developed by several authors [13, 14].

Very recently, Cai and Bu [1] investigated a general hierarchical fixed point problem for a countable family of continuous pseudocontractions, which covers as a special case of the problem considered in [10]. For this purpose, they first established strong convergence of an implicit iterative scheme for solving a hierarchical fixed point problem for a continuous pseudocontractive mapping in a uniformly smooth Banach space.

In this paper, let C be a nonempty closed convex subset of a uniformly smooth Banach space X such that $C \pm C \subset C$. Let $T : C \rightarrow C$ be a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$ and let $f : C \rightarrow C$ be a fixed contractive mapping with contractive coefficient $\beta \in (0, 1)$. Let $F : C \rightarrow C$ be α-strongly accretive and λ-strictly pseudocontractive with $\alpha + \lambda > 1$ and let $A : C \rightarrow C$ be a $\overline{\gamma}$-strongly positive linear bounded operator. First of all, we introduce a relaxed implicit viscosity scheme for solving a hierarchical fixed point problem for a nonexpansive mapping T:

$$x_t = (I - \theta_t F) T x_t + \theta_t [f(x_t) - t(Af(x_t) - Tx_t)], \tag{15}$$

where $\lim_{t \rightarrow 0} \theta_t = 0$. It is proven that as $t \rightarrow 0, \{x_t\}$ converges strongly to a point $z \in \mathrm{Fix}(T)$, which is the unique solution in $\mathrm{Fix}(T)$ to the VIP:

$$\langle (F - f)z, J(z - p)\rangle \leq 0, \quad \forall p \in \mathrm{Fix}(T). \tag{16}$$

On the other hand, let $\{T_n\}_{n=0}^{\infty}$ be a countable family of nonexpansive mappings from C to itself such that $\Omega =$

$\bigcap_{i=0}^{\infty} \mathrm{Fix}(T_i) \neq \emptyset$. We propose a relaxed implicit viscosity iterative algorithm for solving a hierarchical fixed point problem for a countable family of nonexpansive mappings $\{T_n\}$:

$$y_n = \alpha_n f(y_n) + \beta_n x_n + ((1-\beta_n) I - \alpha_n A)(I - \epsilon_n F) T_n y_n,$$
$$x_{n+1} = \sigma_n f(y_n) + (I - \sigma_n A) T_n y_n, \quad \forall n \geq 0, \tag{17}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\epsilon_n\}$, and $\{\sigma_n\}$ are four sequences in $(0,1)$. It is proven that under mild conditions $\{x_n\}$ converges strongly to a point $z \in \Omega$, which is the unique solution in Ω to the VIP:

$$\langle (A-f) z, J(z-p) \rangle \leq 0, \quad \forall p \in \Omega. \tag{18}$$

Furthermore, we also propose a relaxed explicit viscosity iterative algorithm for solving another hierarchical fixed point problem for a countable family of nonexpansive mappings $\{T_n\}$:

$$x_0 \in C \quad \text{chosen arbitrarily,}$$
$$x_{n+1} = (I - \beta_n F) T_n x_n + \beta_n [f(x_n) - \alpha_n (Af(x_n) - T_n x_n)], \quad \forall n \geq 0, \tag{19}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1)$. It is proven that under appropriate assumptions, $\{x_n\}$ converges strongly to a point $z \in \Omega$, which is the unique solution in Ω to the VIP:

$$\langle (F-f) z, J(z-p) \rangle \leq 0, \quad \forall p \in \Omega. \tag{20}$$

The above relaxed viscosity algorithms are based on the well-known viscosity approximation method (see, e.g., [4–6, 9]) and hybrid steepest-descent method (see, e.g., [14–17]). Our results extend, improve, supplement, and develop the recent results announced by many authors.

2. Preliminaries

We list some lemmas that will be used in the sequel. Lemma 2 can be found in [18]. Lemma 3 is an immediate consequence of the subdifferential inequality of the function $(1/2)\|\cdot\|^2$.

Lemma 2. *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1-\alpha_n) s_n + \alpha_n \beta_n + \gamma_n, \quad \forall n \geq 0, \tag{21}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ satisfy the following conditions:

(i) $\{\alpha_n\} \subset [0,1]$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $\limsup_{n\to\infty} \beta_n \leq 0$;

(iii) $\gamma_n \geq 0$ $(\forall n \geq 0)$, $\sum_{n=0}^{\infty} \gamma_n < \infty$.

Then $\limsup_{n\to\infty} s_n = 0$.

Lemma 3. *In a smooth Banach space X, there holds the following inequality:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, J(x+y) \rangle, \quad \forall x, y \in X. \tag{22}$$

Let LIM be a continuous linear functional on l^{∞} and $(a_0, a_1, \ldots) \in l^{\infty}$. We write $\mathrm{LIM}\, a_n$ instead of $\mathrm{LIM}((a_0, a_1, \ldots))$. LIM is said to be Banach limit if LIM satisfies $\|\mathrm{LIM}\| = \mathrm{LIM}\, 1 = 1$ and $\mathrm{LIM}\, a_{n+1} = \mathrm{LIM}\, a_n$ for all $(a_0, a_1, \ldots) \in l^{\infty}$. It is well known that for Banach limit LIM the following holds:

(i) for all $n \geq 1$, $a_n \leq c_n$ implies that $\mathrm{LIM}\, a_n \leq \mathrm{LIM}\, c_n$;

(ii) $\mathrm{LIM}\, a_{n+N} = \mathrm{LIM}\, a_n$ for any fixed positive integer N;

(iii) $\liminf_{n\to\infty} a_n \leq \mathrm{LIM}\, a_n \leq \limsup_{n\to\infty} a_n$ for all $(a_0, a_1, \ldots) \in l^{\infty}$.

It is easy to see that there holds the following conclusion.

Lemma 4 (see [19]). *Let $(a_0, a_1, \ldots) \in l^{\infty}$. If $\mathrm{LIM}\, a_n = 0$, then there exists a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ such that $a_{n_k} \to 0$ as $k \to \infty$.*

Recall that a Banach space X is said to satisfy Opial's condition, if whenever $\{x_n\}$ is a sequence in X which converges weakly to x as $n \to \infty$, then

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|, \quad \forall y \in X, y \neq x. \tag{23}$$

Lemma 5 (Demiclosedness principle; see [20, Theorem 10.3]). *Let X be a reflexive Banach space satisfying Opial's condition, C a nonempty closed convex subset of X, and $T : C \to C$ a nonexpansive mapping. Then the mapping $I - T$ is demiclosed on C, where I is the identity mapping; that is, if $\{x_n\}$ is a sequence of C such that $x_n \rightharpoonup x$ and $(I-T)x_n \to y$, then $(I-T)x = y$.*

The following lemma can be derived by the standard argument and hence its proof will be omitted.

Lemma 6. *Let C be a nonempty closed convex subset of a real smooth Banach space X and let $F : C \to X$ be a mapping.*

(i) *If $F : C \to X$ is α-strongly accretive and λ-strictly pseudocontractive with $\alpha + \lambda \geq 1$, then $I - F$ nonexpansive and F is Lipschitz continuous with constant $1 + 1/\lambda$;*

(ii) *If $F : C \to X$ is α-strongly accretive and λ-strictly pseudocontractive with $\alpha + \lambda > 1$, then for any fixed $\tau \in (0,1)$, $I - \tau F$ is contractive with coefficient $1 - \tau(1 - \sqrt{(1-\alpha)/\lambda})$.*

3. Relaxed Implicit Viscosity Scheme for Hierarchical Fixed Point Problem for a Nonexpansive Mapping

In this section, we introduce our relaxed implicit viscosity scheme for solving hierarchical fixed point problem for a nonexpansive mapping and show the strong convergence theorem. First, we list several useful and helpful lemmas.

Lemma 7 (see [21]). *Let X be a Banach space, C a nonempty closed and convex subset of X, and $T : C \to C$ a continuous* *and strong pseudocontraction. Then T has a unique fixed point in C.*

Lemma 8 (see [19]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the property $a_{n+1} \le (1-\gamma_n)a_n + \gamma_n\beta_n$, $\forall n \ge 0$, where $\{\gamma_n\} \subset (0,1)$ and $\{\beta_n\} \subset \mathbf{R}$ such that* (i) $\sum_{n=0}^{\infty}\gamma_n = \infty$ *and* (ii) $\limsup_{n\to\infty}\beta_n \le 0$. *Then $\{a_n\}$ converges to zero as $n \to \infty$.*

Lemma 9 (see [22]). *Let C be a nonempty closed convex subset of a real Banach space X and $T : C \to C$ a continuous pseudocontractive map. We denote $B = (2I - T)^{-1}$. Then the following holds.*

(i) *The map B is a nonexpansive self-mapping on C.*

(ii) *If $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$, then $\lim_{n\to\infty}\|x_n - Bx_n\| = 0$.*

Lemma 10 (see [23]). *Assume that A is a strongly positive linear bounded operator on a smooth Banach space X with coefficient $\overline{\gamma} > 0$ and $0 < \rho \le \|A\|^{-1}$. Then $\|I - \rho A\| \le 1 - \rho\overline{\gamma}$.*

We now state and prove our first result.

Theorem 11. *Let C be a nonempty closed convex subset of a uniformly smooth Banach space X such that $C \pm C \subset C$. Let $T : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$ and $F : C \to C$ α-strongly accretive and λ-strictly pseudocontractive with $\alpha + \lambda > 1$. Let $f : C \to C$ be a fixed contractive mapping with contractive coefficient $\beta \in (0,\gamma_0)$, $\gamma_0 = 1 - \sqrt{(1-\alpha)/\lambda}$. Let $A : C \to C$ be a $\overline{\gamma}$-strongly positive linear bounded operator with $\overline{\gamma}\beta < 1$. Let $\{x_t\}$ be defined by*

$$x_t = (I - \theta_t F) Tx_t + \theta_t [f(x_t) - t(Af(x_t) - Tx_t)], \quad (24)$$

where $\lim_{t\to 0}\theta_t = 0$. Then, as $t \to 0$, $\{x_t\}$ converges strongly to some fixed point z of T, which is the unique solution in $\mathrm{Fix}(T)$ to the VIP:

$$\langle (F - f) z, J(z - p)\rangle \le 0, \quad \forall p \in \mathrm{Fix}(T). \quad (25)$$

Proof. First, we claim that $\gamma_0 < \alpha$. Indeed, it is known that strongly accretive constant $\alpha \in (0,1)$ and strictly pseudocontractive constant $\lambda \in (0,1)$. Moreover, observe that

$$\sqrt{(1-\alpha)\lambda} < 1 \Longleftrightarrow 1 - \alpha < \sqrt{\frac{1-\alpha}{\lambda}} \Longleftrightarrow \gamma_0 < \alpha. \quad (26)$$

Let us show that the net $\{x_t\}$ is defined well. As a matter of fact, we define the mapping $S_t : C \to C$ as follows:

$$S_t x = (I - \theta_t F) Tx + \theta_t [f(x) - t(Af(x) - Tx)], \quad \forall x \in C. \quad (27)$$

Since $\lim_{t\to 0}\theta_t = 0$, we may assume, without loss of generality, that $\theta_t \in (0,1)$ for all $t \in (0,\epsilon_0)$, where $\epsilon_0 = \min\{(\gamma_0 - \beta)/2(1 - \overline{\gamma}\beta), \|A\|^{-1}\}$. Utilizing Lemmas 6 and 10, we obtain that for each $t \in (0,\epsilon_0)$

$$\begin{aligned}
&\langle S_t x - S_t y, J(x-y)\rangle \\
&= \langle (I - \theta_t F) Tx - (I - \theta_t F) Ty, J(x-y)\rangle \\
&\quad + \theta_t \langle (I - tA) f(x) - (I - tA) f(y), J(x-y)\rangle \\
&\quad + \theta_t t \langle Tx - Ty, J(x-y)\rangle \\
&\le \|(I - \theta_t F) Tx - (I - \theta_t F) Ty\| \|x-y\| \\
&\quad + \theta_t \|(I - tA) f(x) - (I - tA) f(y)\| \|x-y\| \\
&\quad + \theta_t t \|Tx - Ty\| \|x-y\| \\
&\le \left(1 - \theta_t\left(1 - \sqrt{\frac{1-\alpha}{\lambda}}\right)\right)\|Tx - Ty\| \|x-y\| \\
&\quad + \theta_t (1 - t\overline{\gamma}) \|f(x) - f(y)\| \|x-y\| + t\theta_t\|x-y\|^2 \\
&\le (1 - \theta_t\gamma_0)\|x-y\|^2 + \theta_t(1 - t\overline{\gamma})\beta\|x-y\|^2 \\
&\quad + t\theta_t\|x-y\|^2 \\
&= [1 - \theta_t(\gamma_0 - \beta - t(1 - \overline{\gamma}\beta))]\|x-y\|^2 \\
&\le \left[1 - \theta_t\left(\gamma_0 - \beta - \frac{\gamma_0 - \beta}{2(1 - \overline{\gamma}\beta)}(1 - \overline{\gamma}\beta)\right)\right]\|x-y\|^2 \\
&= \left(1 - \frac{1}{2}\theta_t(\gamma_0 - \beta)\right)\|x-y\|^2.
\end{aligned} \quad (28)$$

It follows that for each $t \in (0,\epsilon_0)$, $S_t : C \to C$ is a continuous and strongly pseudocontractive mapping with pseudocontractive coefficient $1 - (1/2)\theta_t(\gamma_0 - \beta)$. Hence, by Lemma 7 we know that there exists a unique fixed point in C, denoted by x_t, which uniquely solves the fixed point equation:

$$x_t = (I - \theta_t F) Tx_t + \theta_t [f(x_t) - t(Af(x_t) - Tx_t)]. \quad (29)$$

Let us show the uniqueness of the solution of VIP (25). Suppose both $z_1 \in \mathrm{Fix}(T)$ and $z_2 \in \mathrm{Fix}(T)$ are solutions to VIP (25). Then we have

$$\begin{aligned}
&\langle (F - f) z_1, J(z_1 - z_2)\rangle \le 0, \\
&\langle (F - f) z_2, J(z_2 - z_1)\rangle \le 0.
\end{aligned} \quad (30)$$

Adding up the above two inequalities, we obtain

$$\langle (F - f) z_1 - (F - f) z_2, J(z_1 - z_2)\rangle \le 0. \quad (31)$$

Note that

$$\begin{aligned}
&\langle (F-f) z_1 - (F-f) z_2, J(z_1 - z_2)\rangle \\
&= \langle Fz_1 - Fz_2, J(z_1 - z_2)\rangle \\
&\quad - \langle f(z_1) - f(z_2), J(z_1 - z_2)\rangle \\
&\geq \alpha \|z_1 - z_2\|^2 - \beta \|z_1 - z_2\|^2 \\
&= (\alpha - \beta)\|z_1 - z_2\|^2 \geq 0.
\end{aligned} \tag{32}$$

Taking into account $\alpha - \beta > 0$, we have $z_1 = z_2$, and hence the uniqueness is proved. We use $\tilde{z}$ to denote the unique solution of VIP (25).

Next, we prove that $\{x_t : t \in (0, \epsilon_0)\}$ is bounded. Indeed, we note that $0 < \theta_t < 1, \forall t \in (0, \epsilon_0)$. Take a fixed $p \in \mathrm{Fix}(T)$ arbitrarily. Utilizing Lemma 10 we deduce that for all $t \in (0, \epsilon_0)$

$$\begin{aligned}
&\|x_t - p\|^2 \\
&\quad = \langle (I - \theta_t F) T x_t + \theta_t [f(x_t) - t(Af(x_t) - Tx_t)] \\
&\qquad - p, J(x_t - p)\rangle \\
&\quad = \langle (I - \theta_t F) T x_t - (I - \theta_t F) Tp, J(x_t - p)\rangle \\
&\qquad + \theta_t \langle (I - tA) f(x_t) - (I - tA) f(p), J(x_t - p)\rangle \\
&\qquad + \theta_t t \langle Tx_t - p, J(x_t - p)\rangle - \theta_t \langle (F - f) p, J(x_t - p)\rangle \\
&\qquad + \theta_t t \langle (I - Af) p, J(x_t - p)\rangle \\
&\quad \leq \|(I - \theta_t F) T x_t - (I - \theta_t F) Tp\| \|x_t - p\| \\
&\qquad + \theta_t \|(I - tA) f(x_t) - (I - tA) f(p)\| \|x_t - p\| \\
&\qquad + \theta_t t \|Tx_t - p\| \|x_t - p\| + \theta_t \|(F - f) p\| \|x_t - p\| \\
&\qquad + \theta_t t \|(I - Af) p\| \|x_t - p\| \\
&\quad \leq \left(1 - \theta_t \left(1 - \sqrt{\frac{1-\alpha}{\lambda}}\right)\right) \|Tx_t - Tp\| \|x_t - p\| \\
&\qquad + \theta_t (1 - t\overline{\gamma}) \|f(x_t) - f(p)\| \|x_t - p\| \\
&\qquad + \theta_t t \|Tx_t - p\| \|x_t - p\| + \theta_t \|(F - f) p\| \|x_t - p\| \\
&\qquad + \theta_t t \|(I - Af) p\| \|x_t - p\| \\
&\quad \leq (1 - \theta_t \gamma_0) \|x_t - p\|^2 + \theta_t (1 - t\overline{\gamma}) \beta \|x_t - p\|^2 \\
&\qquad + \theta_t t \|x_t - p\|^2 + \theta_t \|(F - f) p\| \|x_t - p\| \\
&\qquad + \theta_t t \|(I - Af) p\| \|x_t - p\| \\
&\quad = [1 - \theta_t (\gamma_0 - \beta - t(1 - \overline{\gamma}\beta))] \|x_t - p\|^2 \\
&\qquad + \theta_t \|(F - f) p\| \|x_t - p\| + \theta_t t \|(I - Af) p\| \|x_t - p\| \\
&\quad \leq \left[1 - \theta_t \left(\gamma_0 - \beta - \frac{\gamma_0 - \beta}{2(1 - \overline{\gamma}\beta)}(1 - \overline{\gamma}\beta)\right)\right] \\
&\qquad \times \|x_t - p\|^2 + \theta_t \|(F - f) p\| \|x_t - p\| \\
&\qquad + \theta_t t \|(I - Af) p\| \|x_t - p\| \\
&\quad = \left(1 - \frac{1}{2}\theta_t (\gamma_0 - \beta)\right) \|x_t - p\|^2 + \theta_t \|(F - f) p\| \|x_t - p\| \\
&\qquad + \theta_t t \|(I - Af) p\| \|x_t - p\|,
\end{aligned} \tag{33}$$

which immediately yields

$$\begin{aligned}
\|x_t - p\| &\leq \frac{2}{\gamma_0 - \beta} (\|(F - f) p\| + t \|(I - Af) p\|) \\
&\leq \frac{2}{\gamma_0 - \beta} \left(\|(F - f) p\| + \|A\|^{-1} \|(I - Af) p\|\right).
\end{aligned} \tag{34}$$

Thus $\{x_t : t \in (0, \epsilon_0)\}$ is bounded.

Assume that $\{t_n\} \subset (0, \epsilon_0)$ and $t_n \to 0$ as $n \to \infty$. Set $\theta_n = \theta_{t_n}$ and $x_n := x_{t_n}$, and define $\mu : C \to \mathbf{R}$ by $\mu(x) = \mathrm{LIM}\|x_n - x\|^2, \forall x \in C$, where LIM is a Banach limit on l^∞. Let

$$K = \left\{x \in C : \mu(x) = \min_{y \in C} \mathrm{LIM}\|x_n - y\|^2\right\}. \tag{35}$$

We see easily that K is a nonempty closed convex subset of X. Note that $\|x_n - Tx_n\| = \theta_n \|f(x_n) - t_n(Af(x_n) - Tx_n) - FTx_n\| \to 0$ as $n \to \infty$. In terms of Lemma 9, we know that the mapping $B = (2I - T)^{-1} : C \to C$ is nonexpansive and $\mathrm{Fix}(T) = \mathrm{Fix}(B)$ and $\lim_{n\to\infty}\|x_n - Bx_n\| = 0$, where I denotes the identity operator. It follows that

$$\begin{aligned}
\mu(Bx) &= \mathrm{LIM}\|x_n - Bx\|^2 = \mathrm{LIM}\|Bx_n - Bx\|^2 \\
&\leq \mathrm{LIM}\|x_n - x\|^2 = \mu(x),
\end{aligned} \tag{36}$$

which implies that $B(K) \subset K$; that is, K is invariant under B. Since a uniformly smooth Banach space has the fixed point property for nonexpansive mapping, B has a fixed point, say $z \in K$. Since z is also a minimizer of μ over C, we have that, for $t \in (0, \epsilon_0)$ and $x \in C$,

$$\begin{aligned}
0 &\leq \frac{\mu(z + t(x - Fz)) - \mu(z)}{t} \\
&= \mathrm{LIM}\frac{\|x_n - z + t(Fz - x)\|^2 - \|x_n - z\|^2}{t} \\
&= \mathrm{LIM}\Big(\big(\langle x_n - z, J(x_n - z + t(Fz - x))\rangle \\
&\qquad + t\langle Fz - x, J(x_n - z + t(Fz - x))\rangle \\
&\qquad - \|x_n - z\|^2\big) t^{-1}\Big).
\end{aligned} \tag{37}$$

Since X is uniformly smooth, we conclude that the duality mapping J is norm-to-norm uniformly continuous on any bounded subset of X. Letting $t \to 0$, we find that the two limits above can be interchanged and obtain

$$\text{LIM}\langle x - Fz, J(x_n - z)\rangle \le 0, \quad \forall x \in C. \tag{38}$$

On the other hand, we have

$$\begin{aligned} x_n - z &= (I - \theta_n F) T x_n - (I - \theta_n F) T z \\ &\quad + \theta_n [(I - t_n A) f(x_n) - (I - t_n A) f(z) + t_n (T x_n - z)] \\ &\quad + \theta_n (f - F) z + \theta_n t_n (I - Af) z. \end{aligned} \tag{39}$$

It follows that

$$\begin{aligned} \|x_n - z\|^2 &= \langle (I - \theta_n F) T x_n - (I - \theta_n F) T z, J(x_n - z)\rangle \\ &\quad + \theta_n [\langle (I - t_n A)(f(x_n) - f(z)), J(x_n - z)\rangle \\ &\qquad + t_n \langle T x_n - z, J(x_n - z)\rangle] \\ &\quad + \theta_n \langle (f - F) z, J(x_n - z)\rangle \\ &\quad + \theta_n t_n \langle (I - Af) z, J(x_n - z)\rangle \\ &\le (1 - \theta_n \gamma_0) \|T x_n - T z\| \|x_n - z\| \\ &\quad + \theta_n [(1 - t_n \overline{\gamma}) \|f(x_n) - f(z)\| \|x_n - z\| \\ &\qquad + t_n \|T x_n - z\| \|x_n - z\|] \\ &\quad + \theta_n \langle (f - F) z, J(x_n - z)\rangle \\ &\quad + \theta_n t_n \|(I - Af) z\| \|x_n - z\| \\ &\le (1 - \theta_n \gamma_0) \|x_n - z\|^2 \\ &\quad + \theta_n \left[(1 - t_n \overline{\gamma}) \beta \|x_n - z\|^2 + t_n \|x_n - z\|^2\right] \\ &\quad + \theta_n \langle (f - F) z, J(x_n - z)\rangle \\ &\quad + \theta_n t_n \|(I - Af) z\| \|x_n - z\| \\ &= [1 - \theta_n (\gamma_0 - \beta - t_n (1 - \overline{\gamma}\beta))] \|x_n - z\|^2 \\ &\quad + \theta_n \langle (f - F) z, J(x_n - z)\rangle \\ &\quad + \theta_n t_n \|(I - Af) z\| \|x_n - z\| \\ &\le \left[1 - \theta_n \left(\gamma_0 - \beta - \frac{\gamma_0 - \beta}{2(1 - \overline{\gamma}\beta)} (1 - \overline{\gamma}\beta)\right)\right] \|x_n - z\|^2 \\ &\quad + \theta_n \langle (f - F) z, J(x_n - z)\rangle \\ &\quad + \theta_n t_n \|(I - Af) z\| \|x_n - z\| \\ &= \left(1 - \frac{1}{2}\theta_n (\gamma_0 - \beta)\right) \|x_n - z\|^2 \\ &\quad + \theta_n \langle (f - F) z, J(x_n - z)\rangle \\ &\quad + \theta_n t_n \|(I - Af) z\| \|x_n - z\|. \end{aligned} \tag{40}$$

Therefore,

$$\begin{aligned} \|x_n - z\|^2 &\le \frac{2}{\gamma_0 - \beta} \\ &\quad \times (\langle (f - F) z, J(x_n - z)\rangle + t_n \|(I - Af) z\| \|x_n - z\|). \end{aligned} \tag{41}$$

Combining (38) and (41), we get

$$\text{LIM}\|x_n - z\|^2 \le \frac{2}{\gamma_0 - \beta} \text{LIM}\langle (f - F) z, J(x_n - z)\rangle \le 0, \tag{42}$$

which leads to $\text{LIM}\|x_n - z\|^2 = 0$. Hence there exists a subsequence which is still denoted as $\{x_n\}$ such that $x_n \to z$ as $n \to \infty$.

Next, we prove that z solves VIP (25). Since

$$x_t = (I - \theta_t F) T x_t + \theta_t [f(x_t) - t(Af(x_t) - T x_t)], \tag{43}$$

we can deduce that

$$x_t - T x_t = \theta_t (f(x_t) - F T x_t) + \theta_t t (T x_t - Af(x_t)). \tag{44}$$

Since T is nonexpansive, $I - T$ is accretive. So, from the accretivity of $I - T$, it follows that, for any fixed $p \in \text{Fix}(T)$,

$$\begin{aligned} 0 &\le \langle (I - T) x_t - (I - T) p, J(x_t - p)\rangle \\ &= \langle (I - T) x_t, J(x_t - p)\rangle \\ &= \theta_t \langle f(x_t) - F T x_t, J(x_t - p)\rangle \\ &\quad + \theta_t t \langle T x_t - Af(x_t), J(x_t - p)\rangle \\ &= \theta_t \langle (f - F) x_t, J(x_t - p)\rangle \\ &\quad + \theta_t \langle F x_t - F T x_t, J(x_t - p)\rangle \\ &\quad + \theta_t t \langle T x_t - Af(x_t), J(x_t - p)\rangle. \end{aligned} \tag{45}$$

This implies that

$$\begin{aligned} &\langle (F - f) x_t, J(x_t - p)\rangle \\ &\quad \le \langle F x_t - F T x_t, J(x_t - p)\rangle + t \langle T x_t - Af(x_t), J(x_t - p)\rangle. \end{aligned} \tag{46}$$

Now replacing t with t_n, letting $n \to \infty$, and noticing the boundedness of $\{Tx_{t_n} - Af(x_{t_n})\}$ and the fact that $Fx_{t_n} - FTx_{t_n} \to Fz - FTz = 0$ for $z \in \mathrm{Fix}(T)$, we have that

$$\langle (F - f) z, J(z - p) \rangle \leq 0, \quad \forall p \in \mathrm{Fix}(T). \tag{47}$$

That is, $z \in \mathrm{Fix}(T)$ is a solution of VIP (25). Then $z = \tilde{z}$. In summary, we infer that each cluster point of $\{x_n\}$ is equal to z as $t_n \to 0$. This completes the proof. □

4. Relaxed Viscosity Algorithms for Hierarchical Fixed Point Problems for a Countable Family of Nonexpansive Mappings

In this section, we propose relaxed implicit and explicit viscosity algorithms for solving hierarchical fixed point problems for a countable family of nonexpansive mappings and show strong convergence theorems. For this purpose, we will use the following lemmas in the sequel.

Lemma 12 (see [24]). *Let C be a nonempty closed convex subset of a Banach space X. Let $T_1, T_2, \ldots$ be a sequence of mappings of C into itself. Suppose that $\sum_{n=1}^{\infty} \sup\{\|T_{n+1}x - T_n x\| : x \in C\} < \infty$. Then, for each $y \in C$, $\{T_n y\}$ converges strongly to some point of C. Moreover, let T be a mapping of C into itself defined by $Ty = \lim_{n\to\infty} T_n y$, for all $y \in C$. Then $\lim_{n\to\infty} \sup\{\|Tx - T_n x\| : x \in C\} = 0$.*

Lemma 13 (see [1, Lemma 2.6]). *Let C be a nonempty closed convex subset of a real Banach space X which has uniformly Gateaux differentiable norm. Let $T : C \to C$ be a continuous pseudocontractive mapping with $\mathrm{Fix}(T) \neq \emptyset$ and let $f : C \to C$ be a fixed Lipschitzian strongly pseudocontractive mapping with pseudocontractive coefficient $\beta \in (0,1)$ and Lipschitzian constant $L > 0$. Let $A : C \to C$ be a $\overline{\gamma}$-strongly positive linear bounded operator with coefficient $\overline{\gamma} > 0$. Assume that $C \pm C \subset C$ and that $\{x_t\}$ converges strongly to $z \in \mathrm{Fix}(T)$ as $t \to 0$, where x_t is defined by $x_t = tf(x_t) + (I - tA)Tx_t$. Suppose that $\{x_n\} \subset C$ is bounded and that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Then $\limsup_{n\to\infty} \langle (f - A)z, J(x_n - z)\rangle \leq 0$.*

Theorem 14. *Let C be a nonempty closed convex subset of a uniformly smooth Banach space X such that $C \pm C \subset C$. Let $\{T_i\}_{i=0}^{\infty}$ be a countable family of nonexpansive mappings from C to itself such that $\Omega = \bigcap_{i=0}^{\infty} \mathrm{Fix}(T_i) \neq \emptyset$. Let $F : C \to C$ be α-strongly accretive and λ-strictly pseudocontractive with $\alpha + \lambda > 1$, and let $f : C \to C$ be a fixed contractive mapping with contractive coefficient $\beta \in (0,1)$. Let $A : C \to C$ be a $\overline{\gamma}$-strongly positive linear bounded operator with $\overline{\gamma} \in (\beta, 1 + \beta)$. For arbitrarily given $x_0 \in C$, let the sequence $\{x_n\}$ be generated iteratively by*

$$\begin{aligned} y_n &= \alpha_n f(y_n) + \beta_n x_n \\ &\quad + ((1 - \beta_n) I - \alpha_n A)(I - \epsilon_n F) T_n y_n, \\ x_{n+1} &= \sigma_n f(y_n) + (I - \sigma_n A) T_n y_n, \quad \forall n \geq 0, \end{aligned} \tag{48}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\epsilon_n\}$, and $\{\sigma_n\}$ are four sequences in $(0,1)$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$;

(ii) $\lim_{n\to\infty} (\epsilon_n/\alpha_n) = 0$, $\limsup_{n\to\infty} (\sigma_n/\alpha_n) < \infty$, *and* $\sum_{n=0}^{\infty} (\alpha_n/(\alpha_n + \beta_n)) = \infty$.

Assume that $\sum_{n=0}^{\infty} \sup_{x\in D} \|T_{n+1}x - T_n x\| < \infty$ for any bounded subset D of C, let T be a mapping of C into itself defined by $Tx = \lim_{n\to\infty} T_n x$ for all $x \in C$, and suppose that $\mathrm{Fix}(T) = \bigcap_{i=0}^{\infty} \mathrm{Fix}(T_i)$. Then, $\{x_n\}$ converges strongly to a point z of Ω such that z is a unique solution in Ω to the VIP:

$$\langle (f - A) z, J(p - z) \rangle \leq 0, \quad \forall p \in \Omega. \tag{49}$$

Proof. By condition (i), we may assume, without loss of generality, that $\alpha_n \leq (1 - \beta_n)\|A\|^{-1}$. Since A is a $\overline{\gamma}$-strongly positive linear bounded operator on C, from (11) we have

$$\|A\| = \sup\{|\langle Au, J(u)\rangle| : u \in C, \|u\| = 1\}. \tag{50}$$

Observe that

$$\begin{aligned} \langle ((1 - \beta_n) I - \alpha_n A) u, J(u) \rangle &= 1 - \beta_n - \alpha_n \langle Au, J(u) \rangle \\ &\geq 1 - \beta_n - \alpha_n \|A\| \\ &\geq 0. \end{aligned} \tag{51}$$

It follows that

$$\begin{aligned} &\|(1 - \beta_n) I - \alpha_n A\| \\ &\quad = \sup\{\langle ((1 - \beta_n) I - \alpha_n A) u, J(u) \rangle : u \in C, \|u\| = 1\} \\ &\quad = \sup\{1 - \beta_n - \alpha_n \langle Au, J(u) \rangle : u \in C, \|u\| = 1\} \\ &\quad \leq 1 - \beta_n - \alpha_n \overline{\gamma}. \end{aligned} \tag{52}$$

Next, we show that $\{y_n\}$ is well defined. For each $n \geq 0$, define a mapping $S_n : C \to C$ by

$$S_n x = \alpha_n f(x) + \beta_n x_n + ((1 - \beta_n) I - \alpha_n A)(I - \epsilon_n F) T_n x, \quad \forall x \in C. \tag{53}$$

For every $x, y \in C$, we have

$$\begin{aligned} &\langle S_n x - S_n y, J(x - y) \rangle \\ &\quad = \alpha_n \langle f(x) - f(y), J(x - y) \rangle \\ &\qquad + \langle ((1 - \beta_n) I - \alpha_n A) \\ &\qquad\quad \times ((I - \epsilon_n F) T_n x - (I - \epsilon_n F) T_n y), J(x - y) \rangle \end{aligned}$$

$$
\begin{aligned}
&\le \alpha_n\beta\|x-y\|^2 + (1-\beta_n-\alpha_n\bar{\gamma}) \\
&\quad \times \|(I-\epsilon_n F)T_n x-(I-\epsilon_n F)T_n y\|\,\|x-y\| \\
&\le \alpha_n\beta\|x-y\|^2 \\
&\quad + (1-\beta_n-\alpha_n\bar{\gamma})(1-\epsilon_n\gamma_0)\|T_n x-T_n y\|\,\|x-y\| \\
&\le \alpha_n\beta\|x-y\|^2 + (1-\beta_n-\alpha_n\bar{\gamma})\|x-y\|^2 \\
&= [1-\beta_n-\alpha_n(\bar{\gamma}-\beta)]\,\|x-y\|^2,
\end{aligned} \tag{54}
$$

where $\gamma_0 = 1-\sqrt{(1-\alpha)/\lambda}$. Therefore, S_n is a continuous strong pseudocontraction for each $n \ge 0$. By Lemma 7, we see that there exists a unique fixed point y_n for each $n \ge 0$ such that

$$
y_n = \alpha_n f(y_n) + \beta_n x_n + ((I-\beta_n)I-\alpha_n A)(I-\epsilon_n F)T_n y_n. \tag{55}
$$

That is, the sequence $\{y_n\}$ is well defined. Next, we prove that $\{x_n\}$ is bounded. Take a fixed $p \in \Omega$ arbitrarily. Taking into account $\lim_{n\to\infty}(\epsilon_n/\alpha_n) = 0$, we may assume that there exists a constant $\tau \in (0,1)$ such that $\epsilon_n \le \tau\alpha_n$ for all $n \ge 0$. Then we have

$$
\begin{aligned}
&\|y_n-p\|^2 \\
&= \alpha_n\langle f(y_n)-Ap, J(y_n-p)\rangle \\
&\quad + \beta_n\langle x_n-p, J(y_n-p)\rangle \\
&\quad + \langle((1-\beta_n)I-\alpha_n A)((I-\epsilon_n F)T_n y_n-(I-\epsilon_n F)p), \\
&\qquad J(y_n-p)\rangle \\
&\quad - \epsilon_n\langle((1-\beta_n)I-\alpha_n A)Fp, J(y_n-p)\rangle \\
&\le \alpha_n\langle f(y_n)-f(p), J(y_n-p)\rangle \\
&\quad + \alpha_n\langle f(p)-Ap, J(y_n-p)\rangle \\
&\quad + \beta_n\|x_n-p\|\,\|y_n-p\| \\
&\quad + (1-\beta_n-\alpha_n\bar{\gamma})\|(I-\epsilon_n F)T_n y_n-(I-\epsilon_n F)p\| \\
&\quad \times \|y_n-p\| + \epsilon_n(1-\beta_n-\alpha_n\bar{\gamma})\|Fp\|\,\|y_n-p\| \\
&\le \alpha_n\beta\|y_n-p\|^2 + \alpha_n\langle f(p)-Ap, J(y_n-p)\rangle \\
&\quad + \beta_n\|x_n-p\|\,\|y_n-p\| \\
&\quad + (1-\beta_n-\alpha_n\bar{\gamma})(1-\epsilon_n\gamma_0)\|T_n y_n-p\|\,\|y_n-p\| \\
&\quad + \epsilon_n(1-\beta_n-\alpha_n\bar{\gamma})\|Fp\|\,\|y_n-p\| \\
&\le \alpha_n\beta\|y_n-p\|^2 + (1-\beta_n-\alpha_n\bar{\gamma})\|y_n-p\|^2 \\
&\quad + \alpha_n\langle f(p)-Ap, J(y_n-p)\rangle \\
&\quad + \beta_n\|x_n-p\|\,\|y_n-p\| + \epsilon_n\|Fp\|\,\|y_n-p\| \\
&= (1-\beta_n-\alpha_n(\bar{\gamma}-\beta))\|y_n-p\|^2 \\
&\quad + \beta_n\|x_n-p\|\,\|y_n-p\| \\
&\quad + \alpha_n\|f(p)-Ap\|\,\|y_n-p\| + \epsilon_n\|Fp\|\,\|y_n-p\| \\
&\le (1-\beta_n-\alpha_n(\bar{\gamma}-\beta))\|y_n-p\|^2 \\
&\quad + \beta_n\|x_n-p\|\,\|y_n-p\| \\
&\quad + (\alpha_n+\epsilon_n)(\|f(p)-Ap\|+\|Fp\|)\|y_n-p\| \\
&\le (1-\beta_n-\alpha_n(\bar{\gamma}-\beta))\|y_n-p\|^2 \\
&\quad + \beta_n\|x_n-p\|\,\|y_n-p\| \\
&\quad + \alpha_n(1+\tau)(\|f(p)-Ap\|+\|Fp\|)\|y_n-p\|,
\end{aligned} \tag{56}
$$

which implies that

$$
\begin{aligned}
\|y_n-p\| &\le \frac{\beta_n}{\beta_n+\alpha_n(\bar{\gamma}-\beta)}\|x_n-p\| \\
&\quad + \frac{\alpha_n(\bar{\gamma}-\beta)}{\beta_n+\alpha_n(\bar{\gamma}-\beta)} \\
&\quad \cdot \frac{(1+\tau)(\|f(p)-Ap\|+\|Fp\|)}{\bar{\gamma}-\beta}.
\end{aligned} \tag{57}
$$

Therefore, we have

$$
\begin{aligned}
&\|x_{n+1}-p\| \\
&= \|\sigma_n f(y_n) + (I-\sigma_n A)T_n y_n - p\| \\
&= \|\sigma_n(f(y_n)-f(p)) + (I-\sigma_n A)T_n y_n \\
&\qquad - (I-\sigma_n A)T_n p + \sigma_n(f(p)-Ap)\| \\
&\le \sigma_n\|f(y_n)-f(p)\| \\
&\quad + \|(I-\sigma_n A)(T_n y_n-T_n p)\| + \sigma_n\|f(p)-Ap\| \\
&\le \sigma_n\beta\|y_n-p\| + (1-\sigma_n\bar{\gamma})\|y_n-p\| + \sigma_n\|f(p)-Ap\| \\
&= (1-\sigma_n(\bar{\gamma}-\beta))\|y_n-p\| + \sigma_n\|f(p)-Ap\| \\
&\le (1-\sigma_n(\bar{\gamma}-\beta)) \\
&\quad \times \left[\frac{\beta_n}{\beta_n+\alpha_n(\bar{\gamma}-\beta)}\|x_n-p\| + \frac{\alpha_n(\bar{\gamma}-\beta)}{\beta_n+\alpha_n(\bar{\gamma}-\beta)}\right. \\
&\qquad \left.\cdot\frac{(1+\tau)(\|f(p)-Ap\|+\|Fp\|)}{\bar{\gamma}-\beta}\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\sigma_n \|f(p)-Ap\| \\
&\le (1-\sigma_n(\overline{\gamma}-\beta)) \\
&\quad\times \max\left\{\|x_n-p\|, \frac{(1+\tau)(\|f(p)-Ap\|+\|Fp\|)}{\overline{\gamma}-\beta}\right\} \\
&\quad+\sigma_n \|f(p)-Ap\| \\
&= (1-\sigma_n(\overline{\gamma}-\beta)) \\
&\quad\times \max\left\{\|x_n-p\|, \frac{(1+\tau)(\|f(p)-Ap\|+\|Fp\|)}{\overline{\gamma}-\beta}\right\} \\
&\quad+\sigma_n(\overline{\gamma}-\beta)\frac{\|f(p)-Ap\|}{\overline{\gamma}-\beta} \\
&\le \max\left\{\|x_n-p\|, \frac{(1+\tau)(\|f(p)-Ap\|+\|Fp\|)}{\overline{\gamma}-\beta}, \frac{\|f(p)-Ap\|}{\overline{\gamma}-\beta}\right\} \\
&\le \max\left\{\|x_n-p\|, \frac{(1+\tau)(\|f(p)-Ap\|+\|Fp\|)}{\overline{\gamma}-\beta}\right\}.
\end{aligned} \tag{58}
$$

By induction, we get

$$
\begin{aligned}
&\|x_n-p\| \\
&\quad\le \max\left\{\|x_0-p\|, \frac{(1+\tau)(\|f(p)-Ap\|+\|Fp\|)}{\overline{\gamma}-\beta}\right\}, \\
&\qquad\forall n\ge 0.
\end{aligned} \tag{59}
$$

Therefore, $\{x_n\}$ is bounded and so are the sequences $\{y_n\}$, $\{T_ny_n\}$. We observe that

$$
\begin{aligned}
\|y_n-T_ny_n\| &= \|\alpha_n(f(y_n)-AT_ny_n)+\beta_n(x_n-T_ny_n) \\
&\quad-\epsilon_n((1-\beta_n)I-\alpha_nA)FT_ny_n\| \\
&\le \alpha_n\|f(y_n)-AT_ny_n\|+\beta_n\|x_n-T_ny_n\| \\
&\quad+\epsilon_n\|((1-\beta_n)I-\alpha_nA)FT_ny_n\| \\
&\le \alpha_n\|f(y_n)-AT_ny_n\|+\beta_n\|x_n-T_ny_n\| \\
&\quad+\epsilon_n(1-\beta_n-\alpha_n\overline{\gamma})\|FT_ny_n\| \\
&\le \alpha_n\|f(y_n)-AT_ny_n\|+\beta_n\|x_n-T_ny_n\| \\
&\quad+\epsilon_n\|FT_ny_n\|,
\end{aligned} \tag{60}
$$

which go together with condition (i) and $\epsilon_n \le \tau\alpha_n, \forall n\ge 0$, implying that

$$
\lim_{n\to\infty}\|y_n-T_ny_n\| = 0. \tag{61}
$$

On the other hand, we have

$$
\|y_n-Ty_n\| \le \|y_n-T_ny_n\|+\|T_ny_n-Ty_n\|. \tag{62}
$$

Utilizing Lemma 12, we immediately derive

$$
\lim_{n\to\infty}\|y_n-Ty_n\| = 0. \tag{63}
$$

Let $x_t = tf(x_t)+(I-tA)Tx_t$. Utilizing [1, Lemma 2.5] and Lemma 13, we conclude that $\{x_t\}$ converges strongly to $z \in \mathrm{Fix}(T) = \bigcap_{i=0}^{\infty}\mathrm{Fix}(T_i) = \Omega$ and

$$
\limsup_{n\to\infty}\langle (f-A)z, J(y_n-z)\rangle \le 0. \tag{64}
$$

Finally, we show that $x_n \to z$ as $n\to\infty$. We observe that

$$
\begin{aligned}
&\|y_n-z\|^2 \\
&\quad= \alpha_n\langle f(y_n)-Az, J(y_n-z)\rangle+\beta_n\langle x_n-z, J(y_n-z)\rangle \\
&\qquad+\langle ((1-\beta_n)I-\alpha_nA)(T_ny_n-z), J(y_n-z)\rangle \\
&\qquad-\epsilon_n\langle ((1-\beta_n)I-\alpha_nA)FT_ny_n, J(y_n-z)\rangle \\
&\quad\le (1-\beta_n-\alpha_n\overline{\gamma})\|y_n-z\|^2+\beta_n\|x_n-z\|\|y_n-z\| \\
&\qquad+\alpha_n\langle f(y_n)-f(z), J(y_n-z)\rangle \\
&\qquad+\alpha_n\langle f(z)-Az, J(y_n-z)\rangle \\
&\qquad+\epsilon_n\|((1-\beta_n)I-\alpha_nA)FT_ny_n\|\|y_n-z\| \\
&\quad\le (1-\beta_n-\alpha_n\overline{\gamma})\|y_n-z\|^2 \\
&\qquad+\beta_n\|x_n-z\|\|y_n-z\|+\alpha_n\beta\|y_n-z\|^2 \\
&\qquad+\alpha_n\langle f(z)-Az, J(y_n-z)\rangle \\
&\qquad+\epsilon_n(1-\beta_n-\alpha_n\overline{\gamma})\|FT_ny_n\|\|y_n-z\|
\end{aligned}
$$

$$\leq (1-\beta_n-\alpha_n\overline{\gamma})\|y_n-z\|^2+\frac{\beta_n}{2}\|x_n-z\|^2+\frac{\beta_n}{2}\|y_n-z\|^2$$
$$+\alpha_n\beta\|y_n-z\|^2+\alpha_n\langle f(z)-Az, J(y_n-z)\rangle$$
$$+\epsilon_n\|FT_ny_n\|\|y_n-z\|$$
$$=\left(1-\frac{\beta_n}{2}-\alpha_n(\overline{\gamma}-\beta)\right)\|y_n-z\|^2+\frac{\beta_n}{2}\|x_n-z\|^2$$
$$+\alpha_n\langle f(z)-Az, J(y_n-z)\rangle+\epsilon_n\|FT_ny_n\|\|y_n-z\|, \tag{65}$$

which implies that

$$\|y_n-z\|^2$$
$$\leq \frac{\beta_n}{\beta_n+2\alpha_n(\overline{\gamma}-\beta)}\|x_n-z\|^2$$
$$+\frac{2\alpha_n}{\beta_n+2\alpha_n(\overline{\gamma}-\beta)}\langle f(z)-Az, J(y_n-z)\rangle$$
$$+\frac{2\epsilon_n}{\beta_n+2\alpha_n(\overline{\gamma}-\beta)}\|FT_ny_n\|\|y_n-z\|$$
$$=\left(1-\frac{2\alpha_n(\overline{\gamma}-\beta)}{\beta_n+2\alpha_n(\overline{\gamma}-\beta)}\right)\|x_n-z\|^2+\frac{2\alpha_n(\overline{\gamma}-\beta)}{\beta_n+2\alpha_n(\overline{\gamma}-\beta)}$$
$$\times\left(\frac{\langle f(z)-Az, J(y_n-z)\rangle}{\overline{\gamma}-\beta}+\frac{\epsilon_n}{\alpha_n}\cdot\frac{\|FT_ny_n\|\|y_n-z\|}{\overline{\gamma}-\beta}\right). \tag{66}$$

Furthermore, utilizing Lemma 3 from the last relation we have

$$\|x_{n+1}-z\|^2$$
$$=\|\sigma_n(f(y_n)-f(z))+(I-\sigma_nA)T_ny_n$$
$$-(I-\sigma_nA)T_nz+\sigma_n(f(z)-F(z))\|^2$$
$$\leq\|\sigma_n(f(y_n)-f(z))$$
$$+(I-\sigma_nA)T_ny_n-(I-\sigma_nA)T_nz\|^2$$
$$+2\sigma_n\langle f(z)-Az, J(x_{n+1}-z)\rangle$$
$$\leq[\sigma_n\beta\|y_n-z\|+(1-\sigma_n\overline{\gamma})\|T_ny_n-T_nz\|]^2$$
$$+2\sigma_n\|f(z)-Az\|\|x_{n+1}-z\|$$
$$\leq[\sigma_n\beta\|y_n-z\|+(1-\sigma_n\overline{\gamma})\|y_n-z\|]^2$$
$$+2\sigma_n\|f(z)-Az\|\|x_{n+1}-z\|$$
$$=(1-\sigma_n(\overline{\gamma}-\beta))^2\|y_n-z\|^2$$
$$+2\sigma_n\|f(z)-Az\|\|x_{n+1}-z\|$$
$$\leq\|y_n-z\|^2+2\sigma_n\|f(z)-Az\|\|x_{n+1}-z\|$$
$$\leq\left(1-\frac{2\alpha_n(\overline{\gamma}-\beta)}{\beta_n+2\alpha_n(\overline{\gamma}-\beta)}\right)\|x_n-z\|^2$$
$$+\frac{2\alpha_n(\overline{\gamma}-\beta)}{(\beta_n+2\alpha_n(\overline{\gamma}-\beta))}$$
$$\times\left(\frac{\langle f(z)-Az, J(y_n-z)\rangle}{\overline{\gamma}-\beta}+\frac{\epsilon_n}{\alpha_n}\cdot\frac{\|FT_ny_n\|\|y_n-z\|}{\overline{\gamma}-\beta}\right)$$
$$+2\sigma_n\|f(z)-Az\|\|x_{n+1}-z\|$$
$$=\left(1-\frac{2\alpha_n(\overline{\gamma}-\beta)}{\beta_n+2\alpha_n(\overline{\gamma}-\beta)}\right)\|x_n-z\|^2+\frac{2\alpha_n(\overline{\gamma}-\beta)}{\beta_n+2\alpha_n(\overline{\gamma}-\beta)}$$
$$\times\left\{\frac{\langle f(z)-Az, J(y_n-z)\rangle}{\overline{\gamma}-\beta}+\frac{\epsilon_n}{\alpha_n}\cdot\frac{\|FT_ny_n\|\|y_n-z\|}{\overline{\gamma}-\beta}+\frac{\beta_n+2\alpha_n(\overline{\gamma}-\beta)}{\overline{\gamma}-\beta}\cdot\frac{\sigma_n}{\alpha_n}\cdot\|f(z)-Az\|\|x_{n+1}-z\|\right\}. \tag{67}$$

We note that

$$\frac{2\alpha_n(\overline{\gamma}-\beta)}{\beta_n+2\alpha_n(\overline{\gamma}-\beta)}>\frac{2\alpha_n(\overline{\gamma}-\beta)}{2\beta_n+2\alpha_n}=(\overline{\gamma}-\beta)\frac{\alpha_n}{\alpha_n+\beta_n}. \tag{68}$$

Therefore, condition (ii) leads to $\sum_{n=0}^{\infty}(2\alpha_n(\overline{\gamma}-\beta)/(\beta_n+2\alpha_n(\overline{\gamma}-\beta)))=\infty$. In addition, since $\alpha_n\to 0, \beta_n\to 0, (\epsilon_n/\alpha_n)\to 0$, and $\limsup_{n\to\infty}(\sigma_n/\alpha_n)<\infty$, we get the following from (64)

$$\limsup_{n\to\infty}\left\{\frac{\langle f(z)-Az, J(y_n-z)\rangle}{\overline{\gamma}-\beta}+\frac{\epsilon_n}{\alpha_n}\cdot\frac{\|FT_ny_n\|\|y_n-z\|}{\overline{\gamma}-\beta}+\frac{\beta_n+2\alpha_n(\overline{\gamma}-\beta)}{\overline{\gamma}-\beta}\cdot\frac{\sigma_n}{\alpha_n}\cdot\|f(z)-F(z)\|\|x_{n+1}-z\|\right\}\leq 0. \tag{69}$$

Applying Lemma 2, we have $x_n\to z$ as $n\to\infty$. This completes the proof. □

Remark 15. Put $\alpha_n = \sigma_n = 1/n$ and $\beta_n = \epsilon_n = 1/n^2$. Then $\{\alpha_n\}, \{\beta_n\}, \{\epsilon_n\}$, and $\{\sigma_n\}$ satisfy conditions (i) and (ii) of Theorem 14. But we note that $\alpha_n/\beta_n = n \to \infty$.

Remark 16. In the iterative scheme of Theorem 14, the first iterative step $y_n = \alpha_n f(y_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)(I - \epsilon_n F)T_n y_n$ is a predictor step and the second iterative step $x_{n+1} = \sigma_n f(y_n) + (I - \sigma_n A)T_n y_n$ is a corrector step. Hence our iteration process is the predictor-corrector method.

Remark 17. Theorem 14 extends and improves Theorem 3.1 of [10] to a great extent in the following aspects:

(i) u is replaced by a fixed contractive mapping;

(ii) one continuous pseudocontractive mapping (including nonexpansive mapping) is replaced by a countable family of nonexpansive mappings;

(iii) condition $\alpha_n/\beta_n \to 0$ is weakened to the one $\alpha_n \to 0$ and $\beta_n \to 0$ as $n \to \infty$;

(iv) we add a strongly positive linear bounded operator A and a strongly accretive and strictly pseudocontractive mapping F in our iterative algorithm.

Theorem 18. *Let C be a nonempty closed convex subset of a uniformly smooth Banach space X which has the weakly sequentially continuous duality mapping J. Assume that $C \pm C \subset C$. Let $\{T_i\}_{i=0}^{\infty}$ be a countable family of nonexpansive mappings from C to itself such that $\Omega = \bigcap_{i=0}^{\infty} \mathrm{Fix}(T_i) \neq \emptyset$. Let $F : C \to C$ be α-strongly accretive and λ-strictly pseudocontractive with $\alpha + \lambda > 1$, and let $f : C \to C$ be a fixed contractive mapping with contractive coefficient $\beta \in (0, \gamma_0)$, $\gamma_0 = 1 - \sqrt{(1-\alpha)/\lambda}$. Let $A : C \to C$ be a $\overline{\gamma}$-strongly positive linear bounded operator with $\overline{\gamma}\beta < 1$. For arbitrarily given $x_0 \in C$, let the sequence $\{x_n\}$ be generated iteratively by*

$$x_{n+1} = (I - \beta_n F) T_n x_n + \beta_n [f(x_n) - \alpha_n (Af(x_n) - T_n x_n)], \quad \forall n \geq 0, \tag{70}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1)$ satisfying the following conditions:

(i) $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\beta_n = 0$ *and* $\sum_{n=0}^{\infty}\beta_n = \infty$;

(ii) $\sum_{n=1}^{\infty}|\beta_n - \beta_{n-1}| < \infty$ *or* $\lim_{n\to\infty}\beta_{n-1}/\beta_n = 1$.

Assume that $\sum_{n=0}^{\infty}\sup_{x\in D}\|T_{n+1}x - T_n x\| < \infty$ for any bounded subset D of C, let T be a mapping of C into itself defined by $Tx = \lim_{n\to\infty} T_n x$ for all $x \in C$, and suppose that $\mathrm{Fix}(T) = \bigcap_{i=0}^{\infty}\mathrm{Fix}(T_i)$. Then, $\{x_n\}$ converges strongly to a point z of Ω such that z is a unique solution in Ω to the VIP:

$$\langle (F - f) z, J(z - p) \rangle \leq 0, \quad \forall p \in \Omega. \tag{71}$$

Proof. First, since A is a $\overline{\gamma}$-strongly positive linear bounded operator on C, from (11) we have

$$\|A\| = \sup\{|\langle Au, J(u)\rangle| : u \in C, \|u\| = 1\}. \tag{72}$$

Let us show that $\{x_n\}$ is bounded. Indeed, since $\lim_{n\to\infty}\alpha_n = 0$, without loss of generality, we may assume that $0 < \alpha_n \leq \min\{(\gamma_0 - \beta)/2(1 - \overline{\gamma}\beta), \|A\|^{-1}\}, \forall n \geq 0$. Take $p \in \Omega$. Then it follows that $p = T_n p, \forall n \geq 0$, and

$$\begin{aligned} x_{n+1} - p = {} & (I - \beta_n F) T_n x_n - (I - \beta_n F) T_n p \\ & + \beta_n [(I - \alpha_n A) f(x_n) - (I - \alpha_n A) f(p) \\ & + \alpha_n (T_n x_n - p)] \\ & + \beta_n (f - F) p + \beta_n \alpha_n (I - Af) p. \end{aligned} \tag{73}$$

Hence we deduce the following $0 < \alpha_n \leq \min\{(\gamma_0 - \beta)/2(1 - \overline{\gamma}\beta), \|A\|^{-1}\}$ that

$$\begin{aligned} & \|x_{n+1} - p\| \\ & = \|(I - \beta_n F) T_n x_n - (I - \beta_n F) T_n p \\ & \quad + \beta_n [(I - \alpha_n A) f(x_n) - (I - \alpha_n A) f(p) \\ & \quad + \alpha_n (T_n x_n - p)] \\ & \quad + \beta_n (f - F) p + \beta_n \alpha_n (I - Af) p\| \\ & \leq \|(I - \beta_n F) T_n x_n - (I - \beta_n F) T_n p\| \\ & \quad + \beta_n [\|I - \alpha_n A\| \|f(x_n) - f(p)\| + \alpha_n \|T_n x_n - p\|] \\ & \quad + \beta_n \|(f - F) p\| + \beta_n \alpha_n \|(I - Af) p\| \\ & \leq (1 - \beta_n \gamma_0) \|x_n - p\| \\ & \quad + \beta_n [(1 - \alpha_n \overline{\gamma}) \beta \|x_n - p\| + \alpha_n \|x_n - p\|] \\ & \quad + \beta_n \|(f - F) p\| + \beta_n \alpha_n \|(I - Af) p\| \\ & = [1 - \beta_n (\gamma_0 - \beta - \alpha_n (1 - \overline{\gamma}\beta))] \|x_n - p\| \\ & \quad + \beta_n \|(f - F) p\| + \beta_n \alpha_n \|(I - Af) p\| \\ & \leq \left[1 - \beta_n \left(\gamma_0 - \beta - \frac{\gamma_0 - \beta}{2(1 - \overline{\gamma}\beta)} (1 - \overline{\gamma}\beta)\right)\right] \|x_n - p\| \\ & \quad + \beta_n \|(f - F) p\| + \beta_n \alpha_n \|(I - Af) p\| \\ & \leq \left(1 - \frac{1}{2}\beta_n (\gamma_0 - \beta)\right) \|x_n - p\| \\ & \quad + \beta_n (\|(f - F) p\| + \|(I - Af) p\|) \\ & = \left(1 - \frac{1}{2}\beta_n (\gamma_0 - \beta)\right) \|x_n - p\| \\ & \quad + \frac{1}{2}\beta_n (\gamma_0 - \beta) \frac{2(\|(f - F) p\| + \|(I - Af) p\|)}{\gamma_0 - \beta} \\ & \leq \max\left\{\|x_n - p\|, \frac{2(\|(f - F) p\| + \|(I - Af) p\|)}{\gamma_0 - \beta}\right\}. \end{aligned} \tag{74}$$

By induction

$$\|x_n - p\| \le \max\left\{\|x_0 - p\|, \frac{2\left(\|(f - F)\, p\| + \|(I - Af)\, p\|\right)}{\gamma_0 - \beta}\right\}, \quad \forall n \ge 0. \tag{75}$$

This implies that $\{x_n\}$ is bounded and so are $\{T_n x_n\}, \{f(x_n)\}$ and $\{FT_n x_n\}$.

Now we claim that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{76}$$

Indeed, first of all, (70) can be rewritten as follows:

$$\begin{aligned} y_n &= (I - \alpha_n A)\, f(x_n) + \alpha_n T_n x_n, \\ x_{n+1} &= (I - \beta_n F)\, T_n x_n + \beta_n y_n, \quad \forall n \ge 0. \end{aligned} \tag{77}$$

Observe that

$$\begin{aligned} &\|y_n - y_{n-1}\| \\ &= \|(I - \alpha_n A)\, f(x_n) + \alpha_n T_n x_n \\ &\quad - (I - \alpha_{n-1} A)\, f(x_{n-1}) - \alpha_{n-1} T_{n-1} x_{n-1}\| \\ &= \|\alpha_n (T_n x_n - T_{n-1} x_{n-1}) \\ &\quad + (\alpha_n - \alpha_{n-1})(T_{n-1} x_{n-1} - Af(x_{n-1})) \\ &\quad + (I - \alpha_n A)\, f(x_n) - (I - \alpha_n A)\, f(x_{n-1})\| \\ &\le \alpha_n \|T_n x_n - T_{n-1} x_{n-1}\| \\ &\quad + |\alpha_n - \alpha_{n-1}|\, \|T_{n-1} x_{n-1} - Af(x_{n-1})\| \\ &\quad + \|I - \alpha_n A\|\, \|f(x_n) - f(x_{n-1})\| \\ &\le \alpha_n \left(\|T_n x_n - T_n x_{n-1}\| + \|T_n x_{n-1} - T_{n-1} x_{n-1}\|\right) \\ &\quad + |\alpha_n - \alpha_{n-1}|\, \|T_{n-1} x_{n-1} - Af(x_{n-1})\| \\ &\quad + (1 - \alpha_n \overline{\gamma})\, \beta\, \|x_n - x_{n-1}\| \\ &\le \alpha_n \left(\|x_n - x_{n-1}\| + \|T_n x_{n-1} - T_{n-1} x_{n-1}\|\right) \\ &\quad + |\alpha_n - \alpha_{n-1}|\, \|T_{n-1} x_{n-1} - Af(x_{n-1})\| \\ &\quad + (1 - \alpha_n \overline{\gamma})\, \beta\, \|x_n - x_{n-1}\| \\ &= (\beta - \alpha_n (\overline{\gamma}\beta - 1))\, \|x_n - x_{n-1}\| \\ &\quad + |\alpha_n - \alpha_{n-1}|\, \|T_{n-1} x_{n-1} - Af(x_{n-1})\| \\ &\quad + \alpha_n \|T_n x_{n-1} - T_{n-1} x_{n-1}\|, \end{aligned} \tag{78}$$

and hence

$$\begin{aligned} &\|x_{n+1} - x_n\| \\ &= \|(I - \beta_n F)\, T_n x_n + \beta_n y_n \\ &\quad - (I - \beta_{n-1} F)\, T_{n-1} x_{n-1} - \beta_{n-1} y_{n-1}\| \\ &\le \|\beta_n (y_n - y_{n-1}) + (\beta_n - \beta_{n-1})(y_{n-1} - FT_{n-1} x_{n-1}) \\ &\quad + (I - \beta_n F)\, T_n x_n - (I - \beta_n F)\, T_{n-1} x_{n-1}\| \\ &\le \beta_n \|y_n - y_{n-1}\| + |\beta_n - \beta_{n-1}|\, \|y_{n-1} - FT_{n-1} x_{n-1}\| \\ &\quad + (1 - \beta_n \gamma_0)\, \|T_n x_n - T_{n-1} x_{n-1}\| \\ &\le \beta_n \|y_n - y_{n-1}\| + |\beta_n - \beta_{n-1}|\, \|y_{n-1} - FT_{n-1} x_{n-1}\| \\ &\quad + (1 - \beta_n \gamma_0)\left(\|T_n x_n - T_n x_{n-1}\| + \|T_n x_{n-1} - T_{n-1} x_{n-1}\|\right) \\ &\le \beta_n \|y_n - y_{n-1}\| + |\beta_n - \beta_{n-1}|\, \|y_{n-1} - FT_{n-1} x_{n-1}\| \\ &\quad + (1 - \beta_n \gamma_0)\left(\|x_n - x_{n-1}\| + \|T_n x_{n-1} - T_{n-1} x_{n-1}\|\right) \\ &\le \beta_n \left[(\beta - \alpha_n (\overline{\gamma}\beta - 1))\, \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \right.\\ &\qquad \times \|T_{n-1} x_{n-1} - Af(x_{n-1})\| \\ &\qquad \left. + \alpha_n \|T_n x_{n-1} - T_{n-1} x_{n-1}\| \right] \\ &\quad + |\beta_n - \beta_{n-1}|\, \|y_{n-1} - FT_{n-1} x_{n-1}\| \\ &\quad + (1 - \beta_n \gamma_0)\left(\|x_n - x_{n-1}\| + \|T_n x_{n-1} - T_{n-1} x_{n-1}\|\right) \\ &\le \beta_n \left[(\beta - \alpha_n (\overline{\gamma}\beta - 1))\, \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\, M \right.\\ &\qquad \left. + \alpha_n \|T_n x_{n-1} - T_{n-1} x_{n-1}\|\right] + |\beta_n - \beta_{n-1}|\, M \\ &\quad + (1 - \beta_n \gamma_0)\left(\|x_n - x_{n-1}\| + \|T_n x_{n-1} - T_{n-1} x_{n-1}\|\right) \\ &= \left[1 - \beta_n (\gamma_0 - \beta - \alpha_n (1 - \overline{\gamma}\beta))\right] \|x_n - x_{n-1}\| \\ &\quad + M\left(\beta_n |\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|\right) \\ &\quad + (\beta_n \alpha_n + (1 - \beta_n \gamma_0))\, \|T_n x_{n-1} - T_{n-1} x_{n-1}\| \\ &\le \left[1 - \beta_n \left(\gamma_0 - \beta - \frac{\gamma_0 - \beta}{2(1 - \overline{\gamma}\beta)}(1 - \overline{\gamma}\beta)\right)\right] \\ &\quad \times \|x_n - x_{n-1}\| + M\left(\beta_n |\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|\right) \\ &\quad + 2\, \|T_n x_{n-1} - T_{n-1} x_{n-1}\| \\ &= \left[1 - \frac{1}{2}\beta_n (\gamma_0 - \beta)\right] \|x_n - x_{n-1}\| \\ &\quad + M\left(\beta_n |\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|\right) \\ &\quad + 2\, \|T_n x_{n-1} - T_{n-1} x_{n-1}\|, \end{aligned} \tag{79}$$

where $\sup_{n\geq 0}\{\|T_nx_n - Af(x_n)\| + \|y_n - FT_nx_n\|\} \leq M$ for some $M > 0$ (it is easy to see that $\{y_n\}$ is bounded due to the boundedness of $\{x_n\}$). Utilizing Lemma 2, we conclude that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$ from conditions (i)-(ii) and the property imposed on $\{T_n\}$.

Next let us show that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \quad (80)$$

Indeed, from (76), (77), and $\beta_n \to 0$, it follows that

$$\begin{aligned}\|x_n - T_nx_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T_nx_n\| \\ &= \|x_n - x_{n+1}\| + \beta_n \|y_n - FT_nx_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\end{aligned} \quad (81)$$

That is,

$$\lim_{n\to\infty} \|x_n - T_nx_n\| = 0. \quad (82)$$

Also, it is clear that

$$\|x_n - Tx_n\| \leq \|x_n - T_nx_n\| + \|T_nx_n - Tx_n\|. \quad (83)$$

By Lemma 12, we conclude from (82) and (83) that (80) holds. Let $x_t = (I-\theta_tF)Tx_t + \theta_t[f(x_t) - t(Af(x_t) - Tx_t)]$. According to Theorem 11, we know that $\{x_t\}$ converges strongly to $z \in \mathrm{Fix}(T) = \bigcap_{i=0}^{\infty}\mathrm{Fix}(T_i) = \Omega$, which is the unique solution in Ω to the VIP:

$$\langle (F - f)z, J(z - p)\rangle \leq 0, \quad \forall p \in \Omega. \quad (84)$$

Further, let us show that

$$\limsup_{n\to\infty} \langle (f - F)z, J(x_n - z)\rangle \leq 0. \quad (85)$$

Indeed, take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\begin{aligned}&\limsup_{n\to\infty} \langle (f - F)z, J(x_n - z)\rangle \\ &\quad = \lim_{i\to\infty} \left\langle (f - F)z, J\left(x_{n_i} - z\right)\right\rangle.\end{aligned} \quad (86)$$

Without loss of generality, we may assume that $x_{n_i} \rightharpoonup \tilde{x}$. Utilizing Lemma 5 we obtain from (80) that $\tilde{x} \in \mathrm{Fix}(T)$. Hence from (84) and (86) we get

$$\limsup_{n\to\infty} \langle (f - F)z, J(x_n - z)\rangle = \langle (f - F)z, J(\tilde{x} - z)\rangle \leq 0. \quad (87)$$

As required, let us show that $x_n \to z$ as $n \to \infty$.

As a matter of fact, we observe that

$$\begin{aligned}
&\|x_{n+1} - z\|^2 \\
&= \|(I - \beta_nF)T_nx_n - (I - \beta_nF)T_nz \\
&\quad + \beta_n[(I - \alpha_nA)f(x_n) - (I - \alpha_nA)f(z) \\
&\qquad + \alpha_n(T_nx_n - z)] + \beta_n(f - F)z + \beta_n\alpha_n(I - Af)z\|^2 \\
&\leq \|(I - \beta_nF)T_nx_n - (I - \beta_nF)T_nz \\
&\quad + \beta_n[(I - \alpha_nA)(f(x_n) - f(z)) + \alpha_n(T_nx_n - z)]\|^2 \\
&\quad + 2\beta_n\langle (f - F)z, J(x_{n+1} - z)\rangle \\
&\quad + 2\beta_n\alpha_n\langle (I - Af)z, J(x_{n+1} - z)\rangle \\
&\leq [\|(I - \beta_nF)T_nx_n - (I - \beta_nF)T_nz\| \\
&\quad + \beta_n(\|I - \alpha_nA\|\|f(x_n) - f(z)\| + \alpha_n\|T_nx_n - z\|)]^2 \\
&\quad + 2\beta_n\langle (f - F)z, J(x_{n+1} - z)\rangle \\
&\quad + 2\beta_n\alpha_n\|(I - Af)z\|\|x_{n+1} - z\| \\
&\leq \{(1 - \beta_n\gamma_0)\|T_nx_n - T_nz\| \\
&\quad + \beta_n[(1 - \alpha_n\overline{\gamma})\beta\|x_n - z\| + \alpha_n\|T_nx_n - z\|]\}^2 \\
&\quad + 2\beta_n\langle (f - F)z, J(x_{n+1} - z)\rangle \\
&\quad + 2\beta_n\alpha_n\|(I - Af)z\|\|x_{n+1} - z\| \\
&\leq \{(1 - \beta_n\gamma_0)\|x_n - z\| \\
&\quad + \beta_n[(1 - \alpha_n\overline{\gamma})\beta\|x_n - z\| + \alpha_n\|x_n - z\|]\}^2 \\
&\quad + 2\beta_n\langle (f - F)z, J(x_{n+1} - z)\rangle \\
&\quad + 2\beta_n\alpha_n\|(I - Af)z\|\|x_{n+1} - z\| \\
&= [1 - \beta_n(\gamma_0 - \beta - \alpha_n(1 - \overline{\gamma}\beta))]^2\|x_n - z\|^2 \\
&\quad + 2\beta_n\langle (f - F)z, J(x_{n+1} - z)\rangle \\
&\quad + 2\beta_n\alpha_n\|(I - Af)z\|\|x_{n+1} - z\| \\
&\leq [1 - \beta_n(\gamma_0 - \beta - \alpha_n(1 - \overline{\gamma}\beta))]\|x_n - z\|^2 \\
&\quad + 2\beta_n\langle (f - F)z, J(x_{n+1} - z)\rangle \\
&\quad + 2\beta_n\alpha_n\|(I - Af)z\|\|x_{n+1} - z\| \\
&\leq \left[1 - \beta_n\left(\gamma_0 - \beta - \frac{\gamma_0 - \beta}{2(1 - \overline{\gamma}\beta)}(1 - \overline{\gamma}\beta)\right)\right]\|x_n - z\|^2 \\
&\quad + 2\beta_n\langle (f - F)z, J(x_{n+1} - z)\rangle \\
&\quad + 2\beta_n\alpha_n\|(I - Af)z\|\|x_{n+1} - z\|
\end{aligned}$$

$$= \left[1 - \frac{1}{2}\beta_n(\gamma_0 - \beta)\right]\|x_n - z\|^2 + 2\beta_n\langle(f - F)z, J(x_{n+1} - z)\rangle + 2\beta_n\alpha_n\|(I - Af)z\|\|x_{n+1} - z\| = (1 - \mu_n)\|x_n - z\|^2 + \mu_n\nu_n, \tag{88}$$

where $\mu_n = (1/2)\beta_n(\gamma_0 - \beta)$ and

$$\nu_n = \frac{4(\langle(f - F)z, J(x_{n+1} - z)\rangle + \alpha_n\|(I - Af)z\|\|x_{n+1} - z\|)}{\gamma_0 - \beta}. \tag{89}$$

It can be easily seen from (85) and conditions (i) and (ii) that

$$\sum_{n=0}^{\infty}\mu_n = \infty, \qquad \limsup_{n\to\infty}\nu_n \le 0. \tag{90}$$

In terms of Lemma 8, we infer that $x_n \to z$ as $n \to \infty$. □

Finally, we provide an example to illustrate Theorem 18.

Example 19. Let $X = \mathbf{R}^2$ with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ which are defined by

$$\langle x, y\rangle = ac + bd, \qquad \|x\| = \sqrt{a^2 + b^2}, \tag{91}$$

for all $x, y \in \mathbf{R}^2$ with $x = (a, b)$ and $y = (c, d)$. Let $C = \{(a, a) : a \in \mathbf{R}\}$. Clearly, C is a nonempty closed convex subset of a uniformly smooth Banach space $X = \mathbf{R}^2$ such that $C \pm C \subset C$. Let $\{T_n\}_{n=0}^{\infty}$ be a countable family of nonexpansive mappings from C to itself such that $\Omega = \bigcap_{n=0}^{\infty}\mathrm{Fix}(T_n) \ne \emptyset$, for instance, putting $T_n = (1 - 1/2^{n+1})T$ with $T = \left\{\begin{smallmatrix}3/5 & 2/5\\ 2/5 & 3/5\end{smallmatrix}\right\}$. Then $\|T\| = 1$ and $\|T_n\| = 1 - 1/2^{n+1}, \forall n \ge 0$. It is clear that T_n and T are nonexpansive mappings with $\Omega = \bigcap_{n=0}^{\infty}\mathrm{Fix}(T_n) = \{0\} \ne \emptyset$, and $\{T_n\}$ satisfies the assumption in Theorem 18. Let $F : C \to C$ be α-strongly accretive and λ-strictly pseudocontractive with $\alpha + \lambda > 1$, and $f : C \to C$ be a fixed contractive mapping with contractive coefficient $\beta \in (0, \gamma_0), \gamma_0 = 1 - \sqrt{(1-\alpha)/\lambda}$, for instance, putting $S = \left\{\begin{smallmatrix}2/3 & 1/3\\ 1/3 & 2/3\end{smallmatrix}\right\}$, $F = (1/2)S$, and $f = \left\{\begin{smallmatrix}3/25 & 2/25\\ 2/25 & 3/25\end{smallmatrix}\right\}$, we know that $\|F\| = (1/2)\|S\| = 1/2, \|f\| = 1/5$ and that F is a $(1/2)$-strongly accretive and $(8/9)$-strictly pseudocontractive mapping and f is a $(1/5)$-contraction with $(1/5) \in (0, \gamma_0)$ and $\gamma_0 = 1/4$. Let $A : C \to C$ be a $\overline{\gamma}$-strongly positive linear bounded operator with $\overline{\gamma}\beta < 1$; for instance, putting $A = (7/6)S$, we know that A is a $(7/6)$-strongly positive linear bounded operator with $\overline{\gamma}\beta = (7/6) \times (1/5) < 1$. In this case, from iterative scheme (70) in Theorem 18, we obtain that for any given $x_0 \in C$,

$$\begin{aligned} x_1 &= (I - \beta_0 F)T_0x_0 + \beta_0[f(x_0) - \alpha_0(Af(x_0) - T_0x_0)] \\ &= \left(1 - \frac{1}{2}\beta_0\right)\left(1 - \frac{1}{2^{0+1}}\right)x_0 + \beta_0\left[\frac{1}{5}x_0 - \alpha_0\left(\frac{7}{6}\cdot\frac{1}{5}x_0 - \left(1 - \frac{1}{2^{0+1}}\right)x_0\right)\right] \\ &= \left[\left(1 - \frac{1}{2}\beta_0\right)\left(1 - \frac{1}{2^{0+1}}\right) + \frac{1}{5}\beta_0 - \alpha_0\beta_0\left(\frac{7}{30} - \left(1 - \frac{1}{2^{0+1}}\right)\right)\right] \quad x_0 \in C. \end{aligned} \tag{92}$$

It can be readily seen that

$$x_{n+1} = \left[\left(1 - \frac{1}{2}\beta_n\right)\left(1 - \frac{1}{2^{n+1}}\right) + \frac{1}{5}\beta_n - \alpha_n\beta_n\left(\frac{7}{30} - \left(1 - \frac{1}{2^{n+1}}\right)\right)\right]x_n, \quad \forall n \ge 0. \tag{93}$$

We claim that x_n converges to the unique point 0 in Ω if $\alpha_n = (6/23)\beta_n$ and $\sum_{n=0}^{\infty}\beta_n = \infty$. Indeed, observe that

$$\begin{aligned} \|x_{n+1}\| &= \left[\left(1 - \frac{1}{2}\beta_n\right)\left(1 - \frac{1}{2^{n+1}}\right) + \frac{1}{5}\beta_n - \alpha_n\beta_n\left(\frac{7}{30} - \left(1 - \frac{1}{2^{n+1}}\right)\right)\right]\|x_n\| \\ &\le \left[\left(1 - \frac{1}{2}\beta_n\right) + \frac{1}{5}\beta_n - \alpha_n\beta_n\left(\frac{7}{30} - 1\right)\right]\|x_n\| \\ &= \left(1 - \frac{3}{10}\beta_n + \frac{23}{30}\alpha_n\beta_n\right)\|x_n\| \\ &= \left(1 - \frac{3}{10}\beta_n + \frac{23}{30}\cdot\frac{6}{23}\beta_n\beta_n\right)\|x_n\| \\ &\le \left(1 - \frac{3}{10}\beta_n + \frac{1}{5}\beta_n\right)\|x_n\| \\ &= \left(1 - \frac{1}{10}\beta_n\right)\|x_n\| \le \prod_{i=0}^{n}\left(1 - \frac{1}{10}\beta_i\right)\|x_0\|. \end{aligned} \tag{94}$$

Thus, we conclude from $\sum_{n=0}^{\infty}\beta_n = \infty$ that x_n converges to the unique point 0 in Ω. It is clear that $z = 0$ is a unique solution in Ω for the following variational inequality problem (VIP):

$$\langle(f - F)z, J(p - z)\rangle \le 0, \quad \forall p \in \Omega. \tag{95}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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