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Research Article

New Bilateral Type Generating Function Associated with I-Function

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We aim at establishing a new bilateral type generating function associated with the I-function and a Mellin-Barnes type of contour integral. The results derived here are of general character and can yield a number of (known and new) results in the theory of generating functions.

1. Introduction

Bilinear and bilateral type generating functions are continuous functions associated with a given sequence and have useful applications in many research fields. For this reason, generating functions are very useful in analyzing discrete problems involving sequences of numbers or sequences of functions, in modern combinatorics. A number of generating functions and expansions of such other types of hypergeometric functions in one, two, and more variables have been developed by many authors (see [1–4]; for a very recent work, see also [5]). Here, we present a new bilateral generating function associated with the I-function and Mellin-Barnes type of contour integral, mainly motivated by the work of Srivastava and Panda [4].

For our purpose, we begin by recalling some known functions and throughout this paper we will use the following notations.

Let $\Delta(s,\alpha)$ and $\nabla(s,\alpha)$ stand for the s-parameter sequence $\alpha/s, (\alpha+1)/s, \ldots, (\alpha+s-1)/s$ and $1-(\alpha/s), 1-(\alpha+1)/s, \ldots, 1-(\alpha+s-1)/s$, respectively, for an arbitrary complex number α and for all integers $s \geq 1$.

The H-function introduced by Fox [6, p. 408] will be represented and defined as follows:

$$H^{m,n}_{p,q}\left[x\Big|^{(a_j,A_j)_{1,p}}_{(b_j,B_j)_{1,q}}\right] = \frac{1}{2\pi i}\int_L \phi(\xi)\, x^{\xi} d\xi, \tag{1}$$

where

$$\phi(\xi) = \frac{\prod_{j=1}^{m}\Gamma\left(b_j - B_j\xi\right)\prod_{j=1}^{n}\Gamma\left(1-a_j+A_j\xi\right)}{\prod_{j=m+1}^{q}\Gamma\left(1-b_j+B_j\xi\right)\prod_{j=n+1}^{p}\Gamma\left(a_j-A_j\xi\right)}, \tag{2}$$

where an empty product is interpreted as unity and $0 \leq m \leq q$, $0 \leq n \leq p$, A_j $(j=1,\ldots,p)$, and B_j $(j=1,\ldots,q)$ are positive numbers. L is suitable contour of Barnes type such that the poles of $\Gamma(b_j - B_j\xi)$ $(j=1,\ldots,m)$ lie to the right of it and those of $\Gamma(1-a_j+A_j\xi)$ $(j=1,\ldots,n)$ lie to the left of it. Asymptotic expansions and analytic continuations of the H-function have been discussed by Braaksma [7].

The I-function will be defined and represented as follows [8]:

$$I^{m,n}_{p_i,q_i,r}\left[z\Big|^{(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}}_{(b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i}}\right] = \frac{1}{2\pi i}\int_L \phi(\xi)\, z^{\xi} d\xi, \tag{3}$$

where

$$\phi(\xi) = \frac{\prod_{j=1}^{m}\Gamma\left(b_j-\beta_j\xi\right)\prod_{j=1}^{n}\Gamma\left(1-a_j+\alpha_j\xi\right)}{\sum_{i=1}^{r}\left[\prod_{j=1}^{q_i}\Gamma\left(1-b_{ji}+\beta_{ji}\xi\right)\prod_{j=1}^{p_i}\Gamma\left(a_{ji}-\alpha_{ji}\xi\right)\right]}, \tag{4}$$

and m, n, p_i, q_i are integers satisfying $0 \leq n \leq p_i$, $1 \leq m \leq q_i$ $(i = 1, \ldots, r)$, where r is finite. α_j, β_j, α_{ji}, β_{ji} are positive integers and a_j, b_j, a_{ji}, b_{ji} are complex numbers. The I-function is a generalized form of the well-known Fox H-function [6]. In the sequel, the I-function will be studied under the following conditions of existence:

$$\text{(i)}\quad A_i > 0, \quad |\arg z| < \frac{A_i \pi}{2} \tag{5}$$

$$\text{(ii)}\quad A_i \geq 0, \quad |\arg z| \leq \frac{A_i \pi}{2}, \qquad \Re(B+1) < 0, \tag{6}$$

where

$$\begin{aligned} A_i &= \sum_{j=1}^{n} \alpha_j - \sum_{j=n+1}^{p_i} \alpha_{ji} + \sum_{j=1}^{m} \beta_j \\ &\quad - \sum_{j=m+1}^{q_i} \beta_{ji}, \quad \forall i = (1, 2, \ldots, r), \\ B &= \sum_{j=1}^{m} b_j + \sum_{j=m+1}^{q_i} b_{ji} - \sum_{j=1}^{n} a_j - \sum_{j=n+1}^{p_i} a_{ji} \\ &\quad + \frac{1}{2}(p_i - q_i), \quad \forall i = (1, 2, \ldots, r). \end{aligned} \tag{7}$$

2. Bilateral Generating Functions for I-Functions

In this section, we establish generating functions for the I-function and Mellin-Barnes type of contour integral (3) and (1), respectively.

Theorem 1. *Let M, N, P_i, Q_i, m, n, p, and q be positive integers. Then the following bilateral generating function holds:*

$$\begin{aligned} &\sum_{\omega=0}^{\infty} H^{m,n+s}_{p+s,q+s}\left[x \left|\begin{matrix}(a_j,A_j)_{1,n};(\nabla(s,-\omega),1)_{1,s};(a_j,A_j)_{n+1,p} \\ (b_j,B_j)_{1,q};(\nabla(s,-\omega),0)_{1,s}\end{matrix}\right.\right] \\ &\quad \times I^{M,N+s}_{P_i+s,Q_i,r}\left[y \left|\begin{matrix}(a'_j,\alpha_j)_{1,N};(\nabla(s,\sigma+\omega),1)_{1,s};(a'_{ji},\alpha_{ji})_{N+1,P_i} \\ (b'_j,\beta_j)_{1,M};(b'_{ji},\beta_{ji})_{M+1,Q_i}\end{matrix}\right.\right] \frac{z^\omega}{\omega!} \\ &= \left(1-\frac{z}{s}\right)^{-\sigma} I^{M,N+s}_{P_i+s,Q_i+s,r} \\ &\quad \times \left[y\left(1-\frac{z}{s}\right)^{-s} \left|\begin{matrix}(a'_j,\alpha_j)_{1,N};(\Delta(s,\sigma),1)_{1,s};(a'_{ji},\alpha_{ji})_{N+1,P_i} \\ (b'_j,\beta_j)_{1,M};(b'_{ji},\beta_{ji})_{M+1,Q_i}\end{matrix}\right.\right] \\ &\quad \times H^{m,n+s}_{p+s,q+s}\left[x\left(1-\frac{s}{z}\right)^{-s} \left|\begin{matrix}(a_j,A_j)_{1,n};(\nabla(s,\sigma),1)_{1,s};(a_j,A_j)_{n+1,p} \\ (b_j,B_j)_{1,q};(\nabla(s,\sigma),0)_{1,s}\end{matrix}\right.\right], \end{aligned} \tag{8}$$

where σ is an arbitrary complex number and s is an integer ≥ 0.

Proof. For convenience, let the left-hand side of (8) be denoted by J. Applying the integral representation of (3) to J, then interchanging the order of summation and integration (which can be justified when the integral and the series involved are uniformly absolutely convergent), we get

$$\begin{aligned} J &= \frac{1}{2\pi i} \int_L \phi(\xi) \prod_{k=1}^{s} \Gamma(\Delta(s,\sigma)+\xi) \\ &\quad \times \left\{\sum_{\omega=0}^{\infty} \frac{(\sigma+s\xi)_\omega}{\omega!} H^{m,n+s}_{p+s,q+s} \right. \\ &\quad \left. \times \left[x \left|\begin{matrix}(a_j,A_j)_{1,n};(\nabla(s,-\omega),1)_{1,s};(a_j,A_j)_{n+1,p} \\ (b_j,B_j)_{1,q};(\nabla(s,-\omega),0)_{1,s}\end{matrix}\right.\right] \left(\frac{z}{s}\right)^{\omega}\right\} y^{\xi} d\xi. \end{aligned} \tag{9}$$

Using the known formula (see [4, p. 273, (4.10)]), namely,

$$\begin{aligned} &\sum_{\omega=0}^{\infty} \frac{(\sigma)_\omega}{\omega!} H^{m,n+s}_{p+s,q+s}\left[x \left|\begin{matrix}(a_j,A_j)_{1,n};(\nabla(s,-\omega),1)_{1,s};(a_j,A_j)_{n+1,p} \\ (b_j,B_j)_{1,q};(\nabla(s,-\omega),0)_{1,s}\end{matrix}\right.\right] t^{\omega} \\ &\quad = (1-t)^{-\sigma} H^{m,n+s}_{p+s,q+s} \\ &\quad \times \left[x\left(\frac{t}{t-1}\right)^{s} \left|\begin{matrix}(a_j,A_j)_{1,n};(\nabla(s,\sigma),1)_{1,s};(a_j,A_j)_{n+1,p} \\ (b_j,B_j)_{1,q};(\nabla(s,\sigma),0)_{1,s}\end{matrix}\right.\right] \end{aligned} \tag{10}$$

and the Mellin-Barnes type contour integral of the H-function given by (1).

Finally, in view of (10) and (3), we get the desired assertion (8) of Theorem 1. □

3. Special Cases

In this section, we consider some consequences of the main results derived in the preceding section.

(i) If we put $r = 1$, I-function reduces to Fox H-function [6]. Then the main result (8) takes the following form:

$$\begin{aligned} &\sum_{\omega=0}^{\infty} H^{m,n+s}_{p+s,q+s}\left[x \left|\begin{matrix}(a_j,A_j)_{1,n};(\nabla(s,-\omega),1)_{1,s};(a_j,A_j)_{n+1,p} \\ (b_j,B_j)_{1,q};(\nabla(s,-\omega),0)_{1,s}\end{matrix}\right.\right] \\ &\quad \times H^{M,N+s}_{P+s,Q}\left[y \left|\begin{matrix}(a'_j,\alpha_j)_{1,N};(\nabla(s,\sigma+\omega),1)_{1,s};(a'_{ji},\alpha_j)_{n+1,P} \\ (b'_j,\beta_j)_{1,M};(b'_j,\beta_j)_{M+1,Q}\end{matrix}\right.\right] \frac{z^\omega}{\omega!} \\ &= \left(1-\frac{z}{s}\right)^{-\sigma} H^{m,n+s}_{p+s,q+s} \\ &\quad \times \left[x\left(1-\frac{s}{z}\right)^{-s} \left|\begin{matrix}(a_j,A_j)_{1,n};(\nabla(s,\sigma),1)_{1,s};(a_j,A_j)_{n+1,p} \\ (b_j,B_j)_{1,q};(\nabla(s,\sigma),0)_{1,s}\end{matrix}\right.\right] \\ &\quad \times H^{M,N+s}_{P+s,Q}\left[y\left(1-\frac{z}{s}\right)^{-s} \left|\begin{matrix}(a'_j,\alpha_j)_{1,N};(\Delta(s,\sigma),1)_{1,s};(a'_j,\beta_j)_{N+1,P} \\ (b'_j,\beta_j)_{1,M};(b'_j,\beta_j)_{M+1,Q}\end{matrix}\right.\right]. \end{aligned} \tag{11}$$

(ii) If we put $r = 1$, $M = 1$, $N = P_i = P$, $Q_i = Q+1$, $b'_1 = 0$, $\beta_1 = 1$, $a'_j = 1 - a'_j$, $b'_{ji} = b'_j$, and $\beta_{ji} = \beta_j$,

I-function reduces to Wright's generalized hypergeometric function [9, p. 33, (2.3.8)]. Then the main result (8) takes the following form:

$$\sum_{\omega=0}^{\infty} H_{p+s,q+s}^{m,n+s}\left[x\Big|_{(b_j,B_j)_{1,q};(\nabla(s,-\omega),0)_{1,s}}^{(a_j,A_j)_{1,n};(\nabla(s,-\omega),1)_{1,s};(a_j,A_j)_{n+1,p}}\right]$$
$$\times {}_{P+s}\psi_{Q}\left[\begin{matrix}\left(a'_j,\alpha_j\right)_{1,P};\left(\nabla\left(s,\sigma+\omega\right),1\right)_{1,s} \\ \left(b'_j,\beta_j\right)_{1,Q}\end{matrix};-y\right]\frac{z^{\omega}}{\omega!}$$
$$=\left(1-\frac{z}{s}\right)^{-\sigma} H_{p+s,q+s}^{m,n+s}$$
$$\times\left[x\left(1-\frac{s}{z}\right)^{-s}\Big|_{(b_j,B_j)_{1,q};(\nabla(s,\sigma),0)_{1,s}}^{(a_j,A_j)_{1,n};(\nabla(s,\sigma),1)_{1,s};(a_j,A_j)_{n+1,p}}\right]$$
$$\times {}_{P+s}\psi_{Q}\left[\begin{matrix}\left(a'_j,\alpha_j\right)_{1,P};\left(\nabla\left(s,\sigma\right),1\right)_{1,s} \\ \left(b'_j,\beta_j\right)_{1,Q}\end{matrix};-y\left(1-\frac{z}{s}\right)^{-s}\right]. \tag{12}$$

(iii) If we put $r=1$, $M=1$, $N=P_i=P$, $Q_i=Q+1$, $b'_1=0$, $\beta_1=1$, $a'_j=1-a'_j$, $b'_{ji}=1-b'_j$, and $\alpha_j=\beta_j=\alpha_{ji}=\beta_{ji}=1$, I-function reduces to generalized hypergeometric function [9, p. 33, (2.3.11)]. Then the main result (8) takes the following form:

$$\sum_{\omega=0}^{\infty}\left\{\prod_{k=1}^{s}\Gamma\left(\nabla\left(s,\sigma+\omega\right)\right)\right\}$$
$$\times H_{p+s,q+s}^{m,n+s}\left[x\Big|_{(b_j,B_j)_{1,q};(\nabla(s,-\omega),0)_{1,s}}^{(a_j,A_j)_{1,n};(\nabla(s,-\omega),1)_{1,s};(a_j,A_j)_{n+1,p}}\right]$$
$$\times {}_{P+s}F_{Q}\left[\begin{matrix}\left(a'_j,1\right)_{1,P};\left(\nabla\left(s,\sigma+\omega\right),1\right)_{1,s} \\ \left(b'_j,1\right)_{1,Q}\end{matrix};-y\right]\frac{z^{\omega}}{\omega!}$$
$$=\left(1-\frac{z}{s}\right)^{-\sigma}\left\{\prod_{k=1}^{s}\Gamma\left(\nabla\left(s,\sigma\right)\right)\right\}$$
$$\times H_{p+s,q+s}^{m,n+s}\left[x\left(1-\frac{s}{z}\right)^{-s}\Big|_{(b_j,B_j)_{1,q};(\nabla(s,\sigma),0)_{1,s}}^{(a_j,A_j)_{1,n};(\nabla(s,\sigma),1)_{1,s};(a_j,A_j)_{n+1,p}}\right]$$
$$\times {}_{P+s}F_{Q}\left[\begin{matrix}\left(a'_j,1\right)_{1,P};\left(\nabla\left(s,\sigma\right),1\right)_{1,s} \\ \left(b'_j,1\right)_{1,Q}\end{matrix};y\left(1-\frac{z}{s}\right)^{-s}\right]. \tag{13}$$

4. Concluding Remark

We conclude our present investigation by remarking that the results obtained here are useful in deriving numerous other generating functions involving various special functions due to presence of the I-function given by (8). The I-function, used in our results, is quite basic in nature. Therefore, on specializing the parameters of this function, we may obtain various other special functions such as Fox H-function, Meijer's G-function, Wright's generalized Bessel function, Wright's generalized hypergeometric function, Mac-Robert's E-function, generalized hypergeometric function, Bessel function of first kind, modified Bessel function, Whittaker function, exponential function, and binomial function as its special cases, and, therefore, the result thus derived in this paper is general in character and likely to find certain applications in the theory of special functions.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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