

Research Article

A General Approximation Method for a Kind of Convex Optimization Problems in Hilbert Spaces

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The constrained convex minimization problem is to find a point x^* with the property that $x^* \in C$, and $h(x^*) = \min h(x), \forall x \in C$, where C is a nonempty, closed, and convex subset of a real Hilbert space H , $h(x)$ is a real-valued convex function, and $h(x)$ is not Fréchet differentiable, but lower semicontinuous. In this paper, we discuss an iterative algorithm which is different from traditional gradient-projection algorithms. We firstly construct a bifunction $F_1(x, y)$ defined as $F_1(x, y) = h(y) - h(x)$. And we ensure the equilibrium problem for $F_1(x, y)$ equivalent to the above optimization problem. Then we use iterative methods for equilibrium problems to study the above optimization problem. Based on Jung's method (2011), we propose a general approximation method and prove the strong convergence of our algorithm to a solution of the above optimization problem. In addition, we apply the proposed iterative algorithm for finding a solution of the split feasibility problem and establish the strong convergence theorem. The results of this paper extend and improve some existing results.

1. Introduction

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$. Let C be a nonempty, closed, and convex subset of H . Recall that a mapping T is nonexpansive (see [1]) if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \quad (1)$$

We denote by $F(T) = \{x \in C : Tx = x\}$ the set of fixed points of T . Recall that a self-mapping f on H is a contraction if there exists a constant $\alpha \in (0, 1)$ such that

$$\|f(x) - f(y)\| \leq \alpha \|x - y\|, \quad \forall x, y \in H. \quad (2)$$

Consider the following constrained convex minimization problem:

$$\min_{x \in C} h(x), \quad (3)$$

where $h : C \rightarrow \mathbb{R}$ is a real-valued convex function. Assume that the constrained convex minimization problem (3) is solvable, and let S denote the set of solutions of (3). Assume

that $h(x)$ is lower semicontinuous. Let F_1 be a bifunction from $C \times C$ to \mathbb{R} defined by $F_1(x, y) = h(y) - h(x)$. It is easy to see that $S = EP(F_1)$, where $EP(F_1)$ denotes the set of solutions of equilibrium problem to find $x \in C$ such that

$$F_1(x, y) \geq 0, \quad \forall y \in C. \quad (4)$$

Hence, the optimization problem (3) is equivalent to the equilibrium problem (4).

For solving the convex optimization problem (3), Su and Li [2] introduced the following iterative scheme in a Hilbert space: $x_1 \in H$ and

$$h(y) - h(u_n) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \quad (5)$$

$$x_{n+1} = \alpha_n f(x_n) + \gamma_n u_n + \beta_n x_n,$$

for all $n \in \mathbb{N}$, where $u_n = T_{r_n} x_n$, $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0, 1)$, $\alpha_n + \beta_n + \gamma_n = 1$, and $\{r_n\} \subset (0, \infty)$. Under appropriate conditions, they proved that the sequence $\{x_n\}$ generated by (5) converges strongly to a point $z = P_S f(z)$ of optimization problem (3).

In 2010, combining Yamada’s method [3] and Marino and Xu’s method [4], Tian [5] proposed a general iterative algorithm:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n \mu F) T(x_n), \quad (6)$$

where $f : H \rightarrow H$ is a contraction with coefficient $0 < \alpha < 1$ and $F : H \rightarrow H$ is a k -Lipschitzian and η -strongly monotone operator with $k > 0, \eta > 0$. Let $0 < \mu < 2\eta/k^2, 0 < \gamma < \mu(\eta - (\mu k^2/2))/\alpha = \tau/\alpha$. It is proved that the sequence $\{x_n\}_{n=0}^\infty$ generated by (6) converges strongly to a fixed point $\bar{x} \in \text{Fix}(T)$, which solves the variational inequality

$$\langle (\gamma f - \mu F) \bar{x}, x - \bar{x} \rangle \leq 0, \quad \forall x \in \text{Fix}(T). \quad (7)$$

In 2011, Jung [6] introduced the following general iterative scheme for k -strictly pseudocontractive mapping T :

$$x_{n+1} = \alpha_n \gamma f(x_n) + ((1 - \beta_n)I - \alpha_n \mu F) P_C S'(x_n) + \beta_n x_n, \quad (8)$$

where $S' : C \rightarrow H$ is a mapping defined by $S'x = kx + (1 - k)Tx$ and P_C is the metric projection of H onto C . Under appropriate conditions, he established the strong convergence of the sequence $\{x_n\}$ generated by (8) to a fixed point of T , which is a solution of the variational inequality (7).

In this paper, motivated and inspired by the above results, we introduce a general iterative method: $x_1 \in H$ and

$$h(y) - h(u_n) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C,$$

$$x_{n+1} = \alpha_n \gamma f(x_n) + ((1 - \beta_n)I - \alpha_n \mu F) u_n + \beta_n x_n, \quad n \in \mathbb{N}, \quad (9)$$

for solving the optimization problem (3), where $u_n = T_{r_n} x_n$. Under appropriate conditions, it is proved that the sequence $\{x_n\}$ generated by (9) converges strongly to a point $z \in S$ which solves the variational inequality

$$\langle (\mu F - \gamma f) z, z - x \rangle \leq 0, \quad \forall x \in S. \quad (10)$$

Furthermore, by using the above result we studied the split feasibility problem and obtained the iterative algorithm for solving the split feasibility problem.

2. Preliminaries

In this section we introduce some useful definitions and lemmas which will be used in the proofs for the main results in the next section.

Monotone operators are very useful in the convergence analysis.

Definition 1 (see [7] for comprehensive theory of monotone operators). Let $A : H \rightarrow H$ be an operator.

(i) A is monotone, if and only if

$$\langle x - y, Ax - Ay \rangle \geq 0, \quad \forall x, y \in H. \quad (11)$$

(ii) A is said to be η -strongly monotone, if there exists a positive constant η such that

$$\langle x - y, Ax - Ay \rangle \geq \eta \|x - y\|^2, \quad \forall x, y \in H. \quad (12)$$

(iii) A is said to be v -inverse strongly monotone (v -ism), if there exists a positive constant v such that

$$\langle x - y, Ax - Ay \rangle \geq v \|Ax - Ay\|^2, \quad \forall x, y \in H. \quad (13)$$

It is known that inverse strongly monotone operators have been studied widely (see [7–9]) and applied to solve practical problems in various fields, for instance, in traffic assignment problems (see [10, 11]).

Lemma 2 (see [5]). Let H be a Hilbert space, $f : H \rightarrow H$ a contraction with a coefficient $0 < \rho < 1$, and $F : H \rightarrow H$ a k -Lipschitz continuous and η -strongly monotone operator with constants $k > 0, \eta > 0$. Then, for $0 < \gamma < \mu\eta/\rho$,

$$\langle x - y, (\mu F - \gamma f)x - (\mu F - \gamma f)y \rangle \geq (\mu\eta - \gamma\rho) \|x - y\|^2, \quad \forall x, y \in H. \quad (14)$$

That is, $\mu F - \gamma f$ is strongly monotone with a coefficient $\mu\eta - \gamma\rho$.

Recall the metric (nearest point) projection P_C from a real Hilbert space H to a closed and convex subset C of H which is defined as follows: given $x \in H, P_C x$ is the unique point in C with the property

$$\|x - P_C x\| = \inf \{\|x - y\| : y \in C\}. \quad (15)$$

P_C is characterized as follows.

Lemma 3. Let C be a closed and convex subset of a real Hilbert space H . Given $x \in H$ and $y \in C$, then $y = P_C x$ if and only if the following inequality holds:

$$\langle x - y, y - z \rangle \geq 0, \quad \forall z \in C. \quad (16)$$

For solving the equilibrium problem for a bifunction $F_1 : C \times C \rightarrow \mathbb{R}$, let us assume that F_1 satisfies the following conditions:

- (A1) $F_1(x, x) = 0$ for all $x \in C$;
- (A2) F_1 is monotone; that is, $F_1(x, y) + F_1(y, x) \leq 0$ for all $x, y \in C$;
- (A3) for each $x, y, z \in C, \limsup_{t \rightarrow 0^+} F_1(tz + (1-t)x, y) \leq F_1(x, y)$;
- (A4) for each $x \in C, y \mapsto F_1(x, y)$ is convex and lower semicontinuous.

Lemma 4 (see [12]). Let C be a nonempty, closed, and convex subset of H and let F_1 be a bifunction of $C \times C$ into \mathbb{R} satisfying (A1)–(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that

$$F_1(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0 \quad \forall y \in C. \quad (17)$$

Lemma 5 (see [13]). Assume that $F_1 : C \times C \rightarrow \mathbb{R}$ satisfies (A1)–(A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \rightarrow C$ as follows:

$$T_r(x) = \left\{ z \in C : F_1(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \forall y \in C \right\}. \quad (18)$$

Then, the following hold.

- (1) T_r is single-valued;
- (2) T_r is firmly nonexpansive; that is, $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle$ for any $x, y \in H$;
- (3) $F(T_r) = EP(F_1)$;
- (4) $EP(F_1)$ is closed and convex.

Lemma 6 (see [14]). Assume that $\{a_n\}_{n=0}^\infty$ is a sequence of nonnegative real numbers such that

$$a_{n+1} \leq (1 - \gamma_n) a_n + \gamma_n \delta_n, \quad n \geq 0, \quad (19)$$

where $\{\gamma_n\}_{n=0}^\infty$ is a sequence in $(0, 1)$ and $\{\delta_n\}_{n=0}^\infty$ is a sequence in \mathbb{R} such that

- (i) $\sum_{n=0}^\infty \gamma_n = \infty$;
- (ii) either $\limsup_{n \rightarrow \infty} \delta_n \leq 0$ or $\sum_{n=0}^\infty \gamma_n |\delta_n| < \infty$.

Then $\lim_{n \rightarrow \infty} a_n = 0$.

We adopt the following notation:

- (i) $x_n \rightarrow x$ means that $x_n \rightarrow x$ strongly;
- (ii) $x_n \rightharpoonup x$ means that $x_n \rightarrow x$ weakly.

3. Main Results

Recall that throughout this paper we always use S to denote the solution set of the constrained convex minimization problem (3).

Let H be a real Hilbert space and let C be a nonempty, closed, and convex subset of H . Let $h : C \rightarrow \mathbb{R}$ be a real-valued convex function. Assume that $h(x)$ is lower semicontinuous. Let $f : H \rightarrow H$ be a contraction with a coefficient $0 < \rho < 1$, and let $F : C \rightarrow H$ be a k -Lipschitzian and η -strongly monotone operator with constants $k > 0$, $\eta > 0$. Suppose that $0 < \mu < 2\eta/k^2$, $0 < \gamma < \tau/\rho$, where $\tau = \mu\eta - (\mu^2 k^2/2)$.

Next, we study the following iterative method. For a given arbitrary initial guess $x_1 \in H$, the sequence $\{x_n\}_{n=0}^\infty$ is generated by the following recursive formula:

$$h(y) - h(u_n) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C,$$

$$x_{n+1} = \alpha_n \gamma f(x_n) + ((1 - \beta_n)I - \alpha_n \mu F) u_n + \beta_n x_n, \quad n \in \mathbb{N}, \quad (20)$$

where $u_n = T_{r_n} x_n$ and T_{r_n} is a mapping defined as in Lemma 5.

In order to prove the convergence, we also need the following proposition.

Proposition 7. Let H be a real Hilbert space and let $F : H \rightarrow H$ be a k -Lipschitzian and η -strongly monotone operator with constants $k > 0$, $\eta > 0$. Suppose that $0 < \mu < 2\eta/k^2$, and $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$ satisfy $0 < \alpha_n < \min\{(1 - \beta_n)/2\tau, 1 - \beta_n\}$. Then, the mapping $G(x) = ((1 - \beta_n)I - \alpha_n \mu F)(x)$ is a contraction with a contractive constant $1 - \beta_n - \alpha_n \tau$, where $\tau = \mu\eta - (\mu^2 k^2/2)$.

Proof. Taking $x, y \in H$, we have

$$\begin{aligned} & \|G(x) - G(y)\|^2 \\ &= \|((1 - \beta_n)I - \alpha_n \mu F)(x) - ((1 - \beta_n)I - \alpha_n \mu F)(y)\|^2 \\ &= (1 - \beta_n)^2 \|x - y\|^2 - 2\alpha_n \mu (1 - \beta_n) \langle x - y, F(x) - F(y) \rangle \\ &\quad + \alpha_n^2 \mu^2 \|F(x) - F(y)\|^2 \\ &\leq [(1 - \beta_n)^2 - 2\alpha_n \mu (1 - \beta_n) \eta + \alpha_n^2 \mu^2 k^2] \|x - y\|^2 \\ &\leq (1 - \beta_n)^2 \left[1 - \frac{2\alpha_n \mu \eta}{1 - \beta_n} + \frac{\alpha_n^2 \mu^2 k^2}{1 - \beta_n} \right] \|x - y\|^2. \end{aligned} \quad (21)$$

Thus,

$$\begin{aligned} & \|G(x) - G(y)\| \\ &\leq (1 - \beta_n) \sqrt{1 - \alpha_n \left(\frac{2\mu\eta}{1 - \beta_n} - \frac{\mu^2 k^2}{1 - \beta_n} \right)} \|x - y\| \\ &\leq (1 - \beta_n) \left[1 - \alpha_n \left(\frac{\mu\eta}{1 - \beta_n} - \frac{\mu^2 k^2}{2(1 - \beta_n)} \right) \right] \|x - y\| \\ &= (1 - \beta_n - \alpha_n \tau) \|x - y\|, \end{aligned} \quad (22)$$

where $\tau = \mu\eta - (\mu^2 k^2/2)$. \square

Theorem 8. Let H be a real Hilbert space and let C be a nonempty, closed, and convex subset of H . Let $h : C \rightarrow \mathbb{R}$ be a real-valued convex function. Assume that $h(x)$ is lower semicontinuous. Let $f : H \rightarrow H$ be a contraction with a coefficient $0 < \rho < 1$ and let $F : C \rightarrow H$ be a k -Lipschitzian and η -strongly monotone operator with constants $k > 0$, $\eta > 0$. Suppose that $0 < \mu < 2\eta/k^2$, $0 < \gamma < \tau/\rho$, where $\tau = \mu\eta - (\mu^2 k^2/2)$. Suppose that the optimization problem (3) is consistent and let S denote its solution set. Let $\{x_n\}_{n=0}^\infty$ be generated by $x_1 \in H$ and

$$h(y) - h(u_n) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C,$$

$$x_{n+1} = \alpha_n \gamma f(x_n) + ((1 - \beta_n)I - \alpha_n \mu F) u_n + \beta_n x_n, \quad n \in \mathbb{N}, \quad (23)$$

where $u_n = T_{r_n} x_n$ and T_{r_n} is a mapping defined as in Lemma 5. $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$, $\{r_n\} \subset (0, \infty)$ satisfy the following conditions:

- (C1) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^\infty \alpha_n = \infty$;
- (C2) $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty$;

$$(C3) \limsup_{n \rightarrow \infty} \beta_n < 1;$$

$$(C4) \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty;$$

$$(C5) \liminf_{n \rightarrow \infty} r_n > 0 \text{ and } \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty.$$

Then the sequence $\{x_n\}$ generated by (23) converges strongly to a point $z \in S$, which solves the variational inequality

$$\langle (\mu F - \gamma f)z, z - x \rangle \leq 0, \quad \forall x \in S. \quad (24)$$

Proof. Let F_1 be a bifunction from $C \times C$ to \mathbb{R} defined by $F_1(x, y) = h(y) - h(x)$. We consider the following equilibrium problem, that is, to find $x \in C$, such that

$$F_1(x, y) \geq 0, \quad \forall y \in C. \quad (25)$$

It is obvious that $EP(F_1) = S$, where $EP(F_1)$ denotes the set of solutions of equilibrium problem (25). In addition, it is easy to see that $F_1(x, y)$ satisfies the conditions (A1)–(A4) in Section 2. Then, the iterative method (23) is equivalent to $x_1 \in H$ and

$$F_1(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C,$$

$$x_{n+1} = \alpha_n \gamma f(x_n) + ((1 - \beta_n)I - \alpha_n \mu F)u_n + \beta_n x_n, \quad n \in \mathbb{N}, \quad (26)$$

where $u_n = T_{r_n} x_n$.

It is easy to see the uniqueness of a solution of variational inequality (24). By Lemma 2, $\mu F - \gamma f$ is strongly monotone, so variational inequality (24) has only one solution. Below, we use $z \in S$ to denote the unique solution of (24). Since $z \in S$ solves variational inequality (24), then (24) can be written as

$$\langle (I - \mu F + \gamma f)z - z, z - x \rangle \geq 0, \quad \forall x \in S. \quad (27)$$

So in terms of Lemma 3, it is equivalent to the following fixed-point equation

$$z = P_S(I - \mu F + \gamma f)z = P_{EP(F_1)}(I - \mu F + \gamma f)z. \quad (28)$$

Now, we show that $\{x_n\}$ is bounded. Indeed, picking $p \in S = EP(F_1)$, since $u_n = T_{r_n} x_n$, by Lemma 5, we know that

$$\|u_n - p\| = \|T_{r_n} x_n - T_{r_n} p\| \leq \|x_n - p\|, \quad \forall n \geq 1. \quad (29)$$

From Propositions 7 and (29), we derive that

$$\begin{aligned} & \|x_{n+1} - p\| \\ &= \|\alpha_n \gamma f(x_n) + ((1 - \beta_n)I - \alpha_n \mu F)u_n + \beta_n x_n - p\| \\ &\leq \alpha_n \|\gamma f(x_n) - \mu F(p)\| \\ &\quad + \|((1 - \beta_n)I - \alpha_n \mu F)u_n - ((1 - \beta_n)I - \alpha_n \mu F)p\| \\ &\quad + \beta_n \|x_n - p\| \end{aligned}$$

$$\begin{aligned} &\leq \alpha_n \gamma \|f(x_n) - f(p)\| + \alpha_n \|\gamma f(p) - \mu F(p)\| \\ &\quad + (1 - \beta_n - \alpha_n \tau) \|u_n - p\| + \beta_n \|x_n - p\| \\ &\leq \alpha_n \gamma \rho \|x_n - p\| + \alpha_n \|\gamma f(p) - \mu F(p)\| \\ &\quad + (1 - \beta_n - \alpha_n \tau) \|u_n - p\| + \beta_n \|x_n - p\| \\ &\leq (1 - \alpha_n(\tau - \gamma \rho)) \|x_n - p\| + \alpha_n \|\gamma f(p) - \mu F(p)\| \\ &= (1 - \alpha_n(\tau - \gamma \rho)) \|x_n - p\| \\ &\quad + \alpha_n(\tau - \gamma \rho) \frac{\|\gamma f(p) - \mu F(p)\|}{\tau - \gamma \rho} \\ &\leq \max \left\{ \|x_n - p\|, \frac{\|\gamma f(p) - \mu F(p)\|}{\tau - \gamma \rho} \right\}. \end{aligned} \quad (30)$$

By induction, we have

$$\|x_n - p\| \leq \max \left\{ \|x_0 - p\|, \frac{\|\gamma f(p) - \mu F(p)\|}{\tau - \gamma \rho} \right\}. \quad (31)$$

Hence $\{x_n\}$ is bounded. From (29), we also derive that $\{u_n\}$ is bounded.

Next, we show that $\|x_{n+1} - x_n\| \rightarrow 0$. Consider

$$\begin{aligned} & \|x_{n+1} - x_n\| \\ &= \|\alpha_n \gamma f(x_n) + ((1 - \beta_n)I - \alpha_n \mu F)u_n + \beta_n x_n \\ &\quad - [\alpha_{n-1} \gamma f(x_{n-1}) \\ &\quad + ((1 - \beta_{n-1})I - \alpha_{n-1} \mu F)u_{n-1} + \beta_{n-1} x_{n-1}]\| \\ &\leq \alpha_n \gamma \|f(x_n) - f(x_{n-1})\| + |\alpha_n - \alpha_{n-1}| \gamma \|f(x_{n-1})\| \\ &\quad + \|((1 - \beta_n)I - \alpha_n \mu F)u_n - ((1 - \beta_n)I - \alpha_n \mu F)u_{n-1}\| \\ &\quad + \|(\beta_{n-1} - \beta_n)u_{n-1} + (\alpha_{n-1} - \alpha_n) \mu F(u_{n-1})\| \\ &\quad + |\beta_n - \beta_{n-1}| \|x_{n-1}\| + \beta_n \|x_n - x_{n-1}\| \\ &\leq \alpha_n \gamma \rho \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \gamma \|f(x_{n-1})\| \\ &\quad + (1 - \beta_n - \alpha_n \tau) \|u_n - u_{n-1}\| + |\beta_n - \beta_{n-1}| \|u_{n-1}\| \\ &\quad + |\alpha_n - \alpha_{n-1}| \|\mu F(u_{n-1})\| \\ &\quad + |\beta_n - \beta_{n-1}| \|x_{n-1}\| + \beta_n \|x_n - x_{n-1}\| \\ &= (\alpha_n \gamma \rho + \beta_n) \|x_n - x_{n-1}\| \\ &\quad + |\alpha_n - \alpha_{n-1}| (\gamma \|f(x_{n-1})\| + \|\mu F(u_{n-1})\|) \\ &\quad + (1 - \beta_n - \alpha_n \tau) \|u_n - u_{n-1}\| \\ &\quad + |\beta_n - \beta_{n-1}| (\|u_{n-1}\| + \|x_{n-1}\|). \end{aligned} \quad (32)$$

From $u_{n+1} = T_{r_{n+1}} x_{n+1}$ and $u_n = T_{r_n} x_n$, we note that

$$F_1(u_{n+1}, y) + \frac{1}{r_{n+1}} \langle y - u_{n+1}, u_{n+1} - x_{n+1} \rangle \geq 0, \quad \forall y \in C, \tag{33}$$

$$F_1(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C. \tag{34}$$

Putting $y = u_n$ in (33) and $y = u_{n+1}$ in (34), we have

$$F_1(u_{n+1}, u_n) + \frac{1}{r_{n+1}} \langle u_n - u_{n+1}, u_{n+1} - x_{n+1} \rangle \geq 0, \tag{35}$$

$$F_1(u_n, u_{n+1}) + \frac{1}{r_n} \langle u_{n+1} - u_n, u_n - x_n \rangle \geq 0.$$

So, from (A2), we have

$$\left\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}} \right\rangle \geq 0, \tag{36}$$

and hence

$$\begin{aligned} & \left\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - x_n - \frac{r_n}{r_{n+1}} (u_{n+1} - x_{n+1}) \right\rangle \\ & \geq 0. \end{aligned} \tag{37}$$

Since $\liminf_{n \rightarrow \infty} r_n > 0$, without loss of generality, let us assume that there exists a real number a such that $r_n > a > 0$ for all $n \in \mathbb{N}$. Thus, we have

$$\begin{aligned} & \|u_{n+1} - u_n\|^2 \\ & \leq \left\langle u_{n+1} - u_n, x_{n+1} - x_n + \left(1 - \frac{r_n}{r_{n+1}}\right) (u_{n+1} - x_{n+1}) \right\rangle \\ & \leq \|u_{n+1} - u_n\| \left\{ \|x_{n+1} - x_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right| \|u_{n+1} - x_{n+1}\| \right\}; \end{aligned} \tag{38}$$

thus,

$$\|u_{n+1} - u_n\| \leq \|x_{n+1} - x_n\| + \frac{1}{a} |r_{n+1} - r_n| M_1, \tag{39}$$

where $M_1 = \sup\{\|u_n - x_n\| : n \in \mathbb{N}\}$.

From (32) and (39), we obtain

$$\begin{aligned} & \|x_{n+1} - x_n\| \\ & \leq (\alpha_n \gamma \rho + \beta_n) \|x_n - x_{n-1}\| \\ & \quad + |\alpha_n - \alpha_{n-1}| (\gamma \|f(x_{n-1})\| + \|\mu F(u_{n-1})\|) \\ & \quad + (1 - \beta_n - \alpha_n \tau) \left(\|x_n - x_{n-1}\| + \frac{1}{a} |r_n - r_{n-1}| M_1 \right) \\ & \quad + |\beta_n - \beta_{n-1}| (\|u_{n-1}\| + \|x_{n-1}\|) \\ & \leq (1 - \alpha_n \tau + \alpha_n \gamma \rho) \|x_n - x_{n-1}\| \\ & \quad + |\alpha_n - \alpha_{n-1}| (\gamma \|f(x_{n-1})\| + \|\mu F(u_{n-1})\|) \\ & \quad + \frac{M_1}{a} |r_n - r_{n-1}| + |\beta_n - \beta_{n-1}| (\|u_{n-1}\| + \|x_{n-1}\|) \end{aligned}$$

$$\begin{aligned} & \leq (1 - \alpha_n (\tau - \gamma \rho)) \|x_n - x_{n-1}\| \\ & \quad + M_2 (|\alpha_n - \alpha_{n-1}| + |r_n - r_{n-1}| + |\beta_n - \beta_{n-1}|), \end{aligned} \tag{40}$$

where $M_2 \geq \max\{\gamma \|f(x_{n-1})\| + \|\mu F(u_{n-1})\|, M_1/a, \|u_{n-1}\| + \|x_{n-1}\|\}$, for all $n \in \mathbb{N}$. Hence, by Lemma 6, we have

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \tag{41}$$

Then, from (39), (41), and $|r_{n+1} - r_n| \rightarrow 0$, we have

$$\lim_{n \rightarrow \infty} \|u_{n+1} - u_n\| = 0. \tag{42}$$

Next, we show that $\|x_n - u_n\| \rightarrow 0$.

Indeed, for any $p \in \text{EP}(F_1)$, by Lemma 5, we have

$$\begin{aligned} \|u_n - p\|^2 & = \|T_{r_n} x_n - T_{r_n} p\|^2 \\ & \leq \langle x_n - p, u_n - p \rangle \\ & = \frac{1}{2} (\|x_n - p\|^2 + \|u_n - p\|^2 - \|u_n - x_n\|^2). \end{aligned} \tag{43}$$

This implies that

$$\|u_n - p\|^2 \leq \|x_n - p\|^2 - \|u_n - x_n\|^2. \tag{44}$$

Then from (29) and (44), we derive that

$$\begin{aligned} & \|x_{n+1} - p\|^2 \\ & = \|\alpha_n \gamma f(x_n) + ((1 - \beta_n) I - \alpha_n \mu F) u_n + \beta_n x_n - p\|^2 \\ & \leq (\alpha_n \|\gamma f(x_n) - \mu F(p)\| + \beta_n \|x_n - p\| \\ & \quad + \|((1 - \beta_n) I - \alpha_n \mu F) u_n - ((1 - \beta_n) I - \alpha_n \mu F) p\|)^2 \\ & \leq (\alpha_n \gamma \|f(x_n) - f(p)\| + \alpha_n \|\gamma f(p) - \mu F(p)\| \\ & \quad + (1 - \beta_n - \alpha_n \tau) \|u_n - p\| + \beta_n \|x_n - p\|)^2 \\ & \leq ((\alpha_n \gamma \rho + \beta_n) \|x_n - p\| + (1 - \beta_n - \alpha_n \tau) \|u_n - p\| \\ & \quad + \alpha_n \|\gamma f(p) - \mu F(p)\|)^2 \\ & \leq ((\alpha_n \tau + \beta_n) \|x_n - p\| + (1 - \beta_n - \alpha_n \tau) \|u_n - p\| \\ & \quad + \alpha_n \|\gamma f(p) - \mu F(p)\|)^2 \\ & \leq (\alpha_n \tau + \beta_n) \|x_n - p\|^2 + (1 - \beta_n - \alpha_n \tau) \|u_n - p\|^2 \\ & \quad + \alpha_n^2 \|\gamma f(p) - \mu F(p)\|^2 \\ & \quad + 2 [(\alpha_n \tau + \beta_n) \|x_n - p\| + (1 - \beta_n - \alpha_n \tau) \|u_n - p\|] \\ & \quad \times \alpha_n \|\gamma f(p) - \mu F(p)\| \\ & \leq (\alpha_n \tau + \beta_n) \|x_n - p\|^2 + (1 - \beta_n - \alpha_n \tau) \\ & \quad \times (\|x_n - p\|^2 - \|u_n - x_n\|^2) + \alpha_n^2 \|\gamma f(p) - \mu F(p)\|^2 \\ & \quad + 2 \|x_n - p\| \alpha_n \|\gamma f(p) - \mu F(p)\| \end{aligned}$$

$$\begin{aligned}
 &= \|x_n - p\|^2 - (1 - \beta_n - \alpha_n \tau) \|u_n - x_n\|^2 \\
 &\quad + \alpha_n^2 \|\gamma f(p) - \mu F(p)\|^2 + 2 \|x_n - p\| \alpha_n \|\gamma f(p) - \mu F(p)\| \\
 &\leq (\|x_n - x_{n+1}\| + \|x_{n+1} - p\|)^2 - (1 - \beta_n - \alpha_n \tau) \|u_n - x_n\|^2 \\
 &\quad + \alpha_n^2 \|\gamma f(p) - \mu F(p)\|^2 + 2 \|x_n - p\| \alpha_n \|\gamma f(p) - \mu F(p)\| \\
 &= \|x_n - x_{n+1}\|^2 + \|x_{n+1} - p\|^2 + 2 \|x_n - x_{n+1}\| \|x_{n+1} - p\| \\
 &\quad - (1 - \beta_n - \alpha_n \tau) \|u_n - x_n\|^2 + \alpha_n^2 \|\gamma f(p) - \mu F(p)\|^2 \\
 &\quad + 2 \|x_n - p\| \alpha_n \|\gamma f(p) - \mu F(p)\|.
 \end{aligned} \tag{45}$$

Thus,

$$\begin{aligned}
 &(1 - \beta_n - \alpha_n \tau) \|u_n - x_n\|^2 \\
 &\leq \|x_n - x_{n+1}\|^2 + 2 \|x_n - x_{n+1}\| \|x_{n+1} - p\| \\
 &\quad + \alpha_n^2 \|\gamma f(p) - \mu F(p)\|^2 \\
 &\quad + 2 \|x_n - p\| \alpha_n \|\gamma f(p) - \mu F(p)\|.
 \end{aligned} \tag{46}$$

From (41) and condition (C1), we obtain that

$$(1 - \beta_n - \alpha_n \tau) \|u_n - x_n\|^2 \rightarrow 0, \quad n \rightarrow \infty. \tag{47}$$

From conditions (C1) and (C3), we get that

$$\lim_{n \rightarrow \infty} \|u_n - x_n\| = 0. \tag{48}$$

Next, we show that

$$\limsup_{n \rightarrow \infty} \langle (\gamma f - \mu F)z, x_n - z \rangle \leq 0, \tag{49}$$

where $z \in S$ which solves the variational inequality

$$\langle (\mu F - \gamma f)z, z - x \rangle \leq 0, \quad \forall x \in S. \tag{50}$$

Since $\{x_n\}$ is bounded, without loss of generality, we may assume that $x_{n_i} \rightarrow \tilde{z}$ such that

$$\begin{aligned}
 &\limsup_{n \rightarrow \infty} \langle (\gamma f - \mu F)z, x_n - z \rangle \\
 &= \lim_{i \rightarrow \infty} \langle (\gamma f - \mu F)z, x_{n_i} - z \rangle.
 \end{aligned} \tag{51}$$

From $\|u_n - x_n\| \rightarrow 0$ and $x_{n_i} \rightarrow \tilde{z}$, we obtain that $u_{n_i} \rightarrow \tilde{z}$.

Next, we show that $\tilde{z} \in S = \text{EP}(F_1)$.

Indeed, from $u_n = T_{r_n} x_n$, for any $y \in C$, we obtain

$$F_1(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0. \tag{52}$$

From (A2), we have

$$\frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq F_1(y, u_n) \tag{53}$$

and hence

$$\left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \right\rangle \geq F_1(y, u_{n_i}). \tag{54}$$

Since $(u_{n_i} - x_{n_i})/r_{n_i} \rightarrow 0$ and $u_{n_i} \rightarrow \tilde{z}$, it follows from (A4) that $F_1(y, \tilde{z}) \leq 0$, for any $y \in C$. Let $y_t = ty + (1-t)\tilde{z}$, for all $t \in (0, 1]$, $y \in C$. Since $\{u_{n_i}\} \subset C$, $u_{n_i} \rightarrow \tilde{z}$, and C is closed and convex set, then we get $\tilde{z} \in C$. Since $y \in C$ and $\tilde{z} \in C$, then we have $y_t \in C$. Hence we have $F_1(y_t, \tilde{z}) \leq 0$.

Thus, from (A1) and (A4), we have

$$\begin{aligned}
 0 &= F_1(y_t, y_t) \\
 &= F_1(y_t, ty + (1-t)\tilde{z}) \\
 &\leq tF_1(y_t, y) + (1-t)F_1(y_t, \tilde{z}) \\
 &\leq tF_1(y_t, y),
 \end{aligned} \tag{55}$$

and hence $F_1(y_t, y) \geq 0$. From (A3), we have $F_1(\tilde{z}, y) \geq 0$, for any $y \in C$. Hence $\tilde{z} \in \text{EP}(F_1) = S$.

Therefore,

$$\begin{aligned}
 \limsup_{n \rightarrow \infty} \langle (\gamma f - \mu F)z, x_n - z \rangle &= \lim_{i \rightarrow \infty} \langle (\gamma f - \mu F)z, x_{n_i} - z \rangle \\
 &= \langle (\gamma f - \mu F)z, \tilde{z} - z \rangle \leq 0.
 \end{aligned} \tag{56}$$

Finally, we show that $x_n \rightarrow z$.

As a matter of fact,

$$\begin{aligned}
 x_{n+1} - z &= \alpha_n \gamma f(x_n) + ((1 - \beta_n)I - \alpha_n \mu F)u_n + \beta_n x_n - z \\
 &= \alpha_n \gamma f(x_n) - \alpha_n \mu F(z) + \beta_n x_n - \beta_n z \\
 &\quad + ((1 - \beta_n)I - \alpha_n \mu F)u_n - ((1 - \beta_n)I - \alpha_n \mu F)z \\
 &= \alpha_n \gamma (f(x_n) - f(z)) \\
 &\quad + \alpha_n (\gamma f(z) - \mu F(z)) + \beta_n (x_n - z) \\
 &\quad + ((1 - \beta_n)I - \alpha_n \mu F)u_n - ((1 - \beta_n)I - \alpha_n \mu F)z.
 \end{aligned} \tag{57}$$

So, from (29), we derive

$$\begin{aligned}
 &\|x_{n+1} - z\|^2 \\
 &= \alpha_n \gamma \langle f(x_n) - f(z), x_{n+1} - z \rangle \\
 &\quad + \langle ((1 - \beta_n)I - \alpha_n \mu F)u_n \\
 &\quad - ((1 - \beta_n)I - \alpha_n \mu F)z, x_{n+1} - z \rangle \\
 &\quad + \alpha_n \langle \gamma f(z) - \mu F(z), x_{n+1} - z \rangle + \beta_n \langle x_n - z, x_{n+1} - z \rangle \\
 &\leq \alpha_n \gamma \rho \|x_n - z\| \cdot \|x_{n+1} - z\| \\
 &\quad + (1 - \beta_n - \alpha_n \tau) \|u_n - z\| \cdot \|x_{n+1} - z\| \\
 &\quad + \alpha_n \langle \gamma f(z) - \mu F(z), x_{n+1} - z \rangle \\
 &\quad + \beta_n \|x_n - z\| \cdot \|x_{n+1} - z\|
 \end{aligned}$$

$$\begin{aligned}
 &\leq (1 - \alpha_n(\tau - \gamma\rho)) \|x_n - z\| \cdot \|x_{n+1} - z\| \\
 &\quad + \alpha_n \langle \gamma f(z) - \mu F(z), x_{n+1} - z \rangle \\
 &\leq \frac{1 - \alpha_n(\tau - \gamma\rho)}{2} (\|x_n - z\|^2 + \|x_{n+1} - z\|^2) \\
 &\quad + \alpha_n \langle \gamma f(z) - \mu F(z), x_{n+1} - z \rangle.
 \end{aligned} \tag{58}$$

It follows that

$$\begin{aligned}
 &\|x_{n+1} - z\|^2 \\
 &\leq \frac{1 - \alpha_n(\tau - \gamma\rho)}{1 + \alpha_n(\tau - \gamma\rho)} \|x_n - z\|^2 \\
 &\quad + \frac{2\alpha_n}{1 + \alpha_n(\tau - \gamma\rho)} \langle \gamma f(z) - \mu F(z), x_{n+1} - z \rangle \\
 &\leq (1 - \alpha_n(\tau - \gamma\rho)) \|x_n - z\|^2 \\
 &\quad + \frac{2\alpha_n}{1 + \alpha_n(\tau - \gamma\rho)} \langle \gamma f(z) - \mu F(z), x_{n+1} - z \rangle.
 \end{aligned} \tag{59}$$

Hence,

$$\|x_{n+1} - z\|^2 \leq (1 - \gamma_n) \|x_n - z\|^2 + \gamma_n \delta_n, \tag{60}$$

where $\gamma_n = \alpha_n(\tau - \gamma\rho)$, $\gamma_n \delta_n = (2\alpha_n/(1 + \alpha_n(\tau - \gamma\rho))) \langle \gamma f(z) - \mu F(z), x_{n+1} - z \rangle$.

Since $\alpha_n \rightarrow 0$ and $\sum_{n=0}^\infty \alpha_n = \infty$, we get $\gamma_n \rightarrow 0$ and $\sum_{n=0}^\infty \gamma_n = \infty$. From (49), we get

$$\begin{aligned}
 \limsup_{n \rightarrow \infty} \delta_n &= \limsup_{n \rightarrow \infty} \frac{2}{(1 + \alpha_n(\tau - \gamma\rho))(\tau - \gamma\rho)} \\
 &\quad \times \langle \gamma f(z) - \mu F(z), x_{n+1} - z \rangle \leq 0.
 \end{aligned} \tag{61}$$

Now, applying Lemma 6 to (60) concludes that $x_n \rightarrow z$ as $n \rightarrow \infty$. \square

Remark 9. Our proposed algorithm (23) is more general than Su and Li's algorithm (5), because of introducing nonlinear operator F .

If $\gamma = 1$, $\mu = 1$, and $F = I$, we obtain the following Corollary.

Corollary 10. *Let H be a real Hilbert space and let C be a nonempty, closed, and convex subset of H . Let $h : C \rightarrow \mathbb{R}$ be a real-valued convex function. Assume that $h(x)$ is lower semicontinuous. Let $f : H \rightarrow H$ be a contraction with a coefficient $0 < \rho < 1$. Suppose that the optimization problem (3) is consistent and let S denote its solution set. Let $\{x_n\}_{n=0}^\infty$ be generated by $x_1 \in H$ and*

$$\begin{aligned}
 h(y) - h(u_n) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle &\geq 0, \quad \forall y \in C, \\
 x_{n+1} &= \alpha_n f(x_n) + (1 - \beta_n - \alpha_n) u_n + \beta_n x_n, \quad n \in \mathbb{N},
 \end{aligned} \tag{62}$$

where $u_n = T_{r_n} x_n$. Let $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$, $\{r_n\} \subset (0, \infty)$ satisfy the following conditions:

- (C1) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^\infty \alpha_n = \infty$;
- (C2) $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty$;
- (C3) $\limsup_{n \rightarrow \infty} \beta_n < 1$ and $\sum_{n=1}^\infty |\beta_{n+1} - \beta_n| < \infty$;
- (C4) $\liminf_{n \rightarrow \infty} r_n > 0$ and $\sum_{n=1}^\infty |r_{n+1} - r_n| < \infty$.

Then the sequence $\{x_n\}$ generated by (62) converges strongly to a point $z \in S$, which solves the variational inequality

$$\langle (I - f)z, z - x \rangle \leq 0, \quad \forall x \in S. \tag{63}$$

Remark 11. We would like to point out that the conditions of algorithm (62) are different from those of algorithm (4.3) in [2]. The conditions of the sequence $\{\alpha_n\}$ and the sequence $\{r_n\}$ of parameters in (4.3) are the same as the ones of algorithm (62), but the conditions of the sequence $\{\beta_n\}$ of parameters in (4.3) are $\beta_n = o(\alpha_n)$ and $\sum_{n=1}^\infty |\beta_{n+1} - \beta_n| < \infty$. It is obvious that our conditions are much weaker.

4. Application

In this section, we give an application of Theorem 8 to the split feasibility problem (say SFP, for short) which was introduced by Censor and Elfving [15]. Since its inception in 1994, the split feasibility problem (SFP) has received much attention (see [16–18]) due to its applications to signal processing and image reconstruction, with particular progress in intensity-modulated radiation therapy.

The SFP can mathematically be formulated as the problem of finding a point x with the property

$$x \in C, \quad Bx \in Q, \tag{64}$$

where C and Q are nonempty closed convex subsets of Hilbert spaces H_1 and H_2 , respectively. $B : H_1 \rightarrow H_2$ is a bounded linear operator.

It is clear that x^* is a solution to the split feasibility problem (64) if and only if $x^* \in C$ and $Bx^* - P_Q Bx^* = 0$. We define the proximity function h by

$$h(x) = \frac{1}{2} \|Bx - P_Q Bx\|^2 \tag{65}$$

and consider the following optimization problem:

$$\min_{x \in C} h(x) = \min_{x \in C} \frac{1}{2} \|Bx - P_Q Bx\|^2. \tag{66}$$

Then, x^* solves the split feasibility problem (64) if and only if x^* solves the minimization problem (66) with the minimization being equal to 0. Byrne [19] introduced the so-called CQ algorithm to solve the SFP:

$$x_{n+1} = P_C (I - \gamma B^* (I - P_Q) B) x_n, \quad n \geq 0, \tag{67}$$

where $0 < \gamma < 2/\|B\|^2$. He obtained that the sequence $\{x_n\}$ generated by (67) converges weakly to a solution of the SFP.

In order to obtain strong convergence iterative sequence to solve the SFP (64), we propose the following iterative algorithm by $x_1 \in H_1$ and

$$\begin{aligned} & \frac{1}{2} \|By - P_Q By\|^2 - \frac{1}{2} \|Bu_n - P_Q Bu_n\|^2 \\ & + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \quad (68) \\ & x_{n+1} = \alpha_n \gamma f(x_n) + ((1 - \beta_n)I - \alpha_n \mu F) u_n + \beta_n x_n, \\ & \quad \quad \quad n \in \mathbb{N}, \end{aligned}$$

where $u_n = T_{r_n} x_n$ and $T_{r_n} : H_1 \rightarrow C$ is a mapping defined as in Lemma 5, where $f : H_1 \rightarrow H_1$ is a contraction with a coefficient $0 < \rho < 1$ and $F : C \rightarrow H_1$ is a k -Lipschitzian and η -strongly monotone operator with constants $k > 0$, $\eta > 0$. Suppose that $0 < \mu < 2\eta/k^2$, $0 < \gamma < \tau/\rho$, where $\tau = \mu\eta - (\mu^2 k^2/2)$. We can show that the sequence $\{x_n\}$ generated by (68) converges strongly to a solution of the SFP (64), if the sequence $\{\alpha_n\}$, $\{\beta_n\} \subset (0, 1)$ and the sequence $\{r_n\} \subset (0, \infty)$ of parameters satisfy appropriate conditions.

Applying Theorem 8, we obtain the following result.

Theorem 12. *Assume that the split feasibility problem (64) is consistent. Let the sequence $\{x_n\}$ be generated by (68), where the sequence $\{\alpha_n\}$, $\{\beta_n\} \subset (0, 1)$ and the sequence $\{r_n\} \subset (0, \infty)$ satisfy the conditions (C1)–(C5). Then the sequence $\{x_n\}$ converges strongly to a solution of the split feasibility problem (64).*

Proof. We define the proximity function h by

$$h(x) = \frac{1}{2} \|Bx - P_Q Bx\|^2 \quad (69)$$

and consider the following optimization problem:

$$\min_{x \in C} h(x) = \min_{x \in C} \frac{1}{2} \|Bx - P_Q Bx\|^2, \quad (70)$$

where $h : C \rightarrow \mathbb{R}$ is a real-valued convex function and $h(x)$ is lower semicontinuous. Then, x^* solves the split feasibility problem (64) if and only if x^* solves the minimization problem (66) with the minimization being equal to 0.

Consequently, the iterative scheme (68) is equivalent to

$$\begin{aligned} & x_1 \in H_1, \\ & h(y) - h(u_n) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \\ & x_{n+1} = \alpha_n \gamma f(x_n) + ((1 - \beta_n)I - \alpha_n \mu F) u_n + \beta_n x_n, \quad n \in \mathbb{N}, \quad (71) \end{aligned}$$

where $u_n = T_{r_n} x_n$ and $T_{r_n} : H_1 \rightarrow C$ is a mapping defined as in Lemma 5. Due to Theorem 8, we have the conclusion immediately. \square

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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