Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 148321, 8 pages
http://dx.doi.org/10.1155/2014/148321

Research Article

Linear Maps on Upper Triangular Matrices Spaces Preserving Idempotent Tensor Products

Li Yang,[1] Wei Zhang,[2] and Jinli Xu[3]

[1] *Department of Foundation, Harbin Finance University, Harbin 150030, China*
[2] *School of Mathematical Science, Heilongjiang University, Harbin 150080, China*
[3] *Department of Mathematics, Harbin Institute of Technology, Harbin 150001, China*

Correspondence should be addressed to Wei Zhang; weizhanghlj@163.com

Received 26 October 2013; Accepted 10 December 2013; Published 28 January 2014

Academic Editor: Antonio M. Peralta

Suppose $m, n \geq 2$ are positive integers. Let $\mathcal{T}_n$ be the space of all $n \times n$ complex upper triangular matrices, and let ϕ be an injective linear map on $\mathcal{T}_m \otimes \mathcal{T}_n$. Then $\phi(A \otimes B)$ is an idempotent matrix in $\mathcal{T}_m \otimes \mathcal{T}_n$ whenever $A \otimes B$ is an idempotent matrix in $\mathcal{T}_m \otimes \mathcal{T}_n$ if and only if there exists an invertible matrix $P \in \mathcal{T}_m \otimes \mathcal{T}_n$ such that $\phi(A \otimes B) = P(\xi_1(A) \otimes \xi_2(B))P^{-1}$, $\forall A \in \mathcal{T}_m$, $B \in \mathcal{T}_n$, or when $m = n$, $\phi(A \otimes B) = P(\xi_1(B) \otimes \xi_2(A))P^{-1}$, $\forall A \in \mathcal{T}_m$, $B \in \mathcal{T}_m$, where $\xi_1([a_{ij}]) = [a_{ij}]$ or $\xi_1([a_{ij}]) = [a_{m-i+1,m-j+1}]$ and $\xi_2([b_{ij}]) = [b_{ij}]$ or $\xi_2([b_{ij}]) = [b_{n-i+1,n-j+1}]$.

1. Introduction

Suppose $m, n \geq 2$ are positive integers. Let $\mathcal{M}_n$ be the space of all $n \times n$ complex matrices, and let $\mathcal{T}_n$ be all upper triangular in $\mathcal{M}_n$. For $A \in \mathcal{M}_m$, $B \in \mathcal{M}_n$, we denote by $A \otimes B$ their tensor product (a.k.a. Kronecker product).

Linear preserver problem is a hot area in matrix and operator theory; there are many results about this area (see [1–14]). Specially, the idempotence preservers and the rank one preservers play an important role (see [1, 2]); therefore, it is meaningful to study the idempotence preservers. Chan et al. [3] first characterize linear transformations on $\mathcal{M}_n$ preserving idempotent matrices. Šemrl [4] applying projective geometry gives the form of transformations on rank-1 idempotents. Tang et al. [5] investigate injective linear idempotence preservers on $\mathcal{T}_n$.

In quantum information science, quantum states of a system with n physical states are represented as density matrices, that is, positive semidefinite matrices with trace one. If $A \in \mathcal{M}_m$ and $B \in \mathcal{M}_n$ are two quantum states in two quantum systems, then $A \otimes B$ describes a joint state in bipartite system $\mathcal{M}_m \otimes \mathcal{M}_n$. Recently, many researchers consider the problem combining linear preserver problem with quantum information science. They determine the structure of linear maps on $\mathcal{M}_m \otimes \mathcal{M}_n$ by using information only about the images of matrices possessing tensor product form. One can see [15–18] and their references for some background on linear preserver problems on tensor spaces arising in quantum information science.

Inspired by the above, the purpose of this paper is to study injective linear maps ϕ on $\mathcal{T}_m \otimes \mathcal{T}_n$ satisfying $\phi(A \otimes B)$ is an idempotent matrix whenever $A \otimes B$ is an idempotent matrix in $\mathcal{T}_m \otimes \mathcal{T}_n$. If we remove the assumption that map is injective, then ϕ may have various forms as follows.

Example 1. $A \otimes B \mapsto a_{11} I_m \otimes B$ is a linear idempotent preserver on $\mathcal{T}_m \otimes \mathcal{T}_n$.

Example 2. $A \otimes B \mapsto a_{11}(E_{11}^{(m)} + E_{22}^{(m)}) \otimes (b_{11}E_{11}^{(n)} + b_{12}E_{12}^{(n)} + b_{22}E_{22}^{(n)})$ is a linear idempotent preserver on $\mathcal{T}_m \otimes \mathcal{T}_n$.

We end this section by introducing some notations which will be used in the following sections. Let $\mathbf{C}$ be the complex field, I_k the $k \times k$ identity matrix, 0 the zero matrix whose order is omitted in different matrices just for simplicity, and X^t (resp., rank X) the transpose (resp., rank) of X. $E_{ij}^{(n)}$,

$\forall i, j \in [1,n]$ stands for the $n \times n$ matrix with 1 at the (i,j)th entry and 0 otherwise. Denote by J_k (also J) the matrix $E^{(k)}_{1,k} + E^{(k)}_{2,k-1} + \cdots + E^{(k)}_{k,1}$. Clearly, if $A = [a_{ij}] \in \mathcal{T}_n$, then $JA^tJ = [a_{n-j+1,n-i+1}] \in \mathcal{T}_n$. For positive integers n_1 and n_2 with $n_1 < n_2$, let $[n_1, n_2]$ be the set of all integers between n_1 and n_2. For any $(i,j) \in [1,m] \times [1,n]$, we define ρ by

$$\rho(i,j) = (i-1)n + j. \tag{1}$$

For any $k \in [1, mn]$, we define $\sigma(k) \in [1,m]$ and $\tau(k) \in [1,n]$ such that $k = (\sigma(k)-1)n + \tau(k)$ (it is easy to see that σ and τ are well defined). It is easy to see that

$$E^{(m)}_{rs} \otimes E^{(n)}_{uv} = E^{(mn)}_{(r-1)n+u,(s-1)n+v} = E^{(mn)}_{\rho(r,u),\rho(s,v)}, \tag{2}$$

$$E^{(mn)}_{ij} = E^{(m)}_{\sigma(i)\sigma(j)} \otimes E^{(n)}_{\tau(i)\tau(j)}. \tag{3}$$

We define a partial ordering of $[1,m]\times[1,n]$ by $(a,b) \le (c,d)$ if and only if $a \le c$ and $b \le d$. We say that (a,b) and (c,d) are comparable, if $(a,b) \le (c,d)$ or $(c,d) \le (a,b)$.

2. Preliminary Results

We need the form of injective linear idempotence preserver on $\mathcal{T}_n$, which was obtained in [5].

Lemma 3 (see [5, Theorem 1]). *Let ψ be an injective linear map on $\mathcal{T}_n$. Then $\psi(X)$ is an idempotent matrix in $\mathcal{T}_n$ whenever X is an idempotent matrix in $\mathcal{T}_n$ if and only if there exists an invertible matrix $P \in \mathcal{T}_n$ such that*

$$\psi(X) = P\xi(X)P^{-1}, \quad \forall X \in \mathcal{T}_n, \tag{4}$$

where $\xi(X) = X$ or $\xi(X) = JX^tJ$.

It is clear that $\mathcal{T}_m \otimes \mathcal{T}_n \subsetneq \mathcal{T}_{mn}$. For example, $E^{(4)}_{23} \in \mathcal{T}_4$ and

$$E^{(4)}_{23} \notin \mathcal{T}_2 \otimes \mathcal{T}_2 = \left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & 0 & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix} : a_{ij} \in \mathbf{C} \right\}. \tag{5}$$

It is easy to see that $(\sigma(2), \tau(2)) = (1,2)$ and $(\sigma(3), \tau(3)) = (2,1)$. In fact, we can point out the positions of elements which are in $\mathcal{T}_{mn} \setminus \mathcal{T}_m \otimes \mathcal{T}_n$.

Lemma 4. *$\lambda E^{(mn)}_{ij} \in \mathcal{T}_m \otimes \mathcal{T}_n$ if and only if $(\sigma(i), \tau(i)) \le (\sigma(j), \tau(j))$ or $\lambda = 0$.*

Proof. It is a direct corollary of (3). □

The next Lemma describes the partial ordering we defined in Section 1, which is useful to prove our main Theorem.

Lemma 5 (see [19, Theorem 1]). *Let $m, n \ge 2$, and let A be a matrix with m rows and n columns containing all elements of $[1,m] \times [1,n]$. If every two elements of A in the same row and column are comparable, respectively, then there exist permutation matrices $U \in \mathcal{M}_m$ and $V \in \mathcal{M}_n$ such that*

$$UAV = \begin{bmatrix} (1,1) & (1,2) & \cdots & (1,n-1) & (1,n) \\ (2,1) & (2,2) & \cdots & (2,n-1) & (2,n) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (m-1,1) & (m-1,2) & \cdots & (m-1,n-1) & (m-1,n) \\ (m,1) & (m,2) & \cdots & (m,n-1) & (m,n) \end{bmatrix} \tag{I}$$

or when $m \ge 3$ or $n \ge 3$

$$UAV = \begin{bmatrix} (m,n) & (1,2) & \cdots & (1,n-1) & (1,n) \\ (2,1) & (2,2) & \cdots & (2,n-1) & (2,n) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (m-1,1) & (m-1,2) & \cdots & (m-1,n-1) & (m-1,n) \\ (m,1) & (m,2) & \cdots & (m,n-1) & (1,1) \end{bmatrix} \tag{II}$$

or when $m = n$

$$UAV = \begin{bmatrix} (1,1) & (2,1) & \cdots & (m-1,1) & (m,1) \\ (1,2) & (2,2) & \cdots & (m-1,2) & (m,2) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (1,m-1) & (2,m-1) & \cdots & (m-1,m-1) & (m,m-1) \\ (1,m) & (2,m) & \cdots & (m-1,m) & (m,m) \end{bmatrix} \tag{III}$$

or when $m = n \ge 3$

$$UAV = \begin{bmatrix} (m,m) & (2,1) & \cdots & (m-1,1) & (m,1) \\ (1,2) & (2,2) & \cdots & (m-1,2) & (m,2) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (1,m-1) & (2,m-1) & \cdots & (m-1,m-1) & (m,m-1) \\ (1,m) & (2,m) & \cdots & (m-1,m) & (1,1) \end{bmatrix}. \tag{IV}$$

The following lemmas would make the proof of the main theorem more concise.

Lemma 6. *Suppose $X \in \mathscr{M}_n$ is an idempotent matrix such that $0_s \oplus I_r \oplus 0_{n-r-s} - X$ also is an idempotent matrix. Then there exists an idempotent matrix $X_r \in \mathscr{M}_r$ such that $X = 0_s \oplus X_r \oplus 0_{n-r-s}$.*

Proof. By $X^2 = X$ and $(0_s \oplus I_r \oplus 0_{n-r-s} - X)^2 = 0_s \oplus I_r \oplus 0_{n-r-s} - X$, we have

$$2X = \left(0_s \oplus I_r \oplus 0_{n-r-s}\right) X + X \left(0_s \oplus I_r \oplus 0_{n-r-s}\right). \quad (6)$$

Set

$$X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{bmatrix}, \quad (7)$$

where $X_{11} \in \mathscr{M}_s, X_{22} \in \mathscr{M}_r$. Then (6) implies

$$2 \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{bmatrix} = \begin{bmatrix} 0 & X_{12} & 0 \\ X_{21} & 2X_{22} & X_{23} \\ 0 & X_{32} & 0 \end{bmatrix}. \quad (8)$$

Hence, $X = 0_s \oplus X_{22} \oplus 0_{n-r-s}$; therefore, the lemma holds. □

Lemma 7. *Let $X \in \mathscr{T}_m \otimes \mathscr{T}_n$ be an idempotent matrix such that $E_{ii}^{(m)} \otimes I_n - X$ also is an idempotent matrix. Then there exists an idempotent $Y \in \mathscr{T}_n$ such that $X = E_{ii}^{(m)} \otimes Y$.*

Proof. The proof is similar to that of Lemma 6. □

Lemma 8. *Suppose $r, s \in [1, n-1]$. If for any $\lambda \in \mathbf{C}$, $I_r \oplus 0_{n-r} + \lambda X$ and $0_r \oplus I_s \oplus 0_{n-r-s} + \lambda X$ are idempotent in $\mathscr{T}_n$, then*

$$X = \begin{bmatrix} 0_r & X_1 \\ 0 & 0_s \end{bmatrix} \oplus 0_{n-r-s}. \quad (9)$$

Proof. It follows from $I_r \oplus 0_{n-r} + \lambda X, \forall \lambda \in \mathbf{C}$ is idempotent that

$$\left(I_r \oplus 0_{n-r}\right) X + X \left(I_r \oplus 0_{n-r}\right) = X. \quad (10)$$

Let

$$X = \begin{bmatrix} X_r & X_1 & X_2 \\ 0 & X_s & X_3 \\ 0 & 0 & X_{n-r-s} \end{bmatrix}, \quad (11)$$

where $X_r \in \mathscr{T}_r, X_s \in \mathscr{T}_s$, then (10) implies

$$\begin{bmatrix} 2X_r & X_1 & X_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} X_r & X_1 & X_2 \\ 0 & X_s & X_3 \\ 0 & 0 & X_{n-r-s} \end{bmatrix}. \quad (12)$$

Hence, $X_r = 0, X_s = 0, X_{n-r-s} = 0, X_3 = 0$. Similarly, from $0_r \oplus I_s \oplus 0_{n-r-s} + \lambda X$ being idempotent, we have $X_2 = 0$. □

Lemma 9 (see [6, Page 62, Exercise 1]). *Suppose $A_1, \ldots, A_k \in \mathscr{M}_n$ are idempotent matrices such that, for any $i \neq j \in [1, k]$, $A_i + A_j$ is idempotent. Let $r_i = \operatorname{rank} A_i$. Then there exists an invertible matrix $P \in \mathscr{M}_n$ such that*

$$A_i = P \operatorname{diag}(0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0) P^{-1}, \quad (13)$$

where $\operatorname{diag}(0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0)$ *is the diagonal matrix in which all diagonal entries are zero except those in the $(r_1 + \cdots + r_{i-1} + 1)$st to the $(r_1 + \cdots + r_i)$th rows.*

Similar to Lemma 9, we have the following.

Lemma 10. *Let $A_1, \ldots, A_{mn} \in \mathscr{T}_m \otimes \mathscr{T}_n$ be idempotent matrices of rank-1 such that for any $i \neq j \in [1, mn]$, $A_i + A_j$ is idempotent. Then there exist a permutation π on $[1, mn]$ and an invertible matrix $P \in \mathscr{T}_m \otimes \mathscr{T}_n$ such that*

$$A_i = P E_{\pi(i)\pi(i)}^{(mn)} P^{-1}, \quad i = 1, \ldots, mn. \quad (14)$$

Proof. By $A_i \in \mathscr{T}_m \otimes \mathscr{T}_n$ being an idempotent matrix of rank-1, we can assume

$$A_i = E_{\pi(i)\pi(i)}^{(mn)} + B_i, \quad i = 1, \ldots, mn, \quad (15)$$

where $B_i \in \mathscr{T}_m \otimes \mathscr{T}_n$ with zero diagonal entries. It follows from $A_i + A_j$, $\forall i \neq j$ being is idempotent that $\pi(i) \neq \pi(j)$, $\forall i \neq j$. Hence, π is a permutation on $[1, mn]$. By $A_{\pi^{-1}(1)}^2 = A_{\pi^{-1}(1)}$, we can see $A_{\pi^{-1}(1)} = E_{11}^{(mn)} + \Sigma_{k=2}^{(mn)} \lambda_{1k} E_{1k}^{(mn)}$. Let $P_1 = I_{mn} - \Sigma_{k=2}^{(mn)} \lambda_{1k} E_{1k}^{(mn)} \in \mathscr{T}_m \otimes \mathscr{T}_n$; then $P_1^{-1} = I_{mn} + \Sigma_{k=2}^{(mn)} \lambda_{1k} E_{1k}^{(mn)}$ and

$$A_{\pi^{-1}(1)} = P_1 E_{11}^{(mn)} P_1^{-1}. \quad (16)$$

By $A_{\pi^{-1}(1)} + A_{\pi^{-1}(2)}$ being idempotent, we obtain $P_1^{-1} A_{\pi^{-1}(2)} P_1 = E_{22}^{(mn)} + \Sigma_{k=3}^{(mn)} \lambda_{2k} E_{2k}^{(mn)}$. Let $P_2 = I_{mn} - \Sigma_{k=3}^{(mn)} \lambda_{2k} E_{2k}^{(mn)}$; then

$$\begin{aligned} A_{\pi^{-1}(1)} &= P_2 P_1 E_{11}^{(mn)} P_1^{-1} P_2^{-1}, \\ A_{\pi^{-1}(2)} &= P_2 P_1 E_{22}^{(mn)} P_1^{-1} P_2^{-1}. \end{aligned} \quad (17)$$

Continuing to do this, we can find $P_3, \ldots, P_{mn}$. Let $P = P_{mn} P_{mn-1} \cdots P_2 P_1 \in \mathscr{T}_m \otimes \mathscr{T}_n$; then

$$A_{\pi^{-1}(i)} = P E_{ii}^{(mn)} P^{-1}, \quad i = 1, \ldots, mn. \quad (18)$$

This completes the proof. □

3. The Main Result

The main result of this paper is as follows.

Theorem 11. *Suppose $m, n \geq 2$ are positive integers and ϕ is an injective linear map on $\mathscr{T}_m \otimes \mathscr{T}_n$. Then $\phi(A \otimes B)$*

is an idempotent matrix in $\mathcal{T}_m \otimes \mathcal{T}_n$ whenever $A \otimes B$ is an idempotent matrix in $\mathcal{T}_m \otimes \mathcal{T}_n$ if and only if there exists an invertible matrix $P \in \mathcal{T}_m \otimes \mathcal{T}_n$ such that

$$\phi(A \otimes B) = P(\xi_1(A) \otimes \xi_2(B)) P^{-1}, \quad \forall A \in \mathcal{T}_m,\ B \in \mathcal{T}_n, \tag{i}$$

or when $m = n$

$$\phi(A \otimes B) = P(\xi_1(B) \otimes \xi_2(A)) P^{-1}, \quad \forall A \in \mathcal{T}_m,\ B \in \mathcal{T}_m, \tag{ii}$$

where, for $i = 1, 2$, $\xi_i(X) = X$ or $\xi_i(X) = JX^tJ$.

Proof. The sufficiency is obvious. We will prove the necessity by the following six steps.

Step 1. There exist a permutation π on $[1, mn]$ and an invertible matrix $P \in \mathcal{T}_m \otimes \mathcal{T}_n$ such that

$$\phi\left(E_{kk}^{(mn)}\right) = PE_{\pi(k)\pi(k)}^{(mn)}P^{-1}, \quad \forall k \in [1, mn]. \tag{19}$$

Proof of Step 1. By Lemma 10, we only need to prove that

$$\operatorname{rank}\phi\left(E_{ii}^{(m)} \otimes E_{jj}^{(n)}\right) = 1, \quad \forall i \in [1, m],\ j \in [1, n]. \tag{20}$$

And for any $(i, j) \neq (u, v) \in [1, m] \times [1, n]$,

$$\phi\left(E_{ii}^{(m)} \otimes E_{jj}^{(n)}\right) + \phi\left(E_{uu}^{(m)} \otimes E_{vv}^{(n)}\right) \text{ is an idempotent matrix.} \tag{21}$$

It follows from $E_{11}^{(m)} \otimes I_n, \ldots, E_{mm}^{(m)} \otimes I_n$ and $(E_{ii}^{(m)} + E_{jj}^{(m)}) \otimes I_n$ $\forall i \neq j \in [1, m]$ are idempotent matrices in $\mathcal{T}_m \otimes \mathcal{T}_n$ that $\phi(E_{11}^{(m)} \otimes I_n), \ldots, \phi(E_{mm}^{(m)} \otimes I_n)$ and $\phi(E_{ii}^{(m)} \otimes I_n) + \phi(E_{jj}^{(m)} \otimes I_n)$, $\forall i \neq j \in [1, m]$ are idempotent matrices. We obtain by using Lemma 9 that there exists an invertible matrix $P_1 \in \mathcal{M}_{mn}$ such that

$$\phi\left(E_{ii}^{(m)} \otimes I_n\right) = P_1 \begin{bmatrix} 0_{s_i} & 0 & 0 \\ 0 & I_{r_i} & 0 \\ 0 & 0 & 0_{mn-s-r_i} \end{bmatrix} P_1^{-1}, \tag{22}$$

where $r_i = \operatorname{rank}\phi(E_{ii}^{(m)} \otimes I_n)$ and $s_i = r_1 + \cdots + r_{i-1}$. For any $j \in [1, n]$, it follows from $E_{ii}^{(m)} \otimes E_{jj}^{(n)}$ and $E_{ii}^{(m)} \otimes (I_n - E_{jj}^{(n)})$ being idempotent matrices in $\mathcal{T}_m \otimes \mathcal{T}_n$ that $\phi(E_{ii}^{(m)} \otimes E_{jj}^{(n)})$ and $\phi(E_{ii}^{(m)} \otimes I_n) - \phi(E_{ii}^{(m)} \otimes E_{jj}^{(n)})$ are idempotent matrices; we obtain by (22) and Lemma 6 that

$$\phi\left(E_{ii}^{(m)} \otimes E_{jj}^{(n)}\right) = P_1 \begin{bmatrix} 0_{s_i} & 0 & 0 \\ 0 & X_j & 0 \\ 0 & 0 & 0_{mn-s_i-r_i} \end{bmatrix} P_1^{-1}, \quad \forall j \in [1, n], \tag{23}$$

where $X_j \in \mathcal{M}_{r_i}$ is an idempotent matrix. For any $j \neq l \in [1, n]$, $E_{ii}^{(m)} \otimes (E_{jj}^{(n)} + E_{ll}^{(n)})$ is an idempotent matrix in $\mathcal{T}_m \otimes \mathcal{T}_n$; we have $\phi(E_{ii}^{(m)} \otimes E_{jj}^{(n)}) + \phi(E_{ii}^{(m)} \otimes E_{ll}^{(n)})$ is an idempotent matrix. It follows from (23) that $X_j + X_l$ is an idempotent matrix. If $r_i < n$, by Lemma 9, we can obtain that there exists some $j_0 \in [1, n]$ such that $X_{j_0} = 0$. This, together with (23), implies $\phi(E_{ii}^{(m)} \otimes E_{j_0 j_0}^{(n)}) = 0$, which is a contradiction to the fact that ϕ is injective. Hence, $r_i = n$, $\forall i \in [1, n]$. By Lemma 9, there exists an invertible matrix $Q_i \in \mathcal{M}_n$ such that $X_j = Q_i E_{jj}^{(n)} Q_i^{-1}$, $\forall j \in [1, n]$. Let $Q = \operatorname{diag}(Q_1, \ldots, Q_m)$; it follows from (23) that

$$\phi\left(E_{ii}^{(m)} \otimes E_{jj}^{(n)}\right) = P_1 Q\left(E_{ii}^{(m)} \otimes E_{jj}^{(n)}\right) Q^{-1} P_1^{-1}, \quad \forall i \in [1, m], \quad k \in [1, n]. \tag{24}$$

Hence, (20) and (21) hold. This completes the proof of Step 1.

By Step 1, we may assume that for any $i \in [1, m]$, $j \in [1, n]$,

$$\phi\left(E_{ii}^{(m)} \otimes E_{jj}^{(n)}\right) = E_{\pi(\rho(i,j))\pi(\rho(i,j))}^{(mn)}. \tag{25}$$

From this, together with (3), we can also write

$$\phi\left(E_{ii}^{(m)} \otimes E_{jj}^{(n)}\right) = E_{\sigma(\pi(\rho(i,j)))\sigma(\pi(\rho(i,j)))}^{(m)} \otimes E_{\tau(\pi(\rho(i,j)))\tau(\pi(\rho(i,j)))}^{(n)}. \tag{26}$$

Step 2.(i) For any $i \in [1, m]$, $j_1, j_2 \in [1, n]$, $(\sigma(\pi(\rho(i, j_1))), \tau(\pi(\rho(i, j_1))))$ and $(\sigma(\pi(\rho(i, j_2))), \tau(\pi(\rho(i, j_2))))$ are comparable.

(ii) For any $i_1, i_2 \in [1, m]$, $j \in [1, n]$, $(\sigma(\pi(\rho(i_1, j))), \tau(\pi(\rho(i_1, j))))$ and $(\sigma(\pi(\rho(i_2, j))), \tau(\pi(\rho(i_2, j))))$ are comparable.

Proof of Step 2. (i) Suppose there exist some $i_0 \in [1, m]$ and $j_1 < j_2 \in [1, n]$ such that $(\sigma(\pi(\rho(i_0, j_1))), \tau(\pi(\rho(i_0, j_1))))$ and $(\sigma(\pi(\rho(i_0, j_2))), \tau(\pi(\rho(i_0, j_2))))$ are not comparable. Without loss of generality, we may assume that

$$\begin{gathered} \sigma(\pi(\rho(i_0, j_1))) < \sigma(\pi(\rho(i_0, j_2))), \\ \tau(\pi(\rho(i_0, j_1))) > \tau(\pi(\rho(i_0, j_2))). \end{gathered} \tag{27}$$

It follows that

$$\begin{aligned} \pi(\rho(i_0, j_1)) &= (\sigma(\pi(\rho(i_0, j_1))) - 1)n + \tau(\pi(\rho(i_0, j_1))) \\ &\leq (\sigma(\pi(\rho(i_0, j_1))) - 1)n + n \\ &\leq (\sigma(\pi(\rho(i_0, j_1))))n \\ &\leq (\sigma(\pi(\rho(i_0, j_2))) - 1)n \\ &< (\sigma(\pi(\rho(i_0, j_2))) - 1)n + \tau(\pi(\rho(i_0, j_2))) \\ &= \pi(\rho(i_0, j_2)). \end{aligned} \tag{28}$$

For any $x \in \mathbf{C}$, by $E^{(m)}_{i_0 i_0} \otimes (E^{(n)}_{j_1 j_1} + xE^{(n)}_{j_1 j_2})$ and $E^{(m)}_{i_0 i_0} \otimes (E^{(n)}_{j_2 j_2} + xE^{(n)}_{j_1 j_2})$ being idempotent matrices in $\mathcal{T}_m \otimes \mathcal{T}_n$, we obtain by (25) that

$$E^{(mn)}_{\pi(\rho(i_0,j_1))\pi(\rho(i_0,j_1))} + x\phi\left(E^{(m)}_{i_0 i_0} \otimes E^{(n)}_{j_1 j_2}\right),$$
$$E^{(mn)}_{\pi(\rho(i_0,j_2))\pi(\rho(i_0,j_2))} + x\phi\left(E^{(m)}_{i_0 i_0} \otimes E^{(n)}_{j_1 j_2}\right) \quad (29)$$

are idempotent matrices in $\mathcal{T}_m \otimes \mathcal{T}_n$; hence, by Lemma 8

$$\phi\left(E^{(m)}_{i_0 i_0} \otimes E^{(n)}_{j_1 j_2}\right) = \lambda E^{(mn)}_{\pi(\rho(i_0,j_1))\pi(\rho(i_0,j_2))}. \quad (30)$$

From (27), by Lemma 4, we obtain that $\lambda = 0$, which is a contradiction to the fact that ϕ is injective. Using a similar method, we may prove (ii) holds. This completes the proof of Step 2.

Note. It is easy to see that ρ is a bijective map from $[1,m] \times [1,n]$ to $[1,mn]$ with $\rho^{-1} : k \mapsto (\sigma(k), \tau(k))$ from $[1,mn]$ to $[1,m] \times [1,n]$. This, together with π, is a permutation on $[1,mn]$; we obtain that

$$\{(\sigma(\pi\rho(i,j)), \tau(\pi\rho(i,j))) : i \in [1,m], j \in [1,n]\}$$
$$= [1,m] \times [1,n]. \quad (31)$$

Let $a_{ij} = (\sigma(\pi\rho(i,j)), \tau(\pi\rho(i,j)))$; then $[a_{ij}]$ forms an $m \times n$ matrix containing all elements of $[1,m] \times [1,n]$. Step 2 implies that every two elements of $[a_{ij}]$ in the same row and column are comparable, respectively. Thus, applying Lemma 5 to $[a_{ij}]$, we conclude that one of (I)–(IV) holds. If (I) holds, then, for any but fixed $i \in [1,m]$, all $(\sigma(\pi\rho(i,j)), \tau(\pi\rho(i,j)))$, $j = 1,\ldots,n$, are in the same row; that is, $\sigma(\pi(\rho(i,j))) = \sigma(\pi(\rho(i,1)))$, $\forall j \in [1,n]$ and $\{\tau(\pi\rho(i,j)) : j = 1,\ldots,n\} = [1,n]$; hence, it follows from (26) that

$$\phi\left(E^{(m)}_{ii} \otimes I_n\right) = E^{(m)}_{\sigma(\pi(\rho(i,1)))\sigma(\pi(\rho(i,1)))} \otimes I_n. \quad (32)$$

Similarly, if (III) holds, then $m = n$ and, for any but fixed $j \in [1,m]$, all $(\sigma(\pi\rho(i,j)), \tau(\pi\rho(i,j)))$, $i = 1,\ldots,m$, are in the same row; that is, $\sigma(\pi\rho(i,j)) = \sigma(\pi\rho(1,j))$, $\forall i \in [1,m]$ and $\{\tau(\pi\rho(i,j)) : i = 1,\ldots,m\} = [1,m]$; hence, it follows from (26) that

$$\phi\left(I_m \otimes E^{(m)}_{jj}\right) = E^{(m)}_{\sigma(\pi(\rho(1,j)))\sigma(\pi(\rho(1,j)))} \otimes I_m. \quad (33)$$

We claim that (II) and (IV) do not hold. Indeed, if (II) holds, for convenience, we assume $m \geq 3$ and we first consider the special case $U = I_m$, $V = I_n$ in (II) (one can use a similar method to prove the case of $n \geq 3$). Thus, by (26), we have

$$\phi\left(E^{(m)}_{11} \otimes E^{(m)}_{11}\right) = E^{(m)}_{mm} \otimes E^{(n)}_{nn},$$
$$\phi\left(E^{(m)}_{mm} \otimes E^{(n)}_{nn}\right) = E^{(m)}_{11} \otimes E^{(m)}_{11},$$
$$\phi\left(E^{(m)}_{ii} \otimes E^{(n)}_{jj}\right) = E^{(m)}_{ii} \otimes E^{(n)}_{jj}, \quad \forall (i,j) \neq (1,1), (m,n). \quad (34)$$

Since, for any $x \in \mathbf{C}$, $(E^{(m)}_{11} + xE^{(m)}_{12}) \otimes E^{(n)}_{11}$, $E^{(m)}_{11} \otimes (E^{(n)}_{11} + xE^{(n)}_{12})$ and $(E^{(m)}_{22} + xE^{(m)}_{12}) \otimes E^{(n)}_{11}$, $E^{(m)}_{11} \otimes (E^{(n)}_{22} + xE^{(n)}_{12})$ are idempotent matrices in $\mathcal{T}_m \otimes \mathcal{T}_n$, we obtain by (34) that

$$E^{(m)}_{mm} \otimes E^{(n)}_{nn} + x\phi\left(E^{(m)}_{12} \otimes E^{(n)}_{11}\right),$$
$$E^{(m)}_{mm} \otimes E^{(n)}_{nn} + x\phi\left(E^{(m)}_{11} \otimes E^{(n)}_{12}\right),$$
$$E^{(m)}_{22} \otimes E^{(n)}_{11} + x\phi\left(E^{(m)}_{12} \otimes E^{(n)}_{11}\right),$$
$$E^{(m)}_{11} \otimes E^{(n)}_{22} + x\phi\left(E^{(m)}_{11} \otimes E^{(n)}_{12}\right) \quad (35)$$

are idempotent matrices in $\mathcal{T}_m \otimes \mathcal{T}_n$. This, together with Lemma 8, implies

$$\phi\left(E^{(m)}_{12} \otimes E^{(n)}_{11}\right) = \lambda E^{(mn)}_{(n+1),mn} \neq 0,$$
$$\phi\left(E^{(m)}_{11} \otimes E^{(n)}_{12}\right) = \mu E^{(mn)}_{2,mn} \neq 0. \quad (36)$$

For any $x \in \mathbf{C}$, $(E^{(m)}_{11} + xE^{(m)}_{12}) \otimes (E^{(n)}_{11} + E^{(n)}_{12})$ is an idempotent matrix in $\mathcal{T}_m \otimes \mathcal{T}_n$. Thus, by (34) and (36)

$$\phi\left(E^{(m)}_{12} \otimes E^{(n)}_{12}\right) = \Sigma_k \beta_k E^{(mn)}_{k,mn}. \quad (37)$$

Similarly, since for any $x \in \mathbf{C}$, $(E^{(m)}_{22} + xE^{(m)}_{12}) \otimes E^{(n)}_{22}$, $(E^{(m)}_{11} + xE^{(m)}_{12}) \otimes E^{(n)}_{22}$ and $E^{(m)}_{22} \otimes (E^{(n)}_{22} + xE^{(n)}_{12})$, $E^{(m)}_{22} \otimes (E^{(n)}_{11} + xE^{(n)}_{12})$ are idempotent matrices in $\mathcal{T}_m \otimes \mathcal{T}_n$, we have

$$\phi\left(E^{(m)}_{12} \otimes E^{(n)}_{22}\right) = \lambda' E^{(mn)}_{2,(n+2)} \neq 0,$$
$$\phi\left(E^{(m)}_{22} \otimes E^{(n)}_{12}\right) = \mu' E^{(mn)}_{(n+1),(n+2)} \neq 0. \quad (38)$$

For any $x \in \mathbf{C}$, $(E^{(m)}_{22} + E^{(m)}_{12}) \otimes (E^{(n)}_{22} + E^{(n)}_{12})$ is an idempotent matrix in $\mathcal{T}_m \otimes \mathcal{T}_n$; we obtain

$$\phi\left(E^{(m)}_{12} \otimes E^{(n)}_{12}\right) = \Sigma_k \beta'_k E^{(mn)}_{k,(n+2)}. \quad (39)$$

It follows from (37) and (39) that $\phi(E^{(m)}_{12} \otimes E^{(n)}_{12}) = 0$, which is a contradiction to that ϕ is injective.

For general case, by (II), we can choose a permutation $p_1,\ldots,p_m$ of $[1,m]$ and a permutation $q_1,\ldots,q_n$ of $[1,n]$ such that

$$(\sigma(\pi(\rho(p_1,q_1))), \tau(\pi(\rho(p_1,q_1)))) = (m,n),$$
$$(\sigma(\pi(\rho(p_m,q_n))), \tau(\pi(\rho(p_m,q_n)))) = (1,1),$$
$$\left(\sigma\left(\pi\left(\rho\left(p_i,q_j\right)\right)\right), \tau\left(\pi\left(\rho\left(p_i,q_j\right)\right)\right)\right) = (i,j),$$
$$\forall (i,j) \neq (1,1), (m,n). \quad (40)$$

From this, together with (26), we obtain

$$\phi\left(E^{(m)}_{p_1 p_1} \otimes E^{(m)}_{q_1 q_1}\right) = E^{(m)}_{mm} \otimes E^{(n)}_{nn},$$
$$\phi\left(E^{(m)}_{p_m p_m} \otimes E^{(n)}_{q_n q_n}\right) = E^{(m)}_{11} \otimes E^{(m)}_{11},$$
$$\phi\left(E^{(m)}_{p_i p_i} \otimes E^{(n)}_{q_j q_j}\right) = E^{(m)}_{ii} \otimes E^{(n)}_{jj}, \quad \forall (i,j) \neq (1,1), (m,n). \quad (41)$$

Using a similar method as the above, we can drive a contradiction. Similarly, we may prove (IV) does not hold.

If (32) holds, we may assume

$$\phi\left(E_{ii}^{(m)}\otimes I_n\right)=E_{g(i)g(i)}^{(m)}\otimes I_n, \tag{42}$$

where g is a permutation on $[1,m]$. If (33) holds, we may assume

$$\phi\left(I_m\otimes E_{ii}^{(m)}\right)=E_{g(i)g(i)}^{(m)}\otimes I_m. \tag{43}$$

We next assume (42) to prove (i) of theorem holds and one can use similar methods to prove (ii) of theorem if (43) holds.

Step 3. There exists an invertible matrix $P\in\mathcal{T}_m\otimes\mathcal{T}_n$ such that

$$\phi\left(E_{ii}^{(m)}\otimes X\right)=P\left(E_{g(i)g(i)}^{(m)}\otimes\xi_i(X)\right)P^{-1},\quad \forall i\in[1,m],\ X\in\mathcal{T}_n, \tag{44}$$

where, for $i\in[1,m]$, $\xi_i(X)=X$ or $\xi_i(X)=JX^tJ$.

Proof of Step 3. For any idempotent matrix $A\in\mathcal{T}_n$, since $E_{ii}^{(m)}\otimes A$ and $E_{ii}^{(m)}\otimes(I_n-A)$ are idempotent matrices in $\mathcal{T}_m\otimes\mathcal{T}_n$, we obtain by (42) that

$$\phi\left(E_{ii}^{(m)}\otimes A\right),\qquad E_{g(i)g(i)}^{(m)}\otimes I_n-\phi\left(E_{ii}^{(m)}\otimes A\right) \tag{45}$$

are idempotent matrices in $\mathcal{T}_m\otimes\mathcal{T}_n$. By Lemma 7, we have

$$\phi\left(E_{ii}^{(m)}\otimes A\right)=E_{g(i)g(i)}^{(m)}\otimes\psi_i(A), \tag{46}$$

where $\psi_i(A)\in\mathcal{T}_n$ is an idempotent matrix. By the arbitrariness of A, we can expand ψ_i to be a linear map on $\mathcal{T}_n$. Hence

$$\phi\left(E_{ii}^{(m)}\otimes X\right)=E_{g(i)g(i)}^{(m)}\otimes\psi_i(X),\quad\forall X\in\mathcal{T}_n. \tag{47}$$

It is easy to see that ψ_i is injective and preserving idempotents. Thus, by Lemma 3, there exists an invertible $P_{g(i)}\in\mathcal{T}_n$ such that $\psi_i(X)=P_{g(i)}\xi_i(X)P_{g(i)}^{-1}$, where $\xi_i(X)=X$ or $\xi_i(X)=JX^tJ$. Let $P=\operatorname{diag}(P_1,\dots,P_m)\in\mathcal{T}_m\otimes\mathcal{T}_n$; we complete the proof of this step.

By Step 3, we may assume

$$\phi\left(E_{ii}^{(m)}\otimes X\right)=E_{g(i)g(i)}^{(m)}\otimes\xi_i(X),\quad\forall i\in[1,m],\ X\in\mathcal{T}_n. \tag{48}$$

Step 4. $g(i)=i$ or $g(i)=m-i+1$.

Proof of Step 4. If $m=2$, then this claim is clear. For $m\geq 3$, we prove that if $i<j<k$, then $g(i)<g(j)<g(k)$ or $g(i)>g(j)>g(k)$. Otherwise, we assume $g(i)<g(k)<g(j)$ (other cases can be proven by using similar methods). Since for any $x\in\mathbf{C}$, $(E_{ii}^{(m)}+xE_{ij}^{(m)})\otimes I_n$ and $(E_{jj}^{(m)}+xE_{ij}^{(m)})\otimes I_n$ are idempotent matrices in $\mathcal{T}_m\otimes\mathcal{T}_n$, we have by using (48) that $E_{g(i)g(i)}^{(m)}\otimes I_n+x\phi(E_{ij}^{(m)}\otimes I_n)$ and $E_{g(j)g(j)}^{(m)}\otimes I_n+x\phi(E_{ij}^{(m)}\otimes I_n)$ are idempotent matrices in $T_m\otimes T_n$. This, together with Lemma 8, implies

$$\phi\left(E_{ij}^{(m)}\otimes I_n\right)=E_{g(i)g(j)}^{(m)}\otimes A\quad\text{for some } A\neq 0\in\mathcal{T}_n. \tag{49}$$

Similarly,

$$\begin{aligned}\phi\left(E_{ik}^{(m)}\otimes I_n\right)&=E_{g(i)g(k)}^{(m)}\otimes B\quad\text{for some } B\neq 0\in\mathcal{T}_n,\\ \phi\left(E_{jk}^{(m)}\otimes I_n\right)&=E_{g(k)g(j)}^{(m)}\otimes C\quad\text{for some } C\neq 0\in\mathcal{T}_n.\end{aligned} \tag{50}$$

By $(E_{jj}^{(m)}+E_{ij}^{(m)}+E_{ik}^{(m)}+E_{jk}^{(m)})\otimes I_n$ being an idempotent matrix in $\mathcal{T}_m\otimes\mathcal{T}_n$, we obtain by using (48), (49), and (50) that

$$E_{g(j)g(j)}^{(m)}\otimes I_n+E_{g(i)g(j)}^{(m)}\otimes A+E_{g(i)g(k)}^{(m)}\otimes B+E_{g(k)g(j)}^{(m)}\otimes C \tag{51}$$

is an idempotent matrix in $\mathcal{T}_m\otimes\mathcal{T}_n$; that is,

$$\begin{bmatrix}0&A&B\\0&0&C\\0&0&I_n\end{bmatrix}^2=\begin{bmatrix}0&A&B\\0&0&C\\0&0&I_n\end{bmatrix}. \tag{52}$$

This implies $A=0$, which is a contradiction. Hence, we complete the proof of Step 4.

Step 5. For any $i<j$, $\xi_i=\xi_j$, and there exists $\lambda_{ij}\neq 0$ such that

$$\phi\left(E_{ij}^{(m)}\otimes X\right)=\lambda_{ij}E_{g(i)g(j)}^{(m)}\otimes\xi_i(X),\quad\forall i\neq j,\ X\in\mathcal{T}_n. \tag{53}$$

Proof of Step 5. We prove the case of $g(i)=i$ (one can use a similar method to prove the case of $g(i)=m-i+1$). Hence,

$$\phi\left(E_{ii}^{(m)}\otimes E_{kk}^{(n)}\right)=E_{ii}^{(m)}\otimes E_{kk}^{(n)},\quad\forall i\in[1,m],\ k\in[1,n]. \tag{54}$$

Without loss of generality, we may assume $i=1$, $j=2$. Since for any $x\in\mathbf{C}$, $(E_{11}^{(m)}+xE_{12}^{(m)})\otimes E_{kk}^{(n)}$ and $(E_{22}^{(m)}+xE_{12}^{(m)})\otimes E_{kk}^{(n)}$ are idempotent matrices in $\mathcal{T}_m\otimes\mathcal{T}_n$, we obtain by (54) that $E_{11}^{(m)}\otimes E_{kk}^{(n)}+x\phi(E_{12}^{(m)}\otimes E_{kk}^{(n)})$ and $E_{22}^{(m)}\otimes E_{kk}^{(n)}+x\phi(E_{12}^{(m)}\otimes E_{kk}^{(n)})$ are idempotent matrices in $T_m\otimes T_n$. This, together with Lemma 8, implies $\phi(E_{12}^{(m)}\otimes E_{kk}^{(n)})=\lambda_k E_{12}^{(m)}\otimes E_{kk}^{(n)}$, where $\lambda_k\neq 0$. Let $\Lambda=\operatorname{diag}(\lambda_1,\dots,\lambda_m)$; then

$$\phi\left(E_{12}^{(m)}\otimes I_n\right)=\begin{bmatrix}0&\Lambda\\0&0\end{bmatrix}\oplus 0. \tag{55}$$

Let $Q=\left[\begin{smallmatrix}I_m&-\Lambda\\0&I_m\end{smallmatrix}\right]\oplus I_{(m-2)n}\in\mathcal{T}_m\otimes\mathcal{T}_n$; by (55), one can obtain

$$\begin{aligned}\phi\left(\left(E_{11}^{(m)}+E_{12}^{(m)}\right)\otimes I_n\right)&=\begin{bmatrix}I_n&\Lambda\\0&0\end{bmatrix}\oplus 0\\&=Q\left(E_{11}^{(m)}\otimes I_n\right)Q^{-1},\\ \phi\left(\left(E_{22}^{(m)}-E_{12}^{(m)}\right)\otimes I_n\right)&=\begin{bmatrix}0&-\Lambda\\0&I_n\end{bmatrix}\oplus 0\\&=Q\left(E_{22}^{(m)}\otimes I_n\right)Q^{-1}.\end{aligned} \tag{56}$$

Let $F_1 = E_{11}^{(m)} + E_{12}^{(m)}$, $F_2 = E_{22}^{(m)} - E_{12}^{(m)}$. Then, (56) turn into

$$\phi\left(F_i \otimes I_n\right) = Q\left(E_{ii}^{(m)} \otimes I_n\right)Q^{-1}, \quad i = 1, 2. \tag{57}$$

By (57), using a similar method to Step 3, one can obtain

$$\phi\left(F_i \otimes X\right) = Q\left(E_{ii}^{(m)} \otimes \zeta_i\left(X\right)\right)Q^{-1}, \quad \forall X \in \mathcal{T}_n, \ i = 1, 2, \tag{58}$$

where $\zeta_i(X) = X$ or $\zeta_i(X) = JX^t J$. Hence

$$\begin{aligned}
\phi\left(F_1 \otimes X\right) &= \begin{bmatrix} I_m & -\Lambda \\ 0 & I_m \end{bmatrix}\begin{bmatrix} \zeta_1(X) & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} I_m & \Lambda \\ 0 & I_m \end{bmatrix} \\
&= \begin{bmatrix} \zeta_1(X) & \zeta_1(X)\Lambda \\ 0 & 0 \end{bmatrix}, \quad \forall X \in \mathcal{T}_n, \\
\phi\left(F_2 \otimes X\right) &= \begin{bmatrix} I_m & -\Lambda \\ 0 & I_m \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & \zeta_2(X) \end{bmatrix}\begin{bmatrix} I_m & \Lambda \\ 0 & I_m \end{bmatrix} \\
&= \begin{bmatrix} 0 & -\Lambda\zeta_2(X) \\ 0 & \zeta_2(X) \end{bmatrix}, \quad \forall X \in \mathcal{T}_n.
\end{aligned} \tag{59}$$

Thus,

$$\begin{aligned}
\phi\left(E_{12}^{(m)} \otimes X\right) &= \phi\left(F_1 \otimes X\right) - \phi\left(E_{11}^{(m)} \otimes X\right) \\
&= \begin{bmatrix} \zeta_1(X) - \xi_1(X) & \zeta_1(X)\Lambda \\ 0 & 0 \end{bmatrix}, \\
\phi\left(E_{12}^{(m)} \otimes X\right) &= \phi\left(E_{22}^{(m)} \otimes X\right) - \phi\left(F_2 \otimes X\right) \\
&= \begin{bmatrix} 0 & \Lambda\zeta_2(X) \\ 0 & \xi_2(X) - \zeta_2(X) \end{bmatrix}.
\end{aligned} \tag{60}$$

This implies

$$\begin{bmatrix} \zeta_1(X) - \xi_1(X) & \zeta_1(X)\Lambda \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \Lambda\zeta_2(X) \\ 0 & \xi_2(X) - \zeta_2(X) \end{bmatrix}. \tag{61}$$

From $\zeta_1(X)\Lambda = \Lambda\zeta_2(X), \forall X \in \mathcal{T}_n$, one can easily see that $\Lambda = \lambda_{12} I_n \neq 0$ and $\zeta_1(X) = \zeta_2(X)$; thus $\xi_1(X) = \xi_2(X)$. This completes the proof of Step 5.

By Step 5, we may assume $\xi = \xi_i$.

Step 6. For $m \geq 3$ and $i < j < k$, we have $\lambda_{ij}\lambda_{jk} = \lambda_{ik}$.

Proof of Step 6. From $(E_{jj}^{(m)} + E_{ij}^{(m)} + E_{jk}^{(m)} + E_{ik}^{(m)}) \otimes I_n$ is an idempotent matrix in $\mathcal{T}_m \otimes \mathcal{T}_n$, we have

$$\left(E_{g(j)g(j)}^{(m)} + \lambda_{ij}E_{g(i)g(j)}^{(m)} + \lambda_{jk}E_{g(j)g(k)}^{(m)} + \lambda_{ik}E_{g(i)g(k)}^{(m)}\right) \otimes I_n \tag{62}$$

is an idempotent matrix in $\mathcal{T}_m \otimes \mathcal{T}_n$. It follows from $g(i) = i$ or $g(i) = m - i + 1$ that $\lambda_{ij}\lambda_{jk} = \lambda_{ik}$.

When $g(i) = i$, let $P = \operatorname{diag}(\lambda_{12}, \ldots, \lambda_{1m})$; then

$$\phi\left(E_{ij}^{(m)} \otimes X\right) = PE_{ij}^{(m)}P^{-1} \otimes \xi\left(X\right), \quad \forall i \leq j, \ X \in \mathcal{T}_n. \tag{63}$$

Hence,

$$\begin{aligned}
\phi\left(A \otimes X\right) &= \left(P \otimes I_n\right)\left(A \otimes \xi\left(X\right)\right)\left(P \otimes I_n\right)^{-1}, \\
&\qquad \forall A \in \mathcal{T}_m, \quad X \in \mathcal{T}_n.
\end{aligned} \tag{64}$$

When $g(i) = m - i + 1$, let $P = \operatorname{diag}(\lambda_{1m}, \ldots, \lambda_{12})$; then

$$\phi\left(E_{ij}^{(m)} \otimes X\right) = PE_{n-i+1,n-j+1}^{(m)}P^{-1} \otimes \xi\left(X\right), \quad \forall i \leq j, \ X \in \mathcal{T}_n. \tag{65}$$

Thus

$$\begin{aligned}
\phi\left(A \otimes X\right) &= \left(P \otimes I_n\right)\left(\left(JA^T J\right) \otimes \xi\left(X\right)\right)\left(P \otimes I_n\right)^{-1}, \\
&\qquad \forall A \in \mathcal{T}_m, \quad X \in \mathcal{T}_n.
\end{aligned} \tag{66}$$

This completes the proof of the theorem. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors thank the referee for his/her careful reading of the paper and valuable comments which greatly improved the readability of the paper. Li Yang is supported by Vocational Education Institute in Heilongjiang Province "12th five-year development plan" and the Guiding Function and Practice Research of Mathematical Modeling in Advanced Mathematics Teaching of New Rise Financial Institutions (Grant no. GG0666). Jinli Xu was supported by the National Natural Science Foundation Grants of China (Grant no. 11171294) and the Natural Science Foundation of Heilongjiang Province of China (Grant no. A201013).

References

[1] C.-K. Li and S. Pierce, "Linear preserver problems," *The American Mathematical Monthly*, vol. 108, no. 7, pp. 591–605, 2001.

[2] L. Molnár, *Selected Preserver Problems on Algebraic Structures of Linear Operators and on Function Spaces*, vol. 1895 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 2007.

[3] G. H. Chan, M. H. Lim, and K.-K. Tan, "Linear preservers on matrices," *Linear Algebra and Its Applications*, vol. 93, pp. 67–80, 1987.

[4] P. Šemrl, "Applying projective geometry to transformations on rank one idempotents," *Journal of Functional Analysis*, vol. 210, no. 1, pp. 248–257, 2004.

[5] X. M. Tang, C. G. Cao, and X. Zhang, "Modular automorphisms preserving idempotence and Jordan isomorphisms of triangular matrices over commutative rings," *Linear Algebra and its Applications*, vol. 338, pp. 145–152, 2001.

[6] N. Jacobson, *Lectures in Abstract Algebra*, vol. 2, D. van Nostrand, Toronto, Canada, 1953.

[7] T. Oikhberg and A. M. Peralta, "Automatic continuity of orthogonality preservers on a non-commutative $L^p(\tau)$ space," *Journal of Functional Analysis*, vol. 264, no. 8, pp. 1848–1872, 2013.

[8] M. A. Chebotar, W.-F. Ke, P.-H. Lee, and N.-C. Wong, "Mappings preserving zero products," *Studia Mathematica*, vol. 155, no. 1, pp. 77–94, 2003.

[9] C.-W. Leung, C.-K. Ng, and N.-C. Wong, "Linear orthogonality preservers of Hilbert C^*-modules over C^*-algebras with real rank zero," *Proceedings of the American Mathematical Society*, vol. 140, no. 9, pp. 3151–3160, 2012.

[10] I. V. Konnov and J. C. Yao, "On the generalized vector variational inequality problem," *Journal of Mathematical Analysis and Applications*, vol. 206, no. 1, pp. 42–58, 1997.

[11] D. Huo, B. Zheng, and H. Liu, "Characterizations of nonlinear Lie derivations of $B(X)$," *Abstract and Applied Analysis*, vol. 2013, Article ID 245452, 7 pages, 2013.

[12] H. Yao and B. Zheng, "Zero triple product determined matrix algebras," *Journal of Applied Mathematics*, vol. 2012, Article ID 925092, 18 pages, 2012.

[13] J. Xu, B. Zheng, and L. Yang, "The automorphism group of the Lie ring of real skew-symmetric matrices," *Abstract and Applied Analysis*, vol. 2013, Article ID 638230, 8 pages, 2013.

[14] X. Song, C. Cao, and B. Zheng, "A note on k-potence preservers on matrix spaces over complex field," *Abstract and Applied Analysis*, vol. 2013, Article ID 581683, 8 pages, 2013.

[15] A. Fosner, Z. Haung, C. K. Li, and N. S. Sze, "Linear preservers and quantum information science," *Linear and Multilinear Algebra*, vol. 61, no. 10, pp. 1377–1390, 2013.

[16] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press, Cambridge, UK, 2000.

[17] J. Xu, B. Zheng, and H. Yao, "Linear transformations between multipartite quantum systems that map the set of tensor product of idempotent matrices into idempotent matrix set," *Journal of Function Spaces and Applications*, vol. 2013, Article ID 182569, 7 pages, 2013.

[18] B. D. Zheng, J. L. Xu, and A. Fosner, "Linear maps preserving rank of tensor products of matrices," *Linear and Multilinear Algebra*. In press.

[19] J. L. Xu, B. D. Zheng, and C. G. Cao, "A characterization of comparable tableaux ," Submitted to *Journal of Combinatorial Theory A*.