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Research Article

Properties of Generalized Offset Curves and Surfaces

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This paper proposes a definition of generalized offsets for curves and surfaces, which have the variable offset distance and direction, by using the local coordinate system. Based on this definition, some analytic properties and theorems of generalized offsets are put forward. The regularity and the topological property of generalized offsets are simply given by representing the generalized offset as the standard offset. Some examples are provided as well to show the applications of generalized offsets. The conclusions in this paper can be taken as the foundation for further study on extending the standard offset.

1. Introduction

Offset curves/surfaces, also called parallel curves/surfaces, are defined as locus of the points which are at constant distance along the normal vector from the generator curves/surfaces. In the field of computer aided geometric design (CAGD), offset curves and surfaces have got considerable attention since they are widely used in various practical applications such as tolerance analysis, geometric optics, and robot path-planning [1, 2]. The study on the offset of curve and surface has been one of the hotpots in CAGD [3].

In some of the engineering applications, we need to extend the concept of standard offset, which has constant distance along the normal vector from the generator such as geodesic offset where constant distance is replaced by geodesic distance (distance measured from a curve on a surface along the geodesic curve drawn orthogonally to the curve) and generalized offset where offset direction is not necessarily along the normal direction. Generalized offset surfaces were first introduced by Brechner [3] and have been extended further, from the differential geometric as well as algebraic points of view, by Pottmann [4]. Arrondo et al. [5] presented a formula for computing the genus of irreducible generalized offset curves to projective irreducible plane curves with only affine ordinary singularities over an algebraically closed field. Lin and Rokne [6] defined the variable-radius generalized offset parametric curves and surfaces. The envelopes of these variable offset parametric curves and surfaces are computed explicitly. J. R. Sendra and J. Sendra [7] presented a complete algebraic analysis of degeneration and the existence of simple and special components of generalized offsets to irreducible hypersurfaces over algebraically closed fields of characteristic zero. A notion of a similarity surface offset was introduced by Georgiev [8] and applied to different constructions of rational generalized offsets. There are also some literatures on generalized offsets which primarily focus on solving some concrete problems [4, 9, 10]. But the general definition, properties, and complete analytic conclusions for generalized offsets have not yet been presented.

Some algebraic properties on standard offsets are known to classical geometers. The study of algebraic and geometric properties on offsets has been an active research area since it arises in practical applications. Farouki and Neff [11, 12] analyzed the basic geometric and topological properties of plane offset curves and provided algorithm to compute the implicit equation. We expect that generalized offsets would have more interesting properties and practical applications. In this paper, a strict definition of generalized offsets, which have the variable offset distance and direction, is given. The offset distance and direction are determined by the local coordinate systems. Though using the local coordinate systems to define a curve is not new [13], the definition of offset curves and surfaces by the local coordinate systems has never been presented before. According to this definition, similar to the standard offsets, we are concerned with the enumeration of certain fundamental geometric and algebraic characteristics for generalized offsets. The relationships between generalized and standard offsets are discussed.

This paper studies the generalized offsets of curves and surfaces in two primary segments. In each segment, we firstly give the definition and regularity of generalized offsets which can be explicitly expressed by the local coordinate systems, secondly we analyze the relationship between generalized and standard offsets, then we discuss some major properties of generalized offsets, and finally some examples are given to illustrate the applications of generalized offsets. The results in this paper will be the foundation for further study on extending the standard offset. Most analytic and topological properties of the generalized offset are addressed in this paper, which provide a series of fundamental conclusions for further study in the related field of generalized offsets.

2. Generalized Offset Curves

2.1. The Definition and Regularity of Generalized Offset Curves. For a planar parametric curve $\mathbf{r} = \mathbf{r}(t)$, the well-known Frenet [14, 15] equations are given as follows:

$$\frac{d\mathbf{e}}{ds} = -k\mathbf{n}, \qquad \frac{d\mathbf{n}}{ds} = k\mathbf{e}, \tag{1}$$

where k is the curvature, s is the arc length such that $ds/dt = |\mathbf{r}'|$, and $\mathbf{e}$ and $\mathbf{n}$ are, respectively, the unit tangent vector and the normal vector at each point of the curve $\mathbf{r}(t)$. For the convenience of representation, we define a unit vector $\mathbf{z}$ such that

$$\mathbf{e} = \frac{\mathbf{r}'}{|\mathbf{r}'|}, \quad \mathbf{z} = \mathbf{n}\times\mathbf{e}, \quad \mathbf{e} = \mathbf{z}\times\mathbf{n}, \quad \mathbf{n} = \mathbf{e}\times\mathbf{z}. \tag{2}$$

So we have

$$k = \frac{(\mathbf{r}'\times\mathbf{r}'')\cdot\mathbf{z}}{|\mathbf{r}'|^3}. \tag{3}$$

Based on the local coordinate system $(\mathbf{e}, \mathbf{n})$ in the plane, a generalized curve offset with the variable offset distance and offset direction can be defined.

Definition 1. For a planar smooth parametric curve $\mathbf{r} = \mathbf{r}(t)$ with the regular parameter $t \in [0,1]$, its generalized offset curve $\mathbf{r}_o(t)$ with the variable offset distance and offset direction is defined by

$$\mathbf{r}_o(t) = \mathbf{r}(t) + d_1(t)\,\mathbf{e} + d_2(t)\,\mathbf{n}, \tag{4}$$

where $d_1(t)$ and $d_2(t)$ are the functions of t.

Thus the offset direction depends on $d_2(t)\mathbf{n}$ and $d_1(t)\mathbf{e}$, and the offset distance is $\sqrt{d_1^2(t)+d_2^2(t)}$. From the above definition, the related parametric derivatives of $\mathbf{r}_o$ can be obtained by

$$\begin{aligned}\mathbf{r}_o' &= \mathbf{r}' + d_1'\cdot\mathbf{e} + d_1\cdot\mathbf{e}' + d_2'\cdot\mathbf{n} + d_2\cdot\mathbf{n}' \\ &= \left(|\mathbf{r}'| + d_1' + d_2\cdot k\cdot|\mathbf{r}'|\right)\mathbf{e} + \left(d_2' - d_1\cdot k\cdot|\mathbf{r}'|\right)\mathbf{n}.\end{aligned} \tag{5}$$

Let $\alpha = |\mathbf{r}'| + d_1' + d_2\cdot k\cdot|\mathbf{r}'|$ and let $\beta = d_2' - d_1\cdot k\cdot|\mathbf{r}'|$; we get

$$\begin{aligned}\mathbf{r}_o' &= \alpha\cdot\mathbf{e} + \beta\cdot\mathbf{n}, \\ \mathbf{r}_o'' &= \alpha'\cdot\mathbf{e} + \alpha\cdot\mathbf{e}' + \beta'\cdot\mathbf{n} + \beta\cdot\mathbf{n}' \\ &= \left(\alpha' + \beta\cdot k\cdot|\mathbf{r}'|\right)\mathbf{e} + \left(\beta' - \alpha\cdot k\cdot|\mathbf{r}'|\right)\mathbf{n}.\end{aligned} \tag{6}$$

Therefore we have

$$\begin{aligned}\mathbf{e}_o &= \frac{\mathbf{r}_o'}{|\mathbf{r}_o'|} = \frac{\alpha\mathbf{e} + \beta\mathbf{n}}{\sqrt{\alpha^2+\beta^2}}, \\ \mathbf{n}_o &= \mathbf{e}_o\times\mathbf{z} = \frac{\alpha(\mathbf{e}\times\mathbf{z}) + \beta(\mathbf{n}\times\mathbf{z})}{\sqrt{\alpha^2+\beta^2}} = \frac{\alpha\mathbf{n} - \beta\mathbf{e}}{\sqrt{\alpha^2+\beta^2}}, \\ k_o &= \frac{(\mathbf{r}_o'\times\mathbf{r}_o'')\cdot\mathbf{z}}{|\mathbf{r}_o'|^3} \\ &= \frac{(\alpha'\beta - \alpha\beta') + (\alpha^2+\beta^2)\cdot k\cdot|\mathbf{r}'|}{(\alpha^2+\beta^2)^{3/2}}.\end{aligned} \tag{7}$$

Regarding the regularity of generalized offset curves, we have the following theorem.

Theorem 2. *If there exists $t_0 \in [0,1]$ to satisfy the equation*

$$d_1'(t_0)\,d_1(t_0) + d_2'(t_0)\,d_2(t_0) + d_1(t_0)\cdot|\mathbf{r}'(t_0)| = 0, \tag{8}$$

and for any arbitrary small positive number δ,

$$k = \begin{cases} -\dfrac{1}{d_2(t_0)}\left[1 + \dfrac{d_1'(t_0)}{|\mathbf{r}'(t_0)|}\right], & \text{when } |d_2| \geq \delta > 0, \\ \dfrac{1}{d_1(t_0)}\left[\dfrac{d_2'(t_0)}{|\mathbf{r}'(t_0)|}\right], & \text{when } |d_1| \geq \delta > 0,\end{cases} \tag{9}$$

then $\mathbf{r}_o(t_0)$ is a nonregular point of the generalized offset curve $\mathbf{r}_o$.

Proof. Let $\mathbf{r}_o(t_0)$ be any non-regular point of the offset curve $\mathbf{r}_o(t)$; then $|\mathbf{r}_o'(t_0)| = \sqrt{\alpha^2+\beta^2} = 0$. We get $\alpha = \beta = 0$. Thus

$$d_2\cdot|\mathbf{r}'|\cdot k + d_1' + |\mathbf{r}'| = 0, \tag{10}$$

$$-d_1\cdot|\mathbf{r}'|\cdot k + d_2' = 0. \tag{11}$$

We discuss the following two cases.

(i) When $|d_2| \geq \delta > 0$, from (10) it follows that

$$k = -\frac{1}{d_2}\left(1 + \frac{d_1'}{|\mathbf{r}'|}\right). \tag{12}$$

Substituting it into (11), we have

$$d_1' d_1 + d_2' d_2 + d_1 \left|\mathbf{r}'\right| = 0. \tag{13}$$

(ii) When $|d_1| \geq \delta > 0$, from (11) it follows that

$$k = \frac{1}{d_1} \cdot \frac{d_2'}{\left|\mathbf{r}'\right|}. \tag{14}$$

Substituting it into (10), we also have

$$d_1' d_1 + d_2' d_2 + d_1 \left|\mathbf{r}'\right| = 0. \tag{15}$$

Hence we prove Theorem 2. □

2.2. Relationship between Generalized and Standard Offset Curves. We will prove that the generalized offset curve can be represented as the standard offset curve.

Theorem 3. *The generalized offset* $\mathbf{r}_o = \mathbf{r} + d_1\mathbf{e} + d_2\mathbf{n}$ *can be represented as a standard offset:* $\mathbf{r}_o = \mathbf{r}_1 + d\mathbf{n}_1$*, where* $\mathbf{r}_1$ *is a new planar smooth parametric curve, d is constant, and* $\mathbf{n}_1$ *is the unit normal vector of* $\mathbf{r}_1$*.*

Proof. Let

$$\begin{aligned} \mathbf{r}_o &= \mathbf{r} + d_1\mathbf{e} + d_2\mathbf{n} \\ &= (\mathbf{r} + A\mathbf{e} + B\mathbf{n}) + (d_1 - A)\,\mathbf{e} + (d_2 - B)\,\mathbf{n} \\ &\triangleq \mathbf{r}_1 + d\mathbf{n}_1, \end{aligned} \tag{16}$$

where A and B are the functions of t,

$$\begin{aligned} d &= \sqrt{(d_1 - A)^2 + (d_2 - B)^2}, \\ \mathbf{n}_1 &= \frac{d_1 - A}{d}\mathbf{e} + \frac{d_2 - B}{d}\mathbf{n}, \\ \mathbf{r}_1' &= \mathbf{r}' + A'\mathbf{e} + A\mathbf{e}' + B'\mathbf{n} + B\mathbf{n}' \\ &= \left(\left|\mathbf{r}'\right| + A' + B \cdot k \cdot \left|\mathbf{r}'\right|\right)\mathbf{e} \\ &\quad + \left(B' - A \cdot k \cdot \left|\mathbf{r}'\right|\right)\mathbf{n}. \end{aligned} \tag{17}$$

In order to establish the above relationship, the following two conditions must be satisfied.

(i) The inner product of vectors $d\mathbf{n}_1$ and $\mathbf{r}_1'$ must be zero. That is,

$$\begin{aligned} &(d_1 - A)\left(\left|\mathbf{r}'\right| + A' + B \cdot k \cdot \left|\mathbf{r}'\right|\right) \\ &\quad + (d_2 - B)\left(B' - A \cdot k \cdot \left|\mathbf{r}'\right|\right) = 0. \end{aligned} \tag{18}$$

(ii) $(d_1 - A)^2 + (d_2 - B)^2$ must be constant. That is,

$$(d_1 - A)\left(d_1' - A'\right) + (d_2 - B)\left(d_2' - B'\right) = 0. \tag{19}$$

Our goal is to get the values of A and B by solving the above differential equations.

From (18) and (19), we get

$$\begin{aligned} \left(\left|\mathbf{r}'\right| + d_2 \cdot k \cdot \left|\mathbf{r}'\right| + d_1'\right) A = &\left(d_1 \cdot k \cdot \left|\mathbf{r}'\right| - d_2'\right) B \\ &+ \left(d_1 \left|\mathbf{r}'\right| + d_1' d_1 + d_2' d_2\right). \end{aligned} \tag{20}$$

Note that

$$\begin{aligned} &d_1 \left|\mathbf{r}'\right| + d_1' d_1 + d_2' d_2 \\ &\quad = d_1 \left|\mathbf{r}'\right| + d_1' d_1 + d_2 \left(\beta + d_1 \cdot k \cdot \left|\mathbf{r}'\right|\right) \\ &\quad = d_1\alpha + d_2\beta, \end{aligned} \tag{21}$$

and we have

$$\alpha\,(d_1 - A) + \beta\,(d_2 - B) = 0. \tag{22}$$

Analyzing the following four cases, we can get the values of A and B.

(1) When $\alpha = \beta = 0$, A and B only need to satisfy (19). That is,

$$(d_1 - A)^2 + (d_2 - B)^2 = C_1, \tag{23}$$

where $C_1 > 0$ is an arbitrary constant. Thus one of A and B can be determined arbitrarily. For instance, if $A = g(t)$ is given, then

$$B = d_2 \pm \sqrt{C_1 - (d_1 - g(t))^2}. \tag{24}$$

(2) When $\alpha \neq 0$ and $\beta = 0$, from (22) it follows that $A = d_1$ and $A' = d_1'$. Substituting it into (19), we have $(d_2 - B)(d_2' - B') = 0$. Since $d_2 \neq B$, then $d_2' = B'$. Therefore $B = d_2(t) + C_2$, where C_2 is an arbitrary nonzero constant.

(3) When $\alpha = 0$ and $\beta \neq 0$, from (22) it follows that $B = d_2$ and $B' = d_2'$. Substituting it into (19), we have $(d_1 - A)(d_1' - A') = 0$. Since $d_1 \neq A$, then $d_1' = A'$. Therefore $A = d_1(t) + C_3$, where C_3 is an arbitrary nonzero constant.

(4) When $\alpha \neq 0$ and $\beta \neq 0$, by solving (19) and (22), we get

$$A = d_1 - \frac{\beta C}{\sqrt{\alpha^2 + \beta^2}}, \qquad B = d_2 + \frac{\alpha C}{\sqrt{\alpha^2 + \beta^2}}, \tag{25}$$

where C is an arbitrary constant.

Therefore, in any case there exist two functions A and B to guarantee that the generalized offset $\mathbf{r}_o = \mathbf{r} + d_1\mathbf{e} + d_2\mathbf{n}$ can be expressed as the standard offset $\mathbf{r}_o = \mathbf{r}_1 + d\mathbf{n}_1$, where

$$\begin{aligned}\mathbf{r}_1 &= \mathbf{r} + A\mathbf{e} + B\mathbf{n} \\ &= \mathbf{r} + \left(d_1 - \frac{\beta C}{\sqrt{\alpha^2+\beta^2}}\right)\mathbf{e} + \left(d_2 + \frac{\alpha C}{\sqrt{\alpha^2+\beta^2}}\right)\mathbf{n}, \\ d &= \sqrt{(d_1 - A)^2 + (d_2 - B)^2} = |C|, \\ \mathbf{n}_1 &= \frac{\beta}{\sqrt{\alpha^2+\beta^2}}\mathbf{e} - \frac{\alpha}{\sqrt{\alpha^2+\beta^2}}\mathbf{n}.\end{aligned} \tag{26}$$

That is, we can find $\mathbf{r}_1(t)$ so that $\mathbf{r}_o(t)$ becomes the standard offset of $\mathbf{r}_1(t)$.

So far we have proved that the generalized offset can be represented as the standard offset. Based on the current results of standard offsets, we can continue the research on the regularity and integral properties of generalized offset curves. This theorem also helps us to obtain the simpler and conciser expressions. The following paragraph explains the details. □

2.3. Properties of Generalized Offset Curves. Let $\lambda = |r'| + A' + B\cdot k\cdot|r'|$ and $\omega = B' - A\cdot k\cdot|r'|$. From the expression of standard offsets $\mathbf{r}_o = \mathbf{r}_1 + d\mathbf{n}_1$, where

$$\mathbf{r}_1 = \mathbf{r} + A\mathbf{e} + B\mathbf{n}, \qquad \mathbf{n}_1 = \frac{d_1 - A}{d}\mathbf{e} + \frac{d_2 - B}{d}\mathbf{n}, \tag{27}$$

$d = \text{const}$, and d_1, d_2, A, B are the functions of t, we have

$$\begin{aligned}\mathbf{r}_1' &= \lambda\mathbf{e} + \omega\mathbf{n}, \\ k_1 &= \frac{(\lambda'\omega - \lambda\omega') + (\lambda^2+\omega^2)\cdot k\cdot|\mathbf{r}'|}{(\lambda^2+\omega^2)^{3/2}}, \\ \mathbf{e}_1 &= \frac{\mathbf{r}_1'}{|\mathbf{r}_1'|} = \frac{\lambda\mathbf{e} + \omega\mathbf{n}}{\sqrt{\lambda^2+\omega^2}}, \qquad \mathbf{n}_1 = \frac{\lambda\mathbf{n} - \omega\mathbf{e}}{\sqrt{\lambda^2+\omega^2}}.\end{aligned} \tag{28}$$

Moreover

$$\begin{aligned}\mathbf{r}_o' &= (1 + dk_1)\cdot\sqrt{\lambda^2+\omega^2}\cdot\mathbf{e}_1, \\ k_o &= \frac{1}{|1+dk_1|}k_1, \\ \mathbf{e}_o &= \operatorname{sgn}(1+dk_1)\,\mathbf{e}_1, \\ \mathbf{n}_o &= \operatorname{sgn}(1+dk_1)\,\mathbf{n}_1,\end{aligned} \tag{29}$$

where k_o and k_1 are the curvatures of $\mathbf{r}_o$ and $\mathbf{r}_1$ at each point, respectively. In the above case, there is another expression for the nonregular point of $\mathbf{r}_o$. Since

$$|\mathbf{r}_o'| = |1 + dk_1|\cdot|\mathbf{r}_1'| = |1 + dk_1|\cdot\sqrt{\lambda^2+\omega^2}, \tag{30}$$

$\mathbf{r}(t_0)$ is a nonregular point if $k_1 = -1/d$, and λ, ω are not both zero.

Therefore we can study the properties of the generalized offsets by using the similar approaches as what Farouki and Neff [12] had done for the standard offsets.

(i) Evolute

We construct

$$\mathbf{r}_\varepsilon(t) = \mathbf{r}_1(t) - \rho_1(t)\cdot\mathbf{n}_1, \tag{31}$$

where $\rho_1 \equiv \rho_1(t) = 1/k_1 (k_1 \neq 0)$. At the nonregular point $\mathbf{r}(t_0)$, it follows that $\rho_1 = 1/k_1 = -d$; hence we also have

$$\mathbf{r}_o(\tau) = \mathbf{r}_1(\tau) + d\mathbf{n}_1 = \mathbf{r}_1(\tau) - \rho_1\mathbf{n}_1 = \mathbf{r}_\varepsilon(\tau). \tag{32}$$

On the other hand, we have

$$\mathbf{r}_\varepsilon = \mathbf{r} - \frac{d_1 - A(1 + dk_1)}{dk_1}\mathbf{e} - \frac{d_2 - B(1 + dk_1)}{dk_1}\mathbf{n}. \tag{33}$$

Moreover, from

$$\mathbf{r}_o - \frac{1}{k_o}\mathbf{n}_o = \mathbf{r}_1 - \frac{1}{k_1}\mathbf{n}_1, \tag{34}$$

we can get the relations as follows:

$$\begin{aligned}d_1 + \frac{\beta}{k_o\sqrt{\alpha^2+\beta^2}} &= A + \frac{\omega}{k_1\sqrt{\lambda^2+\omega^2}}, \\ d_2 - \frac{\alpha}{k_o\sqrt{\alpha^2+\beta^2}} &= B - \frac{\lambda}{k_1\sqrt{\lambda^2+\omega^2}}.\end{aligned} \tag{35}$$

(ii) Turning point, inflection, and vertex

Let $\mathbf{r}(t) = (x(t), y(t))^T$; then $\mathbf{r}(t_0)$ is called a turning point [16] if $x'(t_0) = 0$ and $y'(t_0) \neq 0$, or $x'(t_0) \neq 0$ and $y'(t_0) = 0$, and $\mathbf{r}(t_0)$ is called an inflection if $k(t_0) = 0$ and $\mathbf{r}(t_0)$ are called a vertex if $dk(t_0)/ds = 0$.

Theorem 4. *If $k_1 \neq -1/d$, and λ, ω are not both zero, then*

(1) *the turning point, inflection, and vertex on $\mathbf{r}_o(t)$ are, respectively, in one-to-one correspondence to those on $\mathbf{r}_1(t)$;*

(2) *the turning point on $\mathbf{r}_o(t)$ is in one-to-one correspondence to that on $\mathbf{r}(t)$ as $\omega = 0$;*

(3) *the inflection on $\mathbf{r}_o(t)$ is in one-to-one correspondence to that on $\mathbf{r}(t)$ as $\lambda'\omega = \lambda\omega'$.*

Proof. Based upon the following relationships

$$\begin{aligned}\mathbf{e}_o &= \operatorname{sgn}(1+dk_1)\cdot\frac{\lambda\mathbf{e} + \omega\mathbf{n}}{\sqrt{\lambda^2+\omega^2}}, \\ k_o &= \frac{1}{|1+dk_1|}\cdot\frac{(\lambda'\omega - \lambda\omega') + (\lambda^2+\omega^2)\cdot k\cdot|r'|}{(\lambda^2+\omega^2)^{3/2}}, \\ \frac{dk_o}{ds_o} &= (1+dk_1)^{-3}\cdot\frac{dk_1}{ds_1},\end{aligned} \tag{36}$$

we can easily prove Theorem 4. □

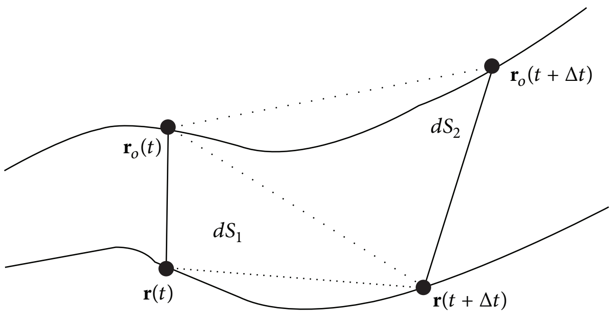

FIGURE 1: The area element between $\mathbf{r}(t)$ and $\mathbf{r}_o(t)$.

(iii) Length and area

We can calculate the lengths l_1 and l_o of the curves $\mathbf{r}_1$ and $\mathbf{r}_o$, respectively. Since $dl_1 = |\mathbf{r}_1'|dt$, then

$$l_1 = \int_0^1 |\mathbf{r}_1'|\,dt = \int_0^1 \sqrt{\lambda^2+\omega^2}dt,$$
$$l_o = \int_0^1 |\mathbf{r}_o'|\,dt = \int_0^1 |1+dk_1|\cdot|\mathbf{r}_1'|\,dt. \quad (37)$$

The area between $\mathbf{r}(t)$ and its generalized offset $\mathbf{r}_o(t)$ is denoted by M (Figure 1). dS_1 and dS_1 are the area elements. See Figure 4.

At first, we compute

$$dS_1 = \frac{|[\mathbf{r}_o(t)-\mathbf{r}(t)]\times[\mathbf{r}(t+\Delta t)-\mathbf{r}(t)]|}{2},$$
$$dS_2 = \frac{|[\mathbf{r}_o(t)-\mathbf{r}_o(t+\Delta t)]\times[\mathbf{r}(t+\Delta t)-\mathbf{r}_o(t+\Delta t)]|}{2}. \quad (38)$$

Since

$$\mathbf{r}(t+\Delta t) = \mathbf{r}(t)+\Delta t\cdot\mathbf{r}'(t)+O(\Delta t^2),$$
$$\mathbf{r}_o(t+\Delta t) = \mathbf{r}_o(t)+\Delta t\cdot\mathbf{r}_o'(t)+O(\Delta t^2), \quad (39)$$

it follows that

$$dS_1 = \frac{\left|[d_1\mathbf{e}+d_2\mathbf{n}]\times[\Delta t\cdot\mathbf{r}'(t)]\right|}{2} = \frac{1}{2}|d_2|\cdot|\mathbf{r}'|\,dt,$$
$$dS_2 = \frac{\left|[\Delta t\cdot\mathbf{r}_o'(t)]\times[\mathbf{r}(t)-\mathbf{r}_o(t)+\Delta t(\mathbf{r}'(t)-\mathbf{r}_o'(t))]\right|}{2}$$
$$= \frac{1}{2}\cdot\Delta t\cdot\left|d_1'd_2-d_2'd_1+d_2\cdot|\mathbf{r}'|+(d_1^2+d_2^2)\cdot k\cdot|\mathbf{r}'|\right|. \quad (40)$$

Therefore $dM = dS_1 + dS_2$, and

$$M = \int_0^1 dM = \frac{1}{2}\left[\int_0^1 |d_2|\cdot|\mathbf{r}'|\,dt + \int_0^1 \left|d_1'd_2-d_2'd_1+d_2\cdot|\mathbf{r}'| + (d_1^2+d_2^2)\cdot k\cdot|\mathbf{r}'|\right|dt\right]. \quad (41)$$

(iv) Topological property

The distance between a regular curve $\mathbf{r}_1 = \mathbf{r}_1(t)$, $t\in[0,1]$ and a point Q in the same plane is defined as follows:

$$\delta(Q,\mathbf{r}_1) = \inf_{t\in[0,1]}|Q-\mathbf{r}_1(t)|. \quad (42)$$

For the standard offset $\mathbf{r}_o = \mathbf{r}_1 + d\mathbf{n}_1$, we have the following theorm.

Theorem 5. *The distance $\delta(\mathbf{r}_o(\tau),\mathbf{r})$ between the point $\mathbf{r}_o(\tau)$ of the generalized offset and the curve $\mathbf{r}=\mathbf{r}(t)$, $t\in[0,1]$ satisfies one of the following conditions:*

$$\delta(\mathbf{r}_o(\tau),\mathbf{r}) = |d| + \sqrt{A^2+B^2},\quad \tau\in(i_k,i_{k+1}),$$
$$\delta(\mathbf{r}_o(\tau),\mathbf{r}) < |d| + \sqrt{A^2+B^2},\quad \tau\in(i_k,i_{k+1}),$$
$$k=0,\ldots,N,\quad N\in Z^+. \quad (43)$$

Each of the open intervals (i_k,i_{k+1}) and $k = 0,\ldots,N.$ is delineated by the self-intersections.

Proof. We have the following:

(1) $\delta(\mathbf{r}_o(\tau),\mathbf{r}_1)\le|d|$, $\tau\in[0,1]$;

(2) $i_0 = 0$, $i_{N+1} = 1$, $i_1,\ldots,i_N \in (0,1)$, $N\in Z^+$ are the self-intersections of $\mathbf{r}_o$.

Then one of the following propositions holds

$$\delta(\mathbf{r}_o(\tau),\mathbf{r}_1) \equiv |d|,\quad \tau\in(i_k,i_{k+1}),$$
$$\delta(\mathbf{r}_o(\tau),\mathbf{r}_1) < |d|,\quad \tau\in(i_k,i_{k+1}),$$
$$k=0,\ldots,N \quad (44)$$

For the generalized offset $\mathbf{r}_0(t)$, we have

$$|\mathbf{r}_1(t)-\mathbf{r}(t)| = \sqrt{A^2+B^2},\quad t\in[0,1]. \quad (45)$$

Considering $\tau\in(i_k,i_{k+1})$, let $p\in\mathbf{r}_1$ such that

$$|\mathbf{r}_o(\tau)-p| = \delta(\mathbf{r}_o(\tau),\mathbf{r}_1), \quad (46)$$

and $q\in\mathbf{r}=\mathbf{r}(t)$, $t\in[0,1]$ is the point with the same parameter t of p on $\mathbf{r}$; then we have

$$|p-q| = \sqrt{A^2+B^2}. \quad (47)$$

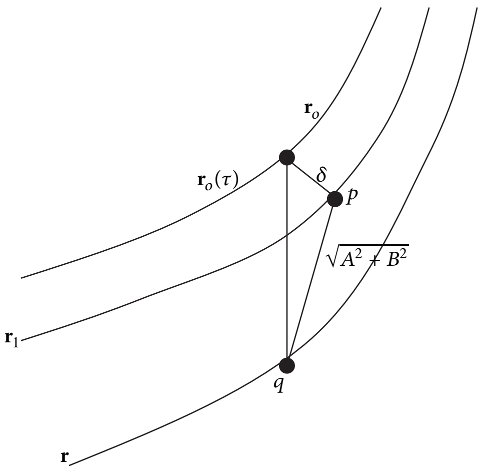

FIGURE 2: Topological property of the curves.

Thus

$$\delta(\mathbf{r}_o(\tau), \mathbf{r}) \leq |\mathbf{r}_o(\tau) - \mathbf{r}| \leq \delta(\mathbf{r}_o(\tau), \mathbf{r}_1) + \sqrt{A^2+B^2}. \quad (48)$$

Therefore we prove Theorem 5, which is shown in Figure 2. □

According to Theorem 5, each of the segments $\{\mathbf{r}_o(t), t \in (i_k, i_{k+1})\}$ of the offset curve among its self-intersections should either be retained or rejected in its entirety when forming the trimmed offset.

2.4. Remark. The curves in three-dimensional space can also be discussed analogously. As we know, a curve is not planar if and only if the torsion of the curve is not zero. Therefore, different from a planar parametric curve, the Frenet equations for a spatial parametric curve $\mathbf{r}(t)$ are

$$\frac{d\mathbf{e}}{ds} = -k\mathbf{n}, \qquad \frac{d\mathbf{n}}{ds} = k\mathbf{e} + \tau\mathbf{z}, \qquad \frac{d\mathbf{z}}{ds} = -\tau\mathbf{n}, \quad (49)$$

where τ is the torsion, and we have

$$\tau = \frac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2}. \quad (50)$$

Based on the local coordinate system $(\mathbf{e}, \mathbf{n}, \mathbf{z})$ in the space, a generalized curve offset with the variable offset distance and offset direction can be defined. The properties of generalized offset curves can be given similarly. Since a torsion item is added in the Frenet equations, the calculations may become more complicated and the conclusions may not be expressed simply.

3. Generalized Offset Surfaces

3.1. The Definition and Regularity of Generalized Offset Surfaces. Note that the symbols used in Section 3 are all redefined.

For a regular parameter surface $\mathbf{r}(u,v) = (x(u,v), y(u,v), z(u,v))$, its two unit tangent vectors in the directions of u and v and its unit normal vector are given by [17]

$$\mathbf{e}_1 = \frac{\mathbf{r}_u(u,v)}{|\mathbf{r}_u(u,v)|}, \qquad \mathbf{e}_2 = \frac{\mathbf{r}_v(u,v)}{|\mathbf{r}_v(u,v)|},$$
$$\mathbf{n} = \frac{\mathbf{r}_u(u,v) \times \mathbf{r}_v(u,v)}{|\mathbf{r}_u(u,v) \times \mathbf{r}_v(u,v)|}, \quad (51)$$

where $\mathbf{r}_u(u,v)$ and $\mathbf{r}_v(u,v)$ are the corresponding partial derivatives of $\mathbf{r}(u,v)$ about parameters u and v. $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{n})$ forms a right-handed system. Based on the local natural coordinate system $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{n})$ of surface $\mathbf{r}(u,v)$, a generalized surface offset $\mathbf{r}^o(u,v)$ with the variable offset direction and the variable offset distance can be defined.

Definition 6. For a regular smooth parametric surface $\mathbf{r}(u,v), (u,v) \in [0,1] \times [0,1]$, the generalized offset surface $\mathbf{r}^o(u,v)$ with the variable offset distance and offset direction is defined by

$$\mathbf{r}^o(u,v) = \mathbf{r}(u,v) + d_1(u,v)\mathbf{e}_1 + d_2(u,v)\mathbf{e}_2 + d_3(u,v)\mathbf{n}, \quad (52)$$

where $d_1(u,v)$, $d_2(u,v)$, and $d_3(u,v)$ are the functions of the variables u and v. The offset direction and the offset distance are determined by $d_1\mathbf{e}_1$, $d_2\mathbf{e}_2$, and $d_3\mathbf{n}$.

For a regular smooth parametric surface $\mathbf{r}(u,v)$, the well-known first and second fundamental quantities and the Gauss curvature [14] are given as follows:

$$E = \mathbf{r}_u^2 = A_1^2, \qquad G = \mathbf{r}_v^2 = A_2^2, \qquad F = \mathbf{r}_u \cdot \mathbf{r}_v,$$
$$D = n \cdot \mathbf{r}_{uu}, \qquad D' = n \cdot \mathbf{r}_{uv}, \qquad D'' = n \cdot \mathbf{r}_{vv}, \quad (53)$$
$$k = \frac{DD'' - D'^2}{EG - F^2}.$$

Let $\mathbf{e}_1 \times \mathbf{n} = \mathbf{e}_1'$, $\mathbf{e}_2 \times \mathbf{n} = \mathbf{e}_2'$, and the angle between $\mathbf{e}_1$ and $\mathbf{e}_2$ is θ; then

$$\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{n}\sin\theta, \qquad \mathbf{e}_1' \times \mathbf{e}_2' = \mathbf{n}\sin\theta,$$
$$\mathbf{e}_1' \cdot \mathbf{e}_2 = -\sin\theta, \qquad \mathbf{e}_2' \cdot \mathbf{e}_1 = \sin\theta. \quad (54)$$

Thus the related parametric partial derivatives of generalized offset surface $\mathbf{r}^o(u,v)$ can be obtained by

$$\begin{aligned}\mathbf{r}_u^o(u,v) &= \mathbf{r}_u + d_{1u}\mathbf{e}_1 + d_1\mathbf{e}_{1u} + d_{2u}\mathbf{e}_2 \\ &\quad + d_2\mathbf{e}_{2u} + d_{3u}\mathbf{n} + d_3\mathbf{n}_u \\ &= \left[A_1 + d_{1u} + \frac{d_2 A_{1v}}{A_2} - \frac{d_3 A_1}{R_1}\right]\mathbf{e}_1 \\ &\quad + \left[d_{2u} - \frac{d_1 A_{1v}}{A_2}\right]\mathbf{e}_2 + \left[d_{3u} + \frac{d_1 A_1}{R_1}\right]\mathbf{n} \\ &\triangleq B_1\mathbf{e}_1 + B_2\mathbf{e}_2 + B_3\mathbf{n},\end{aligned} \quad (55)$$

$$\begin{aligned}\mathbf{r}_v^o(u,v) &= \mathbf{r}_v + d_{1v}\mathbf{e}_1 + d_1\mathbf{e}_{1v} + d_{2v}\mathbf{e}_2 \\ &\quad + d_2\mathbf{e}_{2v} + d_{3v}\mathbf{n} + d_3\mathbf{n}_v \\ &= \left[d_{1v} - \frac{d_2 A_{2u}}{A_1}\right]\mathbf{e}_1 \\ &\quad + \left[A_2 + \frac{d_1 A_{2u}}{A_1} + d_{2v} - \frac{d_3 A_2}{R_2}\right]\mathbf{e}_2 \\ &\quad + \left[d_{3v} + \frac{d_2 A_2}{R_2}\right]\mathbf{n} \\ &\triangleq C_1\mathbf{e}_1 + C_2\mathbf{e}_2 + C_3\mathbf{n},\end{aligned} \tag{56}$$

where $\mathbf{r}_u, d_{1u}, d_{2u}, A_{2u}, \mathbf{e}_{1u}, \mathbf{e}_{2u}, \mathbf{n}_u$ are the corresponding partial derivatives of $\mathbf{r}, d_1, d_2, A_2, \mathbf{e}_1, \mathbf{e}_2, \mathbf{n}$ with respect to u and $\mathbf{r}_v, d_{1v}, d_{2v}, A_{1v}, \mathbf{e}_{1v}, \mathbf{e}_{2v}, \mathbf{n}_v$ are the corresponding partial derivatives of $\mathbf{r}, d_1, d_2, A_1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{n}$ with respect to v. R_1, R_2 are the radii of principal curvature and k is the Gauss curvature. We can get the following equations:

$$\begin{aligned}\mathbf{r}_{uu}^o(u,v) &= B_{1u}\mathbf{e}_1 + B_1\mathbf{e}_{1u} + B_{2u}\mathbf{e}_2 + B_2\mathbf{e}_{2u} \\ &\quad + B_{3u}\mathbf{n} + B_3\mathbf{n}_u \\ &= \left(B_{1u} + B_2\frac{A_{1v}}{A_2} - B_3\frac{A_1}{R_1}\right)\mathbf{e}_1 \\ &\quad + \left(-B_1\frac{A_{1v}}{A_2} + B_{2u}\right) \\ &\quad + \left(B_1\frac{A_1}{R_1} + B_{3u}\right)\mathbf{n}, \\ \mathbf{r}_{uv}^o(u,v) &= \mathbf{r}_{vu}^o(u,v) \\ &= B_{1v}\mathbf{e}_1 + B_1\mathbf{e}_{1v} + B_{2v}\mathbf{e}_2 + B_2\mathbf{e}_{2v} \\ &\quad + B_{3v}\mathbf{n} + B_3\mathbf{n}_v \\ &= \left(B_{1v} - B_2\frac{A_{2u}}{A_1}\right)\mathbf{e}_1 \\ &\quad + \left(B_1\frac{A_{2u}}{A_1} + B_{2v} - B_3\frac{A_2}{R_2}\right)\mathbf{e}_2 \\ &\quad + \left(B_2\frac{A_2}{R_2} + B_{3v}\right)\mathbf{n}, \\ \mathbf{r}_{vv}^o(u,v) &= C_{1v}\mathbf{e}_1 + C_1\mathbf{e}_{1v} + C_{2v}\mathbf{e}_2 + C_2\mathbf{e}_{2v} \\ &\quad + C_{3v}\mathbf{n} + C_3\mathbf{n}_v \\ &= \left(C_{1v} - C_2\frac{A_{2u}}{A_1}\right)\mathbf{e}_1 \\ &\quad + \left(C_1\frac{A_{2u}}{A_1} + C_{2v} - C_3\frac{A_2}{R_2}\right)\mathbf{e}_2 \\ &\quad + \left(C_2\frac{A_2}{R_2} + C_{3v}\right)\mathbf{n}, \\ E_o &= \mathbf{r}_u^{o2} = B_1^2 + B_2^2 + B_3^2 + 2B_1B_2\cos\theta, \\ G_o &= \mathbf{r}_v^{o2} = C_1^2 + C_2^2 + C_3^2 + 2C_1C_2\cos\theta, \\ F_o &= \mathbf{r}_u^o \times \mathbf{r}_v^o \\ &= B_1C_1 + B_2C_2 + B_3C_3 + (B_1C_2 + B_2C_1)\cos\theta.\end{aligned} \tag{57}$$

Thus the unit tangent vectors and unit normal vector of surface offsets $\mathbf{r}^o(u,v)$ are given as follows:

$$\begin{aligned}e_1^o &= \frac{\mathbf{r}_u^o(u,v)}{\left|\mathbf{r}_u^o(u,v)\right|} = \frac{B_1\mathbf{e}_1 + B_2\mathbf{e}_2 + B_3\mathbf{n}}{\sqrt{E_o}}, \\ \mathbf{e}_2^o &= \frac{\mathbf{r}_v^o(u,v)}{\left|\mathbf{r}_v^o(u,v)\right|} = \frac{C_1\mathbf{e}_1 + C_2\mathbf{e}_2 + C_3\mathbf{n}}{\sqrt{G_o}}, \\ \mathbf{n}^o &= \frac{\mathbf{r}_u^o \times \mathbf{r}_v^o}{\left|\mathbf{r}_u^o \times \mathbf{r}_v^o\right|} = \frac{\mathbf{r}_u^o \times \mathbf{r}_v^o}{\sqrt{E_oG_o - F_o^2}} \\ &= \frac{B_1C_3 - B_3C_1}{\sqrt{E_oG_o - F_o^2}}\mathbf{e}_1' + \frac{B_2C_3 - B_3C_2}{\sqrt{E_oG_o - F_o^2}}\mathbf{e}_2' \\ &\quad + \frac{(B_1C_2 - B_2C_1)\sin\theta}{\sqrt{E_oG_o - {F_o}^2}}\mathbf{n} \\ &\triangleq M_1\mathbf{e}_1 + M_2\mathbf{e}_2 + M_3\mathbf{n}, \\ D_o &= \mathbf{n}^o\mathbf{r}_{uu}^o = M_1\left(B_{1u} + B_2\frac{A_{1v}}{A_2} - B_3\frac{A_1}{R_1}\right) \\ &\quad + M_2\left(-B_1\frac{A_{1v}}{A_2} + B_{2u}\right) + M_3\left(B_1\frac{A_1}{R_1} + B_{3u}\right) \\ &\quad + \left[M_1\left(-B_1\frac{A_{1v}}{A_2} + B_{2u}\right)\right. \\ &\qquad \left. + M_2\left(B_{1u} + B_2\frac{A_{1v}}{A_2} - B_3\frac{A_1}{R_1}\right)\right]\cos\theta, \\ D_o' &= \mathbf{n}^o\mathbf{r}_{uv}^o = M_1\left(B_{1v} - B_2\frac{A_{2u}}{A_1}\right) \\ &\quad + M_2\left(B_1\frac{A_{2u}}{A_1} + B_{2v} - B_3\frac{A_2}{R_2}\right) + M_3\left(B_2\frac{A_2}{R_2} + B_{3v}\right) \\ &\quad + \left[M_1\left(B_1\frac{A_{2u}}{A_1} + B_{2v} - B_3\frac{A_2}{R_2}\right)\right. \\ &\qquad \left. + M_2\left(B_{1v} - B_2\frac{A_{2u}}{A_1}\right)\right]\cos\theta, \\ D_o'' &= \mathbf{n}^o\mathbf{r}_{vv}^o = M_1\left(C_{1v} - C_2\frac{A_{2u}}{A_1}\right) \\ &\quad + M_2\left(C_1\frac{A_{2u}}{A_1} + C_{2v} - C_3\frac{A_2}{R_2}\right) + M_3\left(C_2\frac{A_2}{R_2} + C_{3v}\right) \\ &\quad + \left[M_1\left(C_1\frac{A_{2u}}{A_1} + C_{2v} - C_3\frac{A_2}{R_2}\right)\right. \\ &\qquad \left. + M_2\left(C_{1v} - C_2\frac{A_{2u}}{A_1}\right)\right]\cos\theta, \\ k_o &= \frac{D_oD_o'' - D_o'^2}{E_oG_o - F_o^2},\end{aligned} \tag{58}$$

where $E_o, G_o, F_o, D_o, D_o', D_o''$, and k_o are, respectively, the basic quantities and Gauss curvature of surface offset $\mathbf{r}^o(u,v)$. Moreover, we can get the tangent plane and normal line at one particular point of surface $\mathbf{r}^o(u,v)$.

Let $\mathbf{r}^o(u_0,v_0)$ be a nonregular point of $\mathbf{r}^o(u,v)$; then

$$\left|\mathbf{r}_u^o(u_0,v_0)\times\mathbf{r}_v^o(u_0,v_0)\right| = 0. \tag{59}$$

Note that

$$\begin{aligned}\mathbf{r}_u^o\times\mathbf{r}_v^o = &\left(B_1C_3 - B_3C_1\right)\mathbf{e}_1' + \left(B_2C_3 - B_3C_2\right)\mathbf{e}_2'\\ &+\left(B_1C_2 - B_2C_1\right)\mathbf{n}\sin\theta\\ &\triangleq \rho_1\mathbf{e}_1' + \rho_2\mathbf{e}_2' + \rho_3\mathbf{n}\sin\theta.\end{aligned} \tag{60}$$

Regarding the regularity of generalized offset surfaces, we have the following theorem.

Theorem 7. *If*

$$\begin{aligned}&\left(\rho_1^2(u_0,v_0) + \rho_2^2(u_0,v_0) + \rho_3^2(u_0,v_0)\right.\\ &\left.+2\left|\rho_1(u_0,v_0)\rho_2(u_0,v_0)\right|\cos\theta\right)^{1/2} = 0,\end{aligned} \tag{61}$$

then $\mathbf{r}^o(u_0,v_0)$ *is a nonregular point of* $\mathbf{r}^o(u,v)$.

From the above explanation, we can easily prove Theorem 7.

In most cases the local natural coordinate system $(\mathbf{e}_1,\mathbf{e}_2,\mathbf{n})$ at each point of a regular parameter surface $\mathbf{r}(u,v)$ is not the orthonormal coordinate system. In order to discuss the relationship between generalized and standard offset surfaces, we need to do some parameter transformation of surface $\mathbf{r}(u,v)$ firstly. According to the theorem [14], for every point at the regular parameter surface $\mathbf{r}(u,v)$, we can find a neighbourhood and a new parameter system $(\tilde{u},\tilde{v})$ to make the new local natural coordinate system $(\mathbf{e}_{\tilde{u}},\mathbf{e}_{\tilde{v}},\mathbf{n})$ be the orthonormal coordinate system. This theorem guarantees the existence of orthonormal parameter curve net on the regular parameter surface. Therefore for any regular parameter surface, we can make the local natural coordinate system be orthonormal by this means. In the following two paragraphs we suppose that the local natural coordinate system $(\mathbf{e}_1,\mathbf{e}_2,\mathbf{n})$ of a regular surface $\mathbf{r}(u,v)$ is the orthonormal coordinate system.

3.2. Relationship between Generalized and Standard Offset Surfaces. We will prove that the generalized offset surface can be represented as the standard offset surface.

Theorem 8. *The generalized offset* $\mathbf{r}^o = \mathbf{r} + d_1\mathbf{e}_1 + d_2\mathbf{e}_2 + d_3\mathbf{n}$ *can be represented as a standard offset:* $\mathbf{r}_o = \mathbf{r}_1 + d\mathbf{n}_1$, *where* $\mathbf{r}_1$ *is a new regular smooth parametric surface, d is constant, and* $\mathbf{n}_1$ *is the unit normal vector of* $\mathbf{r}_1$.

Proof. Let

$$\begin{aligned}\mathbf{r}^o &= \mathbf{r} + d_1\mathbf{e}_1 + d_2\mathbf{e}_2 + d_3\mathbf{n}\\ &= \left(r + M\mathbf{e}_1 + N\mathbf{e}_2 + P\mathbf{n}\right) + \left(d_1 - M\right)\mathbf{e}_1\\ &\quad+\left(d_2 - N\right)\mathbf{e}_2 + \left(d_3 - P\right)\mathbf{n}\\ &\triangleq \mathbf{r}_1 + d\mathbf{n}_1,\end{aligned} \tag{62}$$

where M, N, and P are the functions of u and v,

$$\begin{aligned}d &= \sqrt{\left(d_1 - M\right)^2 + \left(d_2 - N\right)^2 + \left(d_3 - P\right)^2},\\ \mathbf{n}_1 &= \frac{\left(d_1 - M\right)}{d}\mathbf{e}_1 + \frac{\left(d_2 - N\right)}{d}\mathbf{e}_2 + \frac{\left(d_3 - P\right)}{d}.\end{aligned} \tag{63}$$

The parametric partial derivatives of surface $\mathbf{r}_1(u,v)$ are

$$\begin{aligned}\mathbf{r}_{1u}(u,v) &= \mathbf{r}_u + M_u\mathbf{e}_1 + M\mathbf{e}_{1u} + N_u\mathbf{e}_2\\ &\quad+ N\mathbf{e}_{2u} + P_u\mathbf{n} + P\mathbf{n}_u\\ &= \left[A_1 + M_u + N\frac{A_{1v}}{A_2} - P\frac{A_1}{R_1}\right]\mathbf{e}_1\\ &\quad+\left[N_u - M\frac{A_{1v}}{A_2}\right]\mathbf{e}_2 + \left[P_u + M\frac{A_1}{R_1}\right]\mathbf{n},\end{aligned} \tag{64}$$

$$\begin{aligned}\mathbf{r}_{1v}(u,v) &= \mathbf{r}_v + M_v\mathbf{e}_1 + M\mathbf{e}_{1v} + N_v\mathbf{e}_2\\ &\quad+ N\mathbf{e}_{2v} + P_v\mathbf{n} + P\mathbf{n}_v\\ &= \left[M_v - N\frac{A_{2u}}{A_1}\right]\mathbf{e}_1\\ &\quad+\left[A_2 + M\frac{A_{2u}}{A_1} + N_v - P\frac{A_2}{R_2}\right]\mathbf{e}_2\\ &\quad+\left[P_v + N\frac{A_2}{R_2}\right]\mathbf{n},\end{aligned} \tag{65}$$

where $M_u, M_v, N_u, N_v, P_u, P_v$ are the corresponding partial derivatives.

In order to establish the above relationship, the following two conditions must be satisfied:

(i) $\mathbf{n}_1$ must be the unit normal vector of $\mathbf{r}_1$. That is,

$$\begin{aligned}&\left(A_1 + M_u + N\frac{A_{1v}}{A_2} - P\frac{A_1}{R_1}\right)\left(d_1 - M\right)\\ &\quad+\left(N_u - M\frac{A_{1v}}{A_2}\right)\left(d_2 - N\right)\\ &\quad+\left(P_u + M\frac{A_1}{R_1}\right)\left(d_3 - P\right) = 0,\end{aligned} \tag{66}$$

$$\begin{aligned}&\left(M_v - N\frac{A_{2u}}{A_1}\right)\left(d_1 - M\right)\\ &\quad+\left(A_2 + M\frac{A_{2u}}{A_1} + N_v - P\frac{A_2}{R_2}\right)\left(d_2 - N\right)\\ &\quad+\left(P_v + N\frac{A_2}{R_2}\right)\left(d_3 - P\right) = 0.\end{aligned} \tag{67}$$

(ii) d is constant. That is,

$$(d_1 - M)^2 + (d_2 - N)^2 + (d_3 - P)^2 = \text{const.} \tag{68}$$

It follows that

$$\begin{aligned}(d_1 - M)(d_{1u} - M_u) + (d_2 - N)(d_{2u} - N_u)& \\ + (d_3 - P)(d_{3u} - P_u) = 0,&\end{aligned} \tag{69}$$

$$\begin{aligned}(d_1 - M)(d_{1v} - M_v) + (d_2 - N)(d_{2v} - N_v)& \\ + (d_3 - P)(d_{3v} - P_v) = 0.&\end{aligned} \tag{70}$$

Our goal is to get the values of M, N, and P by solving the above differential equations. From (66) and (69), we get

$$\begin{aligned}&\left(-A_1 - \frac{A_{1v}d_2}{A_2} - d_{1u} + \frac{A_1 d_3}{R_1}\right)M + \left(\frac{A_{1v}d_1}{A_2} - d_{2u}\right)N \\ &\quad + \left(-\frac{A_1 d_1}{R_1} - d_{3u}\right)P \\ &\quad + (A_1 d_1 + d_{1u}d_1 + d_{2u}d_2 + d_{3u}d_3) \\ &\triangleq \alpha_1 M + \beta_1 N + \gamma_1 P + \omega_1 = 0.\end{aligned} \tag{71}$$

From (67) and (70), we get

$$\begin{aligned}&\left(\frac{A_{2u}d_2}{A_1} - d_{1v}\right)M + \left(-\frac{A_{2u}d_1}{A_1} - A_2 - d_{2v} + \frac{A_2 d_3}{R_2}\right)N \\ &\quad + \left(-\frac{A_2 d_2}{R_2} - d_{3v}\right)P \\ &\quad + (A_2 d_2 + d_{1v}d_1 + d_{2v}d_2 + d_{3v}d_3) \\ &\triangleq \alpha_2 M + \beta_2 N + \gamma_2 P + \omega_2 = 0.\end{aligned} \tag{72}$$

Let

$$\begin{aligned}\alpha_1\beta_2 - \alpha_2\beta_1 \triangleq \overline{\alpha}, \qquad \beta_2\gamma_1 - \beta_1\gamma_2 \triangleq \overline{\beta},& \\ \alpha_2\gamma_1 - \alpha_1\gamma_2 \triangleq \overline{\gamma}.&\end{aligned} \tag{73}$$

From (71) and (72), we get

$$\begin{aligned}\overline{\alpha}(d_1 - M) + \overline{\beta}(d_3 - P) = 0,& \\ -\overline{\alpha}(d_2 - N) + \overline{\gamma}(d_3 - P) = 0.&\end{aligned} \tag{74}$$

By solving (69), (70), and (74), we get

$$\begin{aligned}M &= d_1 - \frac{\overline{\beta}\,\overline{C}}{\sqrt{\overline{\alpha}^2 + \overline{\beta}^2 + \overline{\gamma}^2}}, \\ N &= d_2 + \frac{\overline{\gamma}\,\overline{C}}{\sqrt{\overline{\alpha}^2 + \overline{\beta}^2 + \overline{\gamma}^2}}, \\ P &= d_3 + \frac{\overline{\alpha}\,\overline{C}}{\sqrt{\overline{\alpha}^2 + \overline{\beta}^2 + \overline{\gamma}^2}},\end{aligned} \tag{75}$$

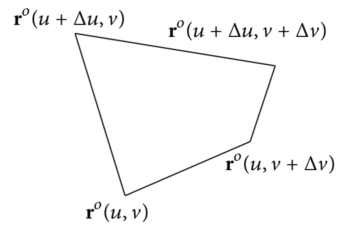

Figure 3: The area element of $\mathbf{r}^o(u, v)$.

where $\overline{C}$ is an arbitrary constant, $\overline{\alpha}$, $\overline{\beta}$, and $\overline{\gamma}$ can not all be zero. Thus

$$\begin{aligned}d &= \sqrt{(d_1 - M)^2 + (d_2 - N)^2 + (d_3 - P)^2} = \overline{C}, \\ \mathbf{n}_1 &= \frac{\overline{\beta}}{\sqrt{\overline{\alpha}^2 + \overline{\beta}^2 + \overline{\gamma}^2}}\mathbf{e}_1 - \frac{\overline{\gamma}}{\sqrt{\overline{\alpha}^2 + \overline{\beta}^2 + \overline{\gamma}^2}}\mathbf{e}_2 \\ &\quad - \frac{\overline{\alpha}}{\sqrt{\overline{\alpha}^2 + \overline{\beta}^2 + \overline{\gamma}^2}}\mathbf{n}.\end{aligned} \tag{76}$$

When $\overline{\alpha} = \overline{\beta} = \overline{\gamma} = 0$, M, N, and P only need to satisfy (69) and (70).

Therefore, in any case there are three functions M, N, and P to guarantee the generalized offset $\mathbf{r}^o = \mathbf{r} + d_1\mathbf{e}_1 + d_2\mathbf{e}_2 + d_3\mathbf{n}$ can be expressed as the standard offset $\mathbf{r}^o(u, v) = \mathbf{r}_1(u, v) + d\mathbf{n}_1(u, v), d = \text{const}$. That is, we can find $\mathbf{r}_1(u, v)$ so that $\mathbf{r}^o(u, v)$ becomes the standard offset of $\mathbf{r}_1(u, v)$.

So far we have proved that the generalized offset can be transformed to the standard offset. Based on the current results of standard offsets, we can continue the research on the properties of generalized offset surfaces. This theorem also helps us to obtain the simpler and conciser expressions. The following paragraph explains the details. □

3.3. Properties of Generalized Offset Surfaces. To study the properties of the generalized offset surfaces, we can use the similar approaches which have been introduced in offset curves. Here we only give the integral and topological properties of generalized offset surfaces.

(i) The area of an offset surface

The area of generalized offset surface

$$\{\mathbf{r}^o(u, v) : (u, v) \in [0,1] \times [0,1]\} \tag{77}$$

is denoted by S. We consider the area element dS, which is shown in Figure 3. Consider the following:

$$dS = \left|(\mathbf{r}_u^o \times \mathbf{r}_v^o)\right| du\, dv. \tag{78}$$

Since

$$(\mathbf{r}_u^o \times \mathbf{r}_v^o)^2 = \mathbf{r}_u^{o2}\mathbf{r}_v^{o2} - (\mathbf{r}_u^o\mathbf{r}_v^o)^2 = E_o G_o - F_o^2, \tag{79}$$

then

$$S = \iint_{[0,1]\times[0,1]} \sqrt{E_o G_o - F_o^2}\, du\, dv. \tag{80}$$

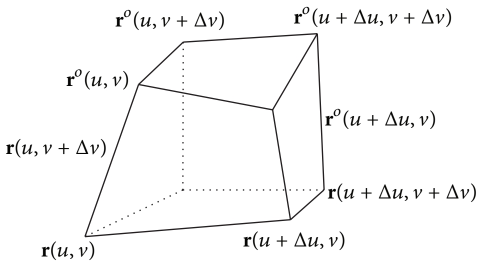

FIGURE 4: The volume element between $\mathbf{r}(u,v)$ and $\mathbf{r}^o(u,v)$.

(ii) The volume between $\mathbf{r}(u,v)$ and $\mathbf{r}^o(u,v)$

The volume between $\mathbf{r}(u,v)$ and its generalized offset $\mathbf{r}^o(u,v)$ is denoted by V. We consider a volume element dV, which is shown in Figure 4. The volume element dV can be divided into five subvolumes:

$$dV = dV_1 + dV_2 + \cdots + dV_5, \quad (81)$$

where

$$\begin{aligned} dV_1 &= [\mathbf{r}(u,v), \mathbf{r}^o(u,v), \mathbf{r}(u+\Delta u,v), \mathbf{r}(u,v+\Delta v)], \\ dV_2 &= [\mathbf{r}^o(u+\Delta u,v), \mathbf{r}^o(u,v), \mathbf{r}(u+\Delta u,v), \\ &\qquad \mathbf{r}^o(u+\Delta u,v+\Delta v)], \\ dV_3 &= [\mathbf{r}(u+\Delta u,v+\Delta v), \mathbf{r}(u+\Delta u,v), \\ &\qquad \mathbf{r}^o(u+\Delta u,v+\Delta v), \mathbf{r}(u,v+\Delta v)], \\ dV_4 &= [\mathbf{r}^o(u,v+\Delta v), \mathbf{r}^o(u,v), \\ &\qquad \mathbf{r}(u,v+\Delta v), \mathbf{r}^o(u+\Delta u,v+\Delta v)], \\ dV_5 &= [\mathbf{r}(u,v+\Delta v), \mathbf{r}^o(u,v), \mathbf{r}(u+\Delta u,v), \\ &\qquad \mathbf{r}^o(u+\Delta u,v+\Delta v)], \end{aligned} \quad (82)$$

and the symbol $[b1, b2, b3, b4]$ denotes the volume of tetrahedron with four vertices $b1, b2, b3$, and $b4$.

After several computations, we get

$$\begin{aligned} dV_1 &= \frac{1}{6}|A_1A_2d_3|\,du\,dv \triangleq w_1du\,dv, \\ dV_2 &= \frac{1}{6}|d_1(B_3C_2 - B_2C_3) + d_2(B_1C_3 - B_3C_1) \\ &\qquad + d_3(B_2C_1 - B_1C_2)|\,du\,dv \triangleq w_2du\,dv, \\ dV_3 &= \frac{1}{6}|A_1A_2d_3|\,du\,dv \triangleq w_1du\,dv, \\ dV_4 &= \frac{1}{6}|d_1(B_2C_3 - B_3C_2) + d_2(B_3C_1 - B_1C_3) \\ &\qquad + d_3(B_1C_2 - B_2C_1)|\,du\,dv \triangleq w_2du\,dv, \\ dV_5 &= \frac{1}{6}|A_1C_3d_2 - A_1C_2d_3 + A_2B_3d_1 \\ &\qquad - A_2B_1d_3|\,du\,dv \triangleq w_5du\,dv. \end{aligned} \quad (83)$$

Therefore, we have

$$\begin{aligned} V &= \iint_0^1 (dV_1 + dV_2 + \cdots + dV_5) \\ &= \iint_{[0,1]\times[0,1]} (2w_1 + 2w_2 + w_5)\,du\,dv. \end{aligned} \quad (84)$$

(ii) Topological property
For a regular parameter surface

$$L_1 = \{\mathbf{r}_1(u,v) : (u,v) \in [0,1]\times[0,1]\}, \quad (85)$$

the distance between a point Q and the surface L_1 is defined as follows:

$$\delta(Q, L_1) = \inf_{(u,v)\in[0,1]\times[0,1]} |Q - \mathbf{r}_1(u,v)|. \quad (86)$$

For the standard offset $\mathbf{r}^o = \mathbf{r}_1 + d\mathbf{n}_1$, $d = \text{const}$, we have the following theorem.

Theorem 9. *The distance $\delta(\mathbf{r}^o(\tau,\eta), C)$ between the point $\mathbf{r}^o(\tau,\eta)$ of the generalized offset and the surface $L = \{\mathbf{r}(u,v) : (u,v) \in [0,1]\times[0,1]\}$ satisfies one of the following conditions:*

$$\begin{aligned} &\delta(\mathbf{r}^o(\tau,\eta), L) = |d| + \sqrt{M^2 + N^2 + P^2}, \\ &\delta(\mathbf{r}^o(\tau,\eta), L) < |d| + \sqrt{M^2 + N^2 + P^2}, \\ &\qquad (\tau,\eta) \in (i_k, i_{k+1}) \times (j_{k'}, j_{k'+1}), \\ &\qquad k, k' = 0, 1, \ldots, \overline{N}, \quad \overline{N} \in Z^+. \end{aligned} \quad (87)$$

Each of the open fields $(i_k, i_{k+1}) \times (j_{k'}, j_{k'+1})$, $k, k' = 0, 1, \ldots, \overline{N}$ is delineated by the self-intersections.

Proof. We have the following:

(1) $\delta(\mathbf{r}^o(\tau,\eta), L_1) \le |d|$, $(\tau,\eta) \in [0,1]\times[0,1]$;

(2) $(i_0, j_0) = (0,0)$, $(i_0, j_{\overline{N}+1}) = (0,1)$, $(i_{\overline{N}+1}, j_0) = (1,0)$, $(i_{\overline{N}+1}, j_{\overline{N}+1}) = (1,1)$, $i_1, \ldots, i_{\overline{N}}, j_1, \ldots, j_{\overline{N}} \in (0,1)$, $\overline{N} \in Z^+$ are the self-intersections of $\mathbf{r}^o$.

Then one of the following propositions holds

$$\begin{aligned} &\delta(\mathbf{r}^o(\tau,\eta), L_1) = |d|, \quad (\tau,\eta) \in (i_k, i_{k+1}) \times (j_{k'}, j_{k'+1}), \\ &\delta(\mathbf{r}^o(\tau,\eta), L_1) < |d|, \quad (\tau,\eta) \in (i_k, i_{k+1}) \times (j_{k'}, j_{k'+1}), \\ &\qquad k, k' = 0, 1, \ldots, \overline{N}. \end{aligned} \quad (88)$$

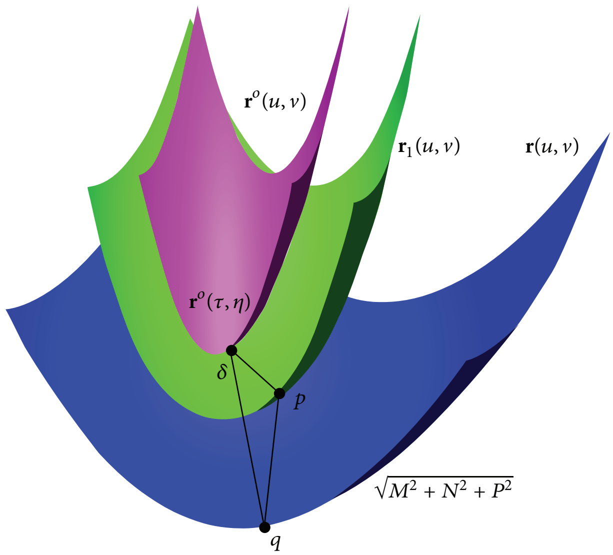

FIGURE 5: Topological property of the surfaces.

For the generalized offset

$$\mathbf{r}^o = \mathbf{r} + d_1\mathbf{e}_1 + d_2\mathbf{e}_2 + d_3\mathbf{n} = \mathbf{r}_1 + d\mathbf{n}_1, \\ \mathbf{r}_1 = \mathbf{r} + M\mathbf{e}_1 + N\mathbf{e}_2 + P\mathbf{n}, \tag{89}$$

we have

$$\mathbf{r}_1(u,v) - \mathbf{r}(u,v) = M\mathbf{e}_1 + N\mathbf{e}_2 + P\mathbf{n}, \\ \left|\mathbf{r}_1(u,v) - \mathbf{r}(u,v)\right| = \sqrt{M^2+N^2+P^2}, \\ (u,v) \in [0,1]\times[0,1]. \tag{90}$$

Considering $(\tau,\eta) \in (i_k, i_{k+1}) \times (j_{k'}, j_{k'+1})$, let $p \in L_1$ such that

$$\left|\mathbf{r}^o(\tau,\eta) - p\right| = \delta\left(\mathbf{r}^o(\tau,\eta), L_1\right), \tag{91}$$

and $q \in L = \{\mathbf{r}(u,v) : (u,v) \in [0,1]\times[0,1]\}$ is the point with the same parameter (u,v) of p on L; then we have

$$\left|p-q\right| = \sqrt{M^2+N^2+P^2}. \tag{92}$$

Thus

$$\delta\left(\mathbf{r}^o(\tau,\eta), L\right) \le \left|\mathbf{r}^o(\tau,\eta) - L\right| \\ \le \delta\left(\mathbf{r}^o(\tau,\eta), L_1\right) + \sqrt{M^2+N^2+P^2}. \tag{93}$$

Therefore we prove the above theorem, which is illustrated in Figure 5. □

According to Theorem 9, each of the segments $\{\mathbf{r}^0(s,t) : (s,t) \in (i_k, i_{k+1}) \times (j_{k'}, j_{k'+1})\}$ of the offset surface among its self-intersections should either be retained or rejected in its entirety when forming the trimmed offset.

4. Applications

Offset for curves and surfaces plays an important role in CAGD. It can be widely used in varieties of applications [18]. Generalized offsets are the extending of standard offsets, which have more flexible properties. By using programming language VC++, some simple examples are given in the following to show how to use generalized offset technique.

Example 10. Generalized offset curves.

In computer aided plane flower design, let a circle parameter curve be the original curve. We choose the union normal direction or deviate a certain angle from the normal direction as the offset direction, and some trigonometric functions as the offset distance. For example, the original circle parameter curve is

$$\mathbf{r}(t) = \{\cos(t), \sin(t)\}, \quad t \in [0, 2\pi]. \tag{94}$$

Then the unit tangent vector at each point of the curve $\mathbf{r}(t)$ is $\mathbf{e}(t) = \{-\sin(t), \cos(t)\}$, and the unit normal vector at each point of the curve $\mathbf{r}(t)$ is $\mathbf{n}(t) = \{\cos(t), \sin(t)\}$. According to the definition of generalized offset curve, we get the new curve

$$\mathbf{r}_o(t) = \mathbf{r}(t) + d_1(t)\,\mathbf{e}(t) + d_2\mathbf{n}(t), \tag{95}$$

where we take $d_1 = |\sin(2t)|$ and $d_1 = |8\cdot\cos(2t)|$.

Thus the generalized offset $\mathbf{r}_o(t)$ is

$$\mathbf{r}_o(t) = \mathbf{r}_1(t) + d\mathbf{n}_1(t) \\ = \{\cos(t) - \sin(t)\cdot|\sin(2t)| \\ + \cos(t)\cdot|8\cdot\cos(2t)|, \\ \sin(t) + \cos(t)\cdot|\sin(2t)| \\ + \sin(t)\cdot|8\cdot\cos(2t)|\} \tag{96}$$

which is shown in Figure 6. By this means, we can get the plane flowers, which have all kinds of beautiful shapes. Pasting them on cloth after machining, we obtain the following pattern shown in Figure 7.

Example 11. Generalized offset surfaces.

The generalized offset surfaces can be widely used in 3D modeling, and more complicated 3D shapes could be defined by using dynamic offset direction and distance. Let a sphere be the original surface. Similar to the plane flower design, we choose the union normal direction or the direction deviating a fixed angle from the normal direction at each point of the sphere as the offset direction, respectively, and some trigonometric functions as the offset distance. We can get the following two models shown in Figures 8 and 9.

We have introduced some applications of generalized offset in 2D graphic design and 3D modeling. Generalized offsets are more flexible since they provide the variable direction and distance. The generalized offset technique is useful especially when the shape design is derived from an

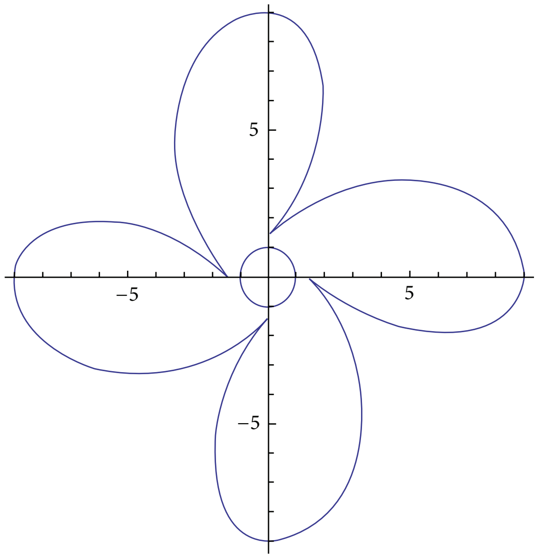

Figure 6: Generalized offset curve.

Figure 7: Flower cloth.

existing graph or a 3D object. Moreover the mathematical expressions of offset curves and surfaces can be simplified by using the concepts and theorems given in this paper.

5. Conclusions

In this paper a strict definition of generalized offsets for curves and surfaces is given. By proving that the generalized offset can be represented as the standard offset, we get a series of conclusions on the properties of generalized offsets. The conclusions given in this paper cover most of the fundamental properties of generalized offsets and can be taken as the foundation for further study on generalized offsets and their application.

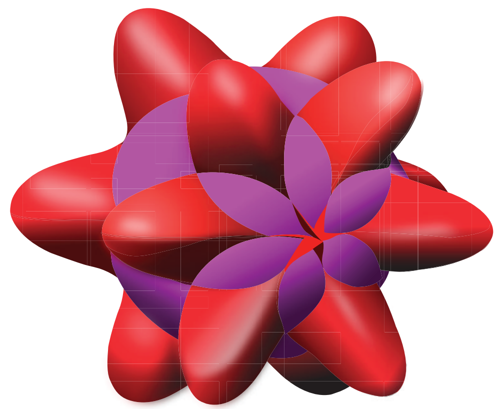

Figure 8: Generalized offset surface with the fixed offset direction.

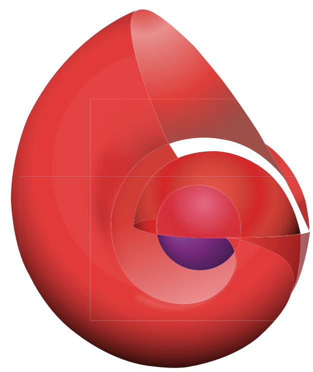

Figure 9: Generalized offset surface with the variable offset direction.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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