Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 123049, 4 pages
http://dx.doi.org/10.1155/2014/123049

Research Article

Algebroid Solutions of Second Order Complex Differential Equations

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Received 28 November 2013; Accepted 17 December 2013; Published 2 January 2014

Academic Editor: Zong-Xuan Chen

Using value distribution theory and maximum modulus principle, the problem of the algebroid solutions of second order algebraic differential equation is investigated. Examples show that our results are sharp.

1. Introduction and Main Results

We use the standard notations and results of the Nevanlinna theory of meromorphic or algebroid functions; see, for example, [1, 2].

In this paper we suppose that second order algebraic differential equation (3) admit at least one nonconstant ν-valued algebroid solution $w(z)$ in the complex plane. We denote by E a subset of $[0,\infty)$ for which $m(E)<\infty$ and by K a positive constant, where $m(E)$ denotes the linear measure of E. E or K does not always mean the same one when they appear in the following.

Let a_{jk} $(j=0,1,\ldots,n;\ k=0,1,\ldots,q_j)$ be entire functions without common zeroes such that $a_{0q_0}\neq 0$. We put

$$Q_j(z,w)=\sum_{k=0}^{q_j}a_{jk}w^k,\quad q_j=\deg_w^{Q_j};\qquad p=\max\left\{q_j+j\colon j=0,1,\ldots,n-1\right\}. \tag{1}$$

Some authors had investigated the problem of the existence of algebroid solutions of complex differential equations, and they obtained many results ([2–10], etc.).

In 1989, Toda [4] considered the existence of algebroid solutions of algebraic differential equation of the form

$$\sum_{j=0}^{n}Q_j(z,w)\left(w'\right)^j=0. \tag{2}$$

He obtained the following.

Theorem A (see [4]). *Let $w(z)$ be a nonconstant ν-valued algebroid solution of the above differential equation and all a_{jk} are polynomials. If $p<n+q_n$, then $w(z)$ is algebraic.*

The purpose of this paper is to investigate algebroid solutions of the following second order differential equation in the complex plane with the aid of the Nevanlinna theory and maximum modulus principle of meromorphic or algebroid functions:

$$\sum_{j=0}^{n}Q_j(z,w)\left(w''\right)^j=0, \tag{3}$$

where $Q_j(z,w)=\sum_{k=0}^{q_j}a_{jk}w^k$, $j=0,1,2,\ldots,n$.

We will prove the following two results.

Theorem 1. *Let $w(z)$ be a nonconstant ν-valued algebroid solution of differential equation (3) and all a_{jk} are polynomials. If $p\leq q_n$, then $w(z)$ is algebraic, $p=\max\{q_j\colon j=0,1,\ldots,n-1\}$.*

Theorem 2. *Let $w(z)$ be a nonconstant ν-valued algebroid solution of differential equation (3) and the orders of all a_{jk} are finite. If $q_0>\max_{1\leq j\leq n-1}\{q_j+j\}$, then the following statements are equivalent:*

(a) $\delta(\infty,w)>0$;

(b) $q_0=q_n+n$;

(c) ∞ *is a Picard exceptional value of* $w(z)$.

2. Some Lemmas

Lemma 3 (see [2]). *Suppose that $w(z)$, $a_i(z)$, $(i = 1, 2, \ldots, p)$ are meromorphic functions, and $a_p(z) \neq 0$. Then one has*

$$m\left(r, \sum_{i=1}^{p} a_i(z) w^i\right) \leq pm(r, w) + \sum_{i=1}^{p} m(r, a_i(z)) + O(1). \tag{4}$$

Examining proof of Lemma 4.5 presented in [2, pp. 192-193], we can verify Lemma 4.

Lemma 4. *Let $w(z)$ be a transcendental algebroid function such that $w(z)$ has only finite number of poles, and let $w(z)$, $w'(z)$, and $w''(z)$ have no poles in $|z| > r_0$. Then, for some constants $C_i > 0$, $i = 1, 2, 3$, and $r \geq r_1 \geq r_0$ it holds:*

$$M(r, w) \leq C_1 + C_2 r + C_3 r^2 M\left(r, w''\right), \tag{5}$$

where $M(r, w) = \max_{|z|=r}\{|w(z)|\}$.

Lemma 5 (see [11]). *The absolute values of roots of equation*

$$z^n + a_1 z^{n-1} + \cdots + a_n = 0 \tag{6}$$

are bounded by

$$\max\left\{n|a_1|, (n|a_2|)^{1/2}, \ldots, (n|a_n|)^{1/n}\right\}. \tag{7}$$

Lemma 6. *Let $w(z)$ be a nonconstant ν-valued algebroid solution of the differential equation (3) and let a_{jk} be a polynomial. If $p < n + q_n$, then*

$$\begin{aligned}\min\{n, q_n - p\} \log^+ M(r, w) &\\ + \max\{0, q_n - p - n\} \log^+ M(r, w) &\\ \leq +O(\log r) + O(1), \quad (r \notin E),&\end{aligned} \tag{8}$$

where $M(r, w) = \max_{|z|=r}\{|w(z)|\}$, K is a positive constant.

Proof. We first prove that the poles of w are contained in the zeroes of $\{a_{nq_n}(z)\}$.

Suppose that z_0 is a pole of w of order τ and z_0 is not the zeroes of $\{a_{nq_n}(z)\}$. Then

$$\begin{aligned}w(z) &= (z - z_0)^{-\tau/\lambda} w_1(z), \quad w_1(z_0) \neq 0, \infty,\\ w''(z) &= (z - z_0)^{-(\tau+2\lambda)/\lambda} w_2(z), \quad w_2(z_0) \neq 0, \infty.\end{aligned} \tag{9}$$

We rewrite differential equation (3) as follows:

$$a_{nq_n}\left(w''\right)^n = \sum_{j=0}^{n-1} Q_j(z, w)\left(w''\right)^j. \tag{10}$$

It follows from (10) that

$$q_n \tau + n(\tau + 2\lambda) \leq p\tau + (n-1)(\tau + 2\lambda). \tag{11}$$

Noting that $p \leq q_n$, we have

$$\tau + 2\lambda < 0. \tag{12}$$

This is a contradiction.

This shows that the poles of w are contained in the zeroes of $\{a_{nq_n}(z)\}$.

We rewrite differential equation (3) as follows:

$$\sum_{j=0}^{n} Q_j(z, w) Q_n(z, w)^{n-j-1}\left(Q_n w''\right)^j = 0. \tag{13}$$

$$\begin{aligned}M\left(r, Q_1 w''\right) &\geq M(r, w)^{q_n} + M\left(r, w''\right)\\ &\quad - \sum_{k=0}^{q_n-1} M\left(r, a_{nk}\right) M(r, w)^k\\ &\geq M(r, w)^{q_n} + \frac{M(r, w) - C_1 - C_2 r}{C_3 r^2}\\ &\quad - \sum_{k=0}^{q_n-1} M\left(r, a_{nk}\right) M(r, w)^k.\end{aligned} \tag{14}$$

For $j = 0, 1, \ldots, n-1$, we have

$$\left|Q_j(z_r, w) Q_n(z_r, w)^{n-j-1}\right| \leq KM(r, w)^{q_j + q_n(n-j-1)}. \tag{15}$$

Applying Lemma 5 to (13) at $z = z_r$,

$$M\left(r, Q_1 w''\right) \leq KM(r, w)^{\max\{h_j:\, j=0,1,\ldots,n-1\}}, \tag{16}$$

where $h_j = (q_j + q_n(n-j-1))/(n-j)$.

From (14) and (15), we have

$$\begin{aligned}M(r, w)^{q_n} &\leq K\left\{M(r, w)^{q_n-1} + r^2 M(r, w)^{\max\{h_j:\, j=0,1,\ldots,n-1\}}\right\}\\ &\leq K\left\{M(r, w)^{q_n-1} + r^2 M(r, w)^{q_n+((p-q_n)/n)}\right\}.\end{aligned} \tag{17}$$

Note that

$$\begin{aligned}h_j &= \frac{q_j + q_n(n-j-1)}{n-j} < \frac{p + q_n(n-j-1) - j}{n-j}\\ &= q_n + \frac{p - q_n}{n} + \frac{j(p - q_n - n)}{n(n-j)}\\ &\leq q_n + \frac{p - q_n}{n}, \quad j = 0, 1, \ldots, n-1.\end{aligned} \tag{18}$$

Dividing the inequality (17) by $M(r, w)^{\max\{q_n-1, q_n+((q_n-p)/n)\}}$, we obtain, for $r \notin E$,

$$M(r, w)^{\min\{1, (q_n-p)/n\}} \leq K_4\left\{1 + \frac{r^2}{M(r, w)^{\max\{0, (q_n-p-n)/n\}}}\right\}, \tag{19}$$

which reduces to our inequality by calculating $\log^+$ of the both sides:

$$\begin{aligned}&\min\{n, q_n - p\} \log^+ M(r, w)\\ &\quad \leq -\max\{0, q_n - p - n\} \log^+ M(r, w) + O(\log r) + O(1).\end{aligned} \tag{20}$$

Lemma 6 is complete. □

3. Proof of Theorem 1

First, we consider $N(r, w)$.

Let z_0 be a pole of w of τ. Let t be the order of zero of $a_{nq_n}(z)$ at z_0.

(i) When the order of the pole of $Q_n(w)(w'')^n$ is not equal to that of other terms of the left-hand side of (10) at z_0, we get

$$q_n\tau + n(\tau + 2\lambda) - t\lambda \leq p\tau; \quad (21)$$

that is,

$$\tau \leq \frac{(t-2n)\lambda}{q_n + n - p}. \quad (22)$$

(ii) When the order of pole of $Q_n(w)(w'')^n$ is equal to that of some term $Q_k(w)(w'')^n$ of the left-hand side of (10) at z_0, we get

$$q_n\tau + n(\tau + 2\lambda) - t\lambda \leq p\tau + n(\tau + 2\lambda); \quad (23)$$

that is,

$$\tau \leq \frac{t\lambda}{q_n - p}. \quad (24)$$

Combining cases (i) and (ii), we obtain

$$N(r, w) \leq K_8 N\left(r, \frac{1}{a_{nq_n}}\right), \quad (25)$$

where K_8 is a positive constant.

Secondly, by Lemma 6, we obtain

$$\min\{n, q_n - p\} m(r, w) \leq K\left[\sum_{i=0}^{p} m(r, a_i) + \sum_{k=0}^{q_n-1} m(r, b_k)\right] + O(\log r), \quad (r \notin E). \quad (26)$$

Combining the inequalities (25) and (26), we have

$$T(r, w) = O(\log r), \quad (r \notin E), \quad (27)$$

which shows that w is an algebraic solution of (3).

This completes the proof of Theorem 1.

4. Proof of Theorem 2

(i) (a) ⇒ (b). Suppose that $\delta(\infty, w) > 0$. If $q_0 > q_n + n$, then we have by (3)

$$w^{q_0} = -\frac{1}{a_{0q_0}}\left\{\sum_{j=1}^{n} Q_j(w) w^j\left(\frac{w''}{w}\right)^j - \sum_{k=0}^{q_0-1} a_{0k}w^k\right\}. \quad (28)$$

Applying Lemma 3 to (28),

$$\begin{aligned} q_0 m(r, w) \leq{}& (q_0 - 1) m(r, w) + \sum_{j,k} m(r, a_{jk}) \\ &+ Km\left(r, \frac{w''}{w}\right) + m\left(r, \frac{1}{a_{0q_0}}\right) + O(1). \end{aligned} \quad (29)$$

Since $w(z)$ is admissible solution, we have

$$m(r, w) = S(r, w), \quad (30)$$

so that

$$\delta(w, \infty) = 0. \quad (31)$$

This is a contradiction. Thus, $q_0 \leq q_n + n$.

If $q_0 < q_n + n$, by Theorem 1, $w(z)$ is nonadmissible. Thus,

$$q_0 = q_n + n. \quad (32)$$

(ii) (b) ⇒ (c). Let $q_0 = q_n + n$. Then, similar to the proof of Lemma 6, we obtain that the poles of $w(z)$ are contained in the set of a_{nq_n} and ∞ is a Picard exceptional value of $w(z)$.

(iii) (c) ⇒ (a). Let ∞ be a Picard exceptional value of $w(z)$. Then $\delta(w, \infty) = 1$.

5. Some Examples

Example 1. The differential equation

$$\nu^2 w^{2\nu-1} w'' - \left[\left((2\nu + 1) w^{4\nu} + 2w^{2\nu} - \nu + 1\right)\right] = 0 \quad (33)$$

has a transcendental algebroid solution $w(z) = (\tan z)^{1/\nu}$. In this case

$$p = q_0 = 4\nu > 1 + q_1 = 2\nu - 1 + 1 = 2\nu. \quad (34)$$

Remark 7. Example 1 shows that the condition in Theorem 1 is sharp.

Example 2. Transcendental algebroid function $w(z) = (\sin z)^{1/2}$ is a 2-valued solution of the following differential equation:

$$16w^6\left(w''\right)^2 + 2w^5 w'' - \left(w^8 - \frac{1}{2}w^6 + 2w^4 - \frac{1}{2}w^2 + 1\right) = 0. \quad (35)$$

In this case

$$q_0 = 8, \qquad n = 2, \qquad q_1 = 5, \qquad q_2 = 6. \quad (36)$$

By Theorem 2, for transcendental algebroid function $w(z) = (\sin z)^{1/2}$, ∞ is a Picard exceptional value.

Remark 8. Example 2 shows that the result in Theorem 2 holds.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This project is project Supported by National Natural Science Foundation (10471065) of China and NSF of Guangdong Province (04010474).

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