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Research Article

Tripled Fixed Point Theorems for Mixed Monotone Kannan Type Contractive Mappings

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We present new results on the existence and uniqueness of tripled fixed points for nonlinear mappings in partially ordered complete metric spaces that extend the results in the previous works: Berinde and Borcut, 2011, Borcut and Berinde, 2012, and Borcut, 2012. An example and an application to support our new results are also included in the paper.

1. Introduction

In some very recent papers, Berinde and Borcut [1], Borcut and Berinde [2], and Borcut [3] have introduced the concepts of *tripled fixed point* and *tripled coincidence point*, respectively, for nonlinear contractive mappings $F : X^3 \to X$ in partially ordered complete metric spaces and obtained existence and uniqueness theorems of tripled fixed points and tripled coincidence points, respectively, for some general classes of contractive type mappings.

The presented theorems in [1–3] extend several existing results in the literature [4–6]. We recall the main concepts needed to present them.

Let $(X, \leq)$ be a partially ordered set and let d be a metric on X such that (X, d) is a complete metric space. Consider on the product space X^3 the following partial order: for $(x, y, z), (u, v, w) \in X^3$,

$$(u, v, w) \leq (x, y, z) \Longleftrightarrow x \geq u,\ \ y \leq v,\ \ z \geq w. \tag{1}$$

Definition 1 (see [1]). Let $(X, \leq)$ be a partially ordered set and $F : X^3 \to X$. We say that F has the mixed monotone property if $F(x, y, z)$ is nondecreasing in x and z and is nonincreasing in y, that is, for any $x, y, z \in X$,

$$\begin{gathered} x_1, x_2 \in X, \quad x_1 \leq x_2 \Longrightarrow F(x_1, y, z) \leq F(x_2, y, z), \\ y_1, y_2 \in X, \quad y_1 \leq y_2 \Longrightarrow F(x, y_1, z) \geq F(x, y_2, z), \\ z_1, z_2 \in X, \quad z_1 \leq z_2 \Longrightarrow F(x, y, z_1) \leq F(x, y, z_2). \end{gathered} \tag{2}$$

Definition 2 (see [1]). An element $(x, y, z) \in X^3$ is called a tripled fixed point of $F : X^3 \to X$ if

$$F(x, y, z) = x, \qquad F(y, x, y) = y, \qquad F(z, y, x) = z. \tag{3}$$

Let (X, d) be a metric space. The mapping $\overline{d} : X^3 \to X$, given by

$$\overline{d}\left[(x, y, z), (u, v, w)\right] = d(x, u) + d(y, v) + d(z, w), \tag{4}$$

defines a metric on X^3, which will be denoted for convenience by d, too.

Definition 3. Let X, Y, Z be nonempty sets and $F : X^3 \to Y$, $G : Y^3 \to Z$. We define the symmetric composition (or, the s-composition, for short) of F and G, by $G * F : X^3 \to Z$,

$$(G * F)(x, y, z) = G(F(x, y, z), F(y, x, y), F(z, y, x)) \quad (x, y, z \in X). \tag{5}$$

For each nonempty set X, denote by P_x the projection mapping

$$P_X : X^3 \longrightarrow X, \quad P(x, y, z) = x \quad \text{for } x, y, z \in X. \tag{6}$$

The symmetric composition has the following properties.

Proposition 4 (associativity). *If $F : X^3 \to Y$, $G : Y^3 \to Z$ and*

$$H : Z \times Z \times Z \longrightarrow W, \quad \text{then } (H * G) * F = H * (G * F). \tag{7}$$

Proposition 5 (identity element). *If $F : X^3 \to Y$, then*

$$F * P_X = P_Y * F = F. \tag{8}$$

Proposition 6 (mixed monotonicity). *If $(X, \leq)$, $(Y, \leq)$, $(Z, \leq)$ are partially ordered sets and the mappings $F : X^3 \to Y$, $G : Y^3 \to Z$ are mixed monotone, then $G * F$ is mixed monotone.*

Proposition 7. *If $(X, \leq)$ is a partially ordered set and F is mixed monotone, then $F^n = F * F^{n-1} = F^{n-1} * F$ is mixed monotone for every n.*

The first main result in [1] is given by the following theorem.

Theorem 8 (see [1]). *Let $(X, \leq)$ be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $F : X^3 \to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exist the constants $j, k, l \in [0, 1)$ with $j+k+l < 1$ for which*

$$d(F(x, y, z), F(u, v, w)) \leq jd(x, u) + kd(y, v) + ld(z, w), \tag{9}$$

for all $x \geq u$, $y \leq v$, $z \geq w$. If there exist x_0, y_0, $z_0 \in X$ such that

$$x_0 \leq F(x_0, y_0, z_0), \qquad y_0 \geq F(y_0, x_0, y_0), \qquad z_0 \leq F(z_0, y_0, x_0), \tag{10}$$

then there exist $x, y, z \in X$ such that

$$x = F(x, y, z), \qquad y = F(y, x, y), \qquad z = F(z, y, x). \tag{11}$$

Remark 9. If we take $j = k = l = \alpha/3$ in Theorem 8, then the contraction condition (9) can be written in a slightly simplified form

$$\begin{aligned} d(F(x, y, z), F(u, v, w)) \\ \leq \frac{\alpha}{3}\left[d(x, u) + d(y, v) + d(z, w)\right]. \end{aligned} \tag{12}$$

Theorem 10 (see [1]). *By adding to the hypotheses of Theorem 8 the condition, for every (x, y, z), $(x_1, y_1, z_1) \in X \times X \times X$, there exists a $(u, v, w) \in X \times X \times X$ that is comparable to (x, y, z) and (x_1, y_1, z_1); then, the tripled fixed point of F is unique.*

Theorem 11 (see [1]). *In addition to the hypotheses of Theorem 8, suppose that x_0, y_0, $z_0 \in X$ are comparable. Then $x = y = z$.*

2. Main Results

Starting from the results presented in the first section, we will obtain new existence and uniqueness theorems for operators which verify a Kannan type contraction condition; see [7].

Denote

$$F_x = F(x, y, z), \qquad F_y = F(y, x, y), \qquad F_z = F(z, y, x). \tag{13}$$

Theorem 12. *Let $(X, \leq)$ be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $F : X^3 \to X$ be a mapping having the mixed monotone property on X. Assume that there exists a $k \in [0, 1)$ such that*

$$\begin{aligned} &d(F(x, y, z), F(u, v, w)) \\ &\quad \leq \frac{k}{8}\left[d(x, F_x) + d(y, F_y) + d(z, F_z) + d(u, F_u)\right. \\ &\qquad \left. + d(v, F_v) + d(w, F_w)\right], \quad \forall x \geq u,\ y \leq v,\ z \geq w. \end{aligned} \tag{14}$$

Also suppose that either

(a) *F is continuous or*

(b) *X has the following property:*

  (i) *if a nondecreasing sequence $\{x_n\} \to x$, then $x_n \leq x$ for all n,*

  (ii) *if a nonincreasing sequence $\{y_n\} \to y$, then $y_n \geq y$ for all n.*

If there exist x_0, y_0, $z_0 \in X$ such that

$$x_0 \leq F(x_0, y_0, z_0), \qquad y_0 \geq F(y_0, x_0, y_0), \qquad z_0 \leq F(z_0, y_0, x_0), \tag{15}$$

then F has a triple fixed point; that is, there exist x, y, $z \in X$ such that

$$x = F(x, y, z), \qquad y = F(y, x, y), \qquad z = F(z, y, x). \tag{16}$$

Proof. Let the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\} \subset X$ be defined by

$$\begin{aligned} x_{n+1} &= F(x_n, y_n, z_n) = F^{n+1}(x_0, y_0, z_0), \\ y_{n+1} &= F(y_n, x_n, y_n) = F^{n+1}(y_0, x_0, y_0), \\ z_{n+1} &= F(z_n, y_n, x_n) = F^{n+1}(z_0, y_0, x_0), \quad (n = 0, 1, \ldots). \end{aligned} \tag{17}$$

Since F^n is mixed monotone for every $n \in \mathbb{N}$, by Proposition 7, it follows by (15) that $\{x_n\}$ and $\{z_n\}$ are nondecreasing and $\{y_n\}$ is nonincreasing. Due to the mixed monotone property of F, it is easy to show that

$$\begin{aligned} x_2 &= F(x_1, y_1, z_1) \geq F(x_0, y_0, z_0) = x_1 \\ y_2 &= F(y_1, x_1, y_1) \leq F(y_0, x_0, y_0) = y_1 \\ z_2 &= F(z_1, y_1, x_1) \geq F(z_0, y_0, x_0) = z_1 \end{aligned} \tag{18}$$

and thus we obtain three sequences satisfying the following conditions:

$$\begin{aligned} &x_0 \leq x_1 \leq \cdots \leq x_n \leq \cdots, \\ &y_0 \geq y_1 \geq \cdots \geq y_n \geq \cdots, \\ &z_0 \leq z_1 \leq \cdots \leq z_n \leq \cdots. \end{aligned} \tag{19}$$

Now, for $n \in \mathbb{N}$, denote

$$\begin{gathered} D_{x_{n+1}} = d\left(x_{n+1}, x_n\right), \qquad D_{y_{n+1}} = d\left(y_{n+1}, y_n\right), \\ D_{z_{n+1}} = d\left(z_{n+1}, z_n\right), \\ D_{n+1} = D_{x_{n+1}} + D_{y_{n+1}} + D_{z_{n+1}}. \end{gathered} \tag{20}$$

Using (14), we get

$$\begin{aligned} D_{x_{n+1}} &= d\left(x_{n+1}, x_n\right) = d\left(F\left(x_n, y_n, z_n\right), F\left(x_{n-1}, y_{n-1}, z_{n-1}\right)\right) \\ &\leq \frac{k}{8}\left[d\left(x_n, F_{x_n}\right) + d\left(y_n, F_{y_n}\right) + d\left(z_n, F_{z_n}\right)\right. \\ &\qquad \left. + d\left(x_{n-1}, F_{x_{n-1}}\right) + d\left(y_{n-1}, F_{y_{n-1}}\right) + d\left(z_{n-1}, F_{z_{n-1}}\right)\right] \\ &= \frac{k}{8}\left[D_{x_n} + D_{y_n} + D_{z_n} + D_{x_{n+1}} + D_{y_{n+1}} + D_{z_{n+1}}\right], \end{aligned} \tag{21}$$

and so

$$D_{x_{n+1}} \leq \frac{k}{8}\left[D_{x_n} + D_{y_n} + D_{z_n} + D_{x_{n+1}} + D_{y_{n+1}} + D_{z_{n+1}}\right]. \tag{22}$$

Similarly, we obtain

$$\begin{aligned} D_{y_{n+1}} &\leq \frac{k}{8}\left[D_{x_n} + 2D_{y_n} + D_{x_{n+1}} + 2D_{y_{n+1}}\right], \\ D_{z_{n+1}} &\leq \frac{k}{8}\left[D_{x_n} + D_{y_n} + D_{z_n} + D_{x_{n+1}} + D_{y_{n+1}} + D_{z_{n+1}}\right]. \end{aligned} \tag{23}$$

By (22) and (23), we get

$$\begin{aligned} D_{n+1} &\leq \frac{k}{8}\left[3D_{x_n} + 4D_{y_n} + 2D_{z_n} + 3D_{x_{n+1}} + 4D_{y_{n+1}} + 2D_{z_{n+1}}\right] \\ &\leq \frac{k}{8}\left[4D_{x_n} + 4D_{y_n} + 4D_{z_n} + 4D_{x_{n+1}} + 4D_{y_{n+1}} + 4D_{z_{n+1}}\right] \\ &\leq \frac{k}{2}\left[D_n + D_{n+1}\right]. \end{aligned} \tag{24}$$

Therefore, for all $n \geq 1$, we have

$$D_{n+1} \leq \alpha \cdot D_n \leq \cdots \leq \alpha^n \cdot D_1, \quad \text{where } \alpha = \frac{k}{2-k} \in [0,1), \quad \text{when } k \in [0,1). \tag{25}$$

Because $D_{x_{n+1}} \leq D_{n+1}$, $D_{y_{n+1}} \leq D_{n+1}$, and $D_{z_{n+1}} \leq D_{n+1}$, we have

$$D_{x_{n+1}} \leq \alpha^n \cdot D_1, \qquad D_{y_{n+1}} \leq \alpha^n \cdot D_1, \qquad D_{z_{n+1}} \leq \alpha^n \cdot D_1. \tag{26}$$

This implies that $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ are Cauchy sequences in X. Indeed, let $m \geq n$; then,

$$\begin{aligned} d\left(x_m, x_n\right) &\leq D_{x_m} + D_{x_{m-1}} + \cdots + D_{x_{n+1}} \\ &\leq \left[\alpha^{m-1} + \alpha^{m-2} + \cdots + \alpha^n\right] \cdot D_1 \\ &= \frac{\alpha^n - \alpha^m}{1-\alpha} \cdot D_1 < \frac{\alpha^n}{1-\alpha} \cdot D_1. \end{aligned} \tag{27}$$

Similarly, we can verify that $\{y_n\}$ and $\{z_n\}$ are also Cauchy sequences.

Since X is a complete metric space, there exist $x, y, z \in X$ such that

$$\lim_{x \to \infty} x_n = x, \qquad \lim_{x \to \infty} y_n = y, \qquad \lim_{x \to \infty} z_n = z. \tag{28}$$

Finally, we claim that

$$x = F\left(x, y, z\right), \qquad y = F\left(y, x, y\right), \qquad z = F\left(z, y, x\right). \tag{29}$$

Suppose first that assumption (a) holds. Hence F is continuous at (x, y, z), and, therefore, for any given $\epsilon/2 > 0$, there exists $\delta > 0$ such that

$$\begin{aligned} d\left(\left(x, y, z\right), \left(u, v, w\right)\right) &= d\left(x, u\right) + d\left(y, v\right) + d\left(z, w\right) \\ &< \delta \Longrightarrow d\left(F\left(x, y, z\right), F\left(u, v, w\right)\right) < \frac{\epsilon}{2}. \end{aligned} \tag{30}$$

Since

$$\lim_{x \to \infty} x_n = x, \qquad \lim_{x \to \infty} y_n = y, \qquad \lim_{x \to \infty} z_n = z, \tag{31}$$

for $\eta = \min(\epsilon/2, \delta/2)$, there exist n_0, m_0, p_0 such that, for $n \geq n_0, m \geq m_0, p \geq p_0$,

$$d\left(x_n, x\right) < \eta, \qquad d\left(y_n, y\right) < \eta, \qquad d\left(z_n, z\right) < \eta. \tag{32}$$

Hence, for $n \in \mathbb{N}$, $n \geq \max\{n_0, m_0, p_0\}$,

$$\begin{aligned} d\left(F\left(x, y, z\right), x\right) &\leq d\left(F\left(x, y, z\right), x_{n+1}\right) + d\left(x_{n+1}, x\right) \\ &= d\left(F\left(x, y, z\right), F\left(x_n, y_n, z_n\right)\right) + d\left(x_{n+1}, x\right) \\ &< \frac{\epsilon}{2} + \eta \leq \epsilon. \end{aligned} \tag{33}$$

This shows that $x = F(x, y, z)$. Similarly, one can show that

$$y = F\left(y, x, y\right), \qquad z = F\left(z, y, x\right). \tag{34}$$

Suppose now that assumption (b) holds. Since $\{x_n\}$, $\{z_n\}$ are nondecreasing and $x_n \to x$, $z_n \to z$, $\{y_n\}$ is nonincreasing and $y_n \to y$, by assumption (b), we have that

$x_n \leq x$, $y_n \geq y$, and $z_n \leq z$, for all n. Then, by triangle inequality and (14), we get

$$\begin{aligned}
&d(x, F(x, y, z)) \\
&\quad \leq d(x, x_{n+1}) + d(x_{n+1}, F(x, y, z)) \\
&\quad = d(x, x_{n+1}) d(F(x_n, y_n, z_n), F(x, y, z)) \\
&\quad \leq d(x, x_{n+1}) + \frac{k}{8}[d(x_n, x_{n+1}) + d(y_n, y_{n+1}) \\
&\qquad + d(z_n, z_{n+1}) + d(x, F(x, y, z)) \\
&\qquad + d(y, F(y, x, y)) + d(z, F(z, y, x))], \\
&d(y, F(y, x, y)) \\
&\quad \leq d(y, y_{n+1}) + \frac{k}{8}[d(x_n, x_{n+1}) + 2d(y_n, y_{n+1}) \\
&\qquad + d(x, F(x, y, z)) + 2d(y, F(y, x, y))], \\
&d(z, F(z, y, x)) \\
&\quad \leq d(z, z_{n+1}) + \frac{k}{8}[d(x_n, x_{n+1}) + d(y_n, y_{n+1}) \\
&\qquad + d(z_n, z_{n+1}) + d(x, F(x, y, z)) \\
&\qquad + d(y, F(y, x, y)) + d(z, F(z, y, x))].
\end{aligned} \tag{35}$$

By summing (35), we obtain

$$\begin{aligned}
&d(x, F(x, y, z)) + d(y, F(y, x, y)) + d(z, F(z, y, x)) \\
&\quad \leq \frac{2}{2-k}[d(x, x_{n+1}) + d(y, y_{n+1}) + d(z, z_{n+1})] \\
&\quad + \frac{k}{4(2-k)}[3d(x_n, x_{n+1}) + 4d(y_n, y_{n+1}) \\
&\qquad + 2d(z_n, z_{n+1})],
\end{aligned} \tag{36}$$

and by letting $n \to \infty$ in the previous inequality, one obtains

$$d(x, F(x, y, z)) + d(y, F(y, x, y)) + d(z, F(z, y, x)) \leq 0, \tag{37}$$

which proves that $x = F(x, y, z)$, $y = F(y, x, y)$, $z = F(z, y, x)$. □

3. Uniqueness of Tripled Fixed Points

In [1–3], the authors also considered some additional conditions that ensure the uniqueness of the tripled fixed point or that, for the tripled fixed point (x, y, z), we have $x = y = z$.

Similarly, one can prove that the tripled fixed point in Theorem 12 is in fact unique, provided that the product space X^3 endowed with the partial order mentioned earlier has an additional property, as shown in the next theorem.

Theorem 13. *By adding to the hypotheses of Theorem 12 the condition, for every (x, y, z), $(x_1, y_1, z_1) \in X \times X \times X$, there exists a $(u, v, w) \in X \times X \times X$ which is comparable to (x, y, z) and (x_1, y_1, z_1); then, the tripled fixed point of F is unique.*

Proof. If $(x^*, y^*, z^*) \in X \times X \times X$ is another tripled fixed point of F, then we show that

$$d((x, y, z), (x^*, y^*, z^*)) = 0, \tag{38}$$

where

$$\lim_{x \to \infty} x_n = x, \qquad \lim_{x \to \infty} y_n = y, \qquad \lim_{x \to \infty} z_n = z, \tag{39}$$

are as in the proof of Theorem 12. We consider two cases.

Case 1. If (x, y, z) are comparable to (x^*, y^*, z^*) with respect to the ordering in X^3, then, for every $n = 0, 1, 2, \ldots$,

$$\begin{aligned}
&(F^n(x, y, z), F^n(y, x, y), F^n(z, y, x)) = (x, y, z) \\
&\text{that is comparable to } (F^n(x^*, y^*, z^*), F^n(y^*, x^*, y^*), \\
&\qquad F^n(z^*, y^*, x^*)) = (x^*, y^*, z^*).
\end{aligned} \tag{40}$$

Also, using the process of obtaining (26), we get

$$\begin{aligned}
&d((x, y, z), (x^*, y^*, z^*)) \\
&\quad = d(x, x^*) + d(y, y^*) + d(z, z^*) \\
&\quad = d(F^n(x, y, z), F^n(x^*, y^*, z^*)) \\
&\qquad + d(F^n(y, x, y), F^n(y^*, x^*, y^*)) \\
&\qquad + d(F^n(z, y, x), F^n(z^*, y^*, x^*)) \\
&\quad \leq \alpha^n[d(x, x^*) + d(y, y^*) + d(z, z^*)] \\
&\quad = \alpha^n d((x, y, z), (y^*, x^*, z^*)), \quad \alpha \in [0, 1).
\end{aligned} \tag{41}$$

Now letting $n \to \infty$, this implies that $d((x, y, z), (y^*, x^*, z^*)) = 0$.

Case 2. If (x, y, z) are not comparable to (x^*, y^*, z^*), then there exists an upper bound or a lower bound $(u, v, w) \in X \times X \times X$ of (x, y, z) and (x^*, y^*, z^*). Then, for all $n = 1, 2, \ldots$,

$$\begin{aligned}
&(F^n(u, v, w), F^n(v, u, v), F^n(w, v, u)) \text{ is comparable to} \\
&(F^n(x, y, z), F^n(y, x, y), F^n(z, y, x)) = (x, y, z) \text{ and to} \\
&(F^n(x^*, y^*, z^*), F^n(y^*, x^*, y^*), F^n(z^*, y^*, x^*)) \\
&\quad = (x^*, y^*, z^*).
\end{aligned} \tag{42}$$

We have

$$
\begin{aligned}
d&\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix},\begin{pmatrix} x^* \\ y^* \\ z^* \end{pmatrix}\right)\\
&= d\left(\begin{pmatrix} F^n(x,y,z) \\ F^n(y,x,y) \\ F^n(z,y,x) \end{pmatrix},\begin{pmatrix} F^n(x^*,y^*,z^*) \\ F^n(y^*,x^*,y^*) \\ F^n(z^*,y^*,x^*) \end{pmatrix}\right)\\
&\leq d\left(\begin{pmatrix} F^n(x,y,z) \\ F^n(y,x,y) \\ F^n(z,y,x) \end{pmatrix},\begin{pmatrix} F^n(u,v,w) \\ F^n(v,u\cdot v) \\ F^n(w,v,u) \end{pmatrix}\right)\\
&\quad + d\left(\begin{pmatrix} F^n(u,v,w) \\ F^n(v,u,v) \\ F^n(w,v,u) \end{pmatrix},\begin{pmatrix} F^n(x^*,y^*,z^*) \\ F^n(y^*,x^*,y^*) \\ F^n(z^*,y^*,x^*) \end{pmatrix}\right)\\
&\leq \alpha^n\{[d(x,u)+d(y,v)+d(z,w)]\\
&\quad + [d(u,x^*)+d(v,y^*)+d(w,z^*)]\} \longrightarrow 0\\
&\quad \text{as } n \longrightarrow \infty, \text{ and so } d\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix},\begin{pmatrix} x^* \\ y^* \\ z^* \end{pmatrix}\right) = 0.
\end{aligned}
\tag{43}
$$

□

Theorem 14. *In addition to the hypotheses of Theorem 12, suppose that x_0, y_0, $z_0 \in X$ are comparable. Then $x = y = z$.*

Proof. Recall that x_0, y_0, z_0, $\in X$ are such that

$$
\begin{gathered}
x_0 \leq F(x_0,y_0,z_0), \qquad y_0 \geq F(y_0,x_0,y_0),\\
z_0 \leq F(z_0,y_0,x_0).
\end{gathered}
\tag{44}
$$

Now, if $x_0 \leq y_0$ and $z_0 \leq y_0$, we claim that, for all $n \in \mathbb{N}$, $x_n \leq y_n$ and $z_n \leq y_n$. Indeed, by the mixed monotone property of F,

$$
\begin{aligned}
x_1 &= F(x_0,y_0,z_0) \leq F(y_0,x_0,y_0) = y_1,\\
z_1 &= F(z_0,y_0,x_0) \leq F(y_0,x_0,y_0) = y_1.
\end{aligned}
\tag{45}
$$

Assume that $x_n \leq y_n$ and $z_n \leq y_n$ for some n. Now, consider

$$
\begin{aligned}
x_{n+1} &= F^{n+1}(x_0,y_0,z_0)\\
&= F(F^n(x_0,y_0,z_0), F^n(y_0,x_0,y_0), F^n(z_0,y_0,x_0))\\
&= F(x_n,y_n,z_n) \leq F(y_n,x_n,y_n) = y_{n+1},
\end{aligned}
\tag{46}
$$

similarly for z_n. Hence,

$$
\begin{aligned}
d(x,y) &\leq d(x,x_{n+1}) + d(y,x_{n+1})\\
&\leq d(x,x_{n+1}) + d(x_{n+1},y_{n+1}) + d(y,y_{n+1})\\
&= d(x,F^{n+1}(x_0,y_0,z_0))\\
&\quad + d[F(F^n(x_0,y_0,z_0), F^n(y_0,x_0,y_0),\\
&\qquad F^n(z_0,y_0,x_0)),\\
&\qquad F(F^n(y_0,x_0,y_0), F^n(x_0,y_0,x_0),\\
&\qquad F^n(y_0,x_0,y_0))] + d(y,y_{n+1}) \longrightarrow 0
\end{aligned}
\tag{47}
$$

as $n \rightarrow \infty$. This implies that $d(x,y) = 0$ and hence we have $x = y$.

Similarly, we obtain that $d(x,z) = 0$ and $d(y,z) = 0$. The other remaining cases for x_0, y_0, z_0 are similar. □

4. An Example

Let $X = [0,1]$ be endowed with the usual metric $d(x,y) = |x-y|$ and let $F : X^3 \rightarrow X$ be given by $F(x,y,z) = 11/80$, for $(x,y,z) \in [0,4/5) \times [0,1]^2$, and $F(x,y,z) = 1/20$, for $(x,y,z) \in [4/5,1] \times [0,1]^2$.

Then F satisfies the Kannan type contractive condition (14) with $k = 14/15 < 1$ but does not satisfy the Banach type contractive condition (9).

Let us first prove the first part of the assertion above. It suffices to completely cover the following limit case.

Case 1 ($x \in [4/5,1]$, u, y, z, v, $w \in [0,4/5)$). In this case, condition (14) reduces to

$$
\begin{aligned}
\left|\frac{1}{20}-\frac{11}{80}\right| \leq \frac{k}{8}\bigg[&\left|x-\frac{1}{20}\right| + \left|y-\frac{11}{80}\right| + \left|z-\frac{11}{80}\right|\\
&+ \left|u-\frac{11}{80}\right| + \left|v-\frac{11}{80}\right| + \left|w-\frac{11}{80}\right|\bigg].
\end{aligned}
\tag{48}
$$

Since $x \in [4/5,1]$, we have

$$
\left|x-\frac{1}{20}\right| \geq \left|\frac{4}{5}-\frac{1}{20}\right| = \frac{3}{4}
\tag{49}
$$

and hence the minimum value of the right hand side of (48) is greater or equal to $k/8 \cdot 3/4$.

Therefore, in order to have (48) satisfied for all $x \in [4/5,1]$ and u, y, z, v, $w \in [0,4/5)$, with $x \geq u$, $y \leq v$, $z \geq w$, that is,

$$
\left|\frac{1}{20}-\frac{11}{80}\right| \leq \frac{k}{8} \cdot \frac{3}{4},
\tag{50}
$$

it suffices to take k such that $14/15 \leq k < 1$.

Note that, for the remaining cases to be discussed, the right hand side of (14) will be greater than the value obtained in Case 1.

Case 2 ($x, v \in [4/5, 1]$ and $u, y, z, w \in [0, 4/5)$). In this case, for example, the minimum value of the right hand side of (14) will be greater or equal to $k/8 \cdot 6/4$.

Note also that, in the cases $x, u \in [4/5, 1]$ or $x, u \in [0, 4/5)$, the left hand side of (14) is always zero and so (14) is satisfied for all values of $y, z, v, w \in [0, 1]$.

This proves that, indeed, F satisfies (14) with $k = 14/15 < 1$.

F is not continuous but X satisfies assumption (b) in Theorem 12. Moreover, by taking $x_0 = 0$, $y_0 = 1/5$, and $z_0 = 1/8$, one can easily check that (15) is fulfilled.

Thus, the assumptions in Theorem 12 are satisfied and hence F does admit tripled fixed points. By Theorem 13, we actually conclude that F has a unique tripled fixed point, $(11/80, 11/80, 11/80)$.

Now let us show that F does not satisfy (9).

Assume the contrary, that is, that F does satisfy (9) and let $\epsilon > 0$ such that $u = 4/5 - \epsilon \in [0, 4/5)$, and then take $x = 4/5$ and $y = z$, $v = w \in [0, 1]$ arbitrary in (9) to obtain the inequality

$$\frac{7}{80} \le i \cdot \epsilon, \qquad \epsilon > 0. \tag{51}$$

Now by letting $\epsilon \to 0$ in (51), we reach a contradiction. This proves that, indeed, F does not satisfy (9).

Remark 15. For various particular cases of the results established in this section and for possible further developments, we refer to [2, 4–6, 8–28].

5. Applications

In this section, we present an application of tripled fixed point theorems for establishing existence and uniqueness results for the solutions of the nonlinear integral equation

$$\begin{aligned} x(t) = & \int_0^T G(t,s)\left[f(s,x(s)) + g(s,x(s)) + h(s,x(s))\right] ds \\ & + a(t), \quad t \in [0,T], \ T > 0. \end{aligned} \tag{52}$$

We consider the space $X = C([0,T], \mathbb{R})$ of continuous real functions defined on the interval $[0,T]$, endowed with metric

$$d(u,v) = \max_{0 \le t \le T} |u(t) - v(t)|, \quad \text{for } u, v \in X. \tag{53}$$

Define the partial order "$\le$" on X by

$$\begin{aligned} \text{for } x, y \in X, \quad x \le y \Longleftrightarrow x(t) \le y(t), \\ \text{for any } t \in [0,T]. \end{aligned} \tag{54}$$

Thus, $(X, d, \le)$ is a partially ordered complete metric space.

For (52) we consider the following assumptions:

(i) $f, g, h : [0,T] \times \mathbb{R} \to \mathbb{R}$ are continuous;

(ii) $a : [0,T] \times \ \to \mathbb{R}$ is continuous;

(iii) $G : [0,T] \times [0,T] \to [0, \infty)$ is continuous;

(iv) there exist the constants $\lambda_1, \lambda_2, \lambda_3 > 0$, such that, for all $x, y \in \mathbb{R}$, $x \le y$ we have

$$\begin{aligned} & 0 \le f(s,y) - f(s,x) \le \lambda_1 (y - x), \\ & 0 \le g(s,x) - g(s,y) \le \lambda_2 (y - x), \\ & 0 \le h(s,y) - h(s,x) \le \lambda_3 (y - x); \end{aligned} \tag{55}$$

(v) we suppose that $(\lambda_1 + \lambda_2 + \lambda_3)\sup_{0 \le t \le T} \int_0^T G(t,s) ds < 1$;

(vi) there exist continuous functions $\alpha, \beta, \gamma : [0,T] \to \mathbb{R}$ such that

$$\begin{aligned} \alpha \le & \int_0^T G(t,s)\left[f(s,\alpha(s)) + g(s,\beta(s)) + h(s,\gamma(s))\right] ds \\ & + a(t) \\ \beta \ge & \int_0^T G(t,s)\left[f(s,\beta(s)) + g(s,\alpha(s)) + h(s,\beta(s))\right] ds \\ & + a(t) \\ \gamma \le & \int_0^T G(t,s)\left[f(s,\gamma(s)) + g(s,\beta(s)) + h(s,\alpha(s))\right] ds \\ & + a(t). \end{aligned} \tag{56}$$

Theorem 16. *Under assumptions (i)–(vi),* (52) *has a unique solution in* $C([0,1], \mathbb{R})$.

Proof. We consider the operator $F : X^3 \to X$ defined by

$$\begin{aligned} & F(x_1, x_2, x_3)(t) \\ & \quad = \int_0^T G(t,s)\left[f(s,x_1(s)) + g(s,x_2(s)) + h(s,x_3(s))\right] ds \\ & \qquad + a(t), \quad t \in [0,T], \end{aligned} \tag{57}$$

for any $x_1, x_2, x_3 \in X$.

We prove that the operator F fulfills the conditions of Theorem 8.

First, we prove that F has the mixed-monotone property. Let $x_1, y_1 \in X$, with $x_1 \le y_1$ and $t \in [0,T]$, then, we have

$$\begin{aligned} & F(y_1, x_2, x_3)(t) - F(x_1, x_2, x_3)(t) \\ & \quad = \int_0^T G(t,s)\left[f(s,y_1(s)) - f(s,x_1(s))\right] ds. \end{aligned} \tag{58}$$

Given that $x_1(t) \le y_1(t)$ for all $t \in [0,T]$ and based on our assumption (iv), we have

$$f(s,y)-f(s,x)\geq 0,\quad \forall x,y\in\mathbb{R},\ x\leq y$$

and from (60), we have (59)

$$F(y_1,x_2,x_3)(t)-F(x_1,x_2,x_3)(t)\geq 0,\quad \forall t\in[0,T].$$

That is, $F(x_1,x_2,x_3)(t)\leq F(y_1,x_2,x_3)(t)$.

For x_2, $y_2\in X$, with $x_2\leq y_2$ and $t\in[0,T]$, we have

$$\begin{aligned}&F(x_1,x_2,x_3)(t)-F(x_1,y_2,x_3)(t)\\&\quad=\int_0^T G(t,s)\left[g(s,x_2(s))-g(s,y_2(s))\right]ds.\end{aligned}\tag{60}$$

Given that $x_2(t)\leq y_2(t)$ for all $t\in[0,T]$ and based on our assumption (iv), that is,

$$g(s,x)-g(s,y)\geq 0,\quad \forall x,y\in\mathbb{R},\ x\leq y$$

and from (62), we get (61)

$$F(x_1,x_2,x_3)(t)-F(x_1,y_2,x_3)(t)\geq 0,\quad \forall t\in[0,T].$$

That is, $F(x_1,x_2,x_3)(t)\geq F(x_1,y_2,x_3)(t)$.

Similarly, one proves the same property for the third component and hence we have $F(x_1,x_2,x_3)(t)\leq F(x_1,x_2,y_3)(t)$.

So, F has the mixed-monotone property.

Now, we estimate $d(F(x_1,x_2,x_3),F(y_1,y_2,y_3))$ for $x_1\leq y_1, x_2\geq y_2, x_3\leq y_3$, and with F having the mixed-monotone property, we get

$$\begin{aligned}&d(F(x_1,x_2,x_3),F(y_1,y_2,y_3))\\&\quad=\max_{0\leq t\leq T}\left|F(x_1,x_2,x_3)(t)-F(y_1,y_2,y_3)(t)\right|\\&\quad=\max_{0\leq t\leq T}\left(F(y_1,y_2,y_3)(t)-F(x_1,x_2,x_3)(t)\right).\end{aligned}\tag{62}$$

Now, for all $t\in[0,t]$, by using (iv), we have

$$\begin{aligned}&F(y_1,y_2,y_3)(t)-F(x_1,x_2,x_3)(t)\\&\quad=\int_0^T G(t,s)\left[f(s,y_1(s))-g(s,x_1(s))\right]ds\\&\qquad+\int_0^T G(t,s)\left[g(s,y_2(s))-g(s,x_2(s))\right]ds\\&\qquad+\int_0^T G(t,s)\left[h(s,y_3(s))-h(s,x_3(s))\right]ds\\&\quad\leq\left(\int_0^T G(t,s)\,ds\right)\left(\lambda_1 d(x_1,y_1)+\lambda_2 d(x_2,y_2)\right.\\&\qquad\left.+\lambda_3 d(x_3,y_3)\right)\\&\quad\leq\left(\sup_{0\leq t\leq T}\int_0^T G(t,s)\,ds\right)\left(\lambda_1 d(x_1,y_1)+\lambda_2 d(x_2,y_2)\right.\\&\qquad\left.+\lambda_3 d(x_3,y_3)\right),\end{aligned}\tag{63}$$

which implies

$$\begin{aligned}&d(F(x_1,x_2,x_3),F(y_1,y_2,y_3))\\&\quad\leq\delta_1 d(x_1,y_1)+\delta_2 d(x_2,y_2)+\delta_3 d(x_3,y_3),\end{aligned}\tag{64}$$

where

$$\delta_i=\left(\sup_{0\leq t\leq T}\int_0^T G(t,s)\,ds\right)\lambda_i,\quad i=1,2,3.\tag{65}$$

Using (v), we have $\delta_1+\delta_2+\delta_3<1$. So, F fulfills the conditions of Theorem 8.

Let α, β, γ be the functions appearing in assumption (vi); then, we have

$$\alpha\leq F(\alpha,\beta,\gamma),\qquad \beta\geq F(\beta,\alpha,\beta),\qquad \gamma\leq F(\gamma,\beta,\alpha).\tag{66}$$

If $x_0=\alpha$, $y_0=\beta$, $z_0=\lambda$, then all assumptions of Theorem 8 are fulfilled. So, there exists a tripled fixed point (x_1,x_2,x_3) for the operator F; that is,

$$\begin{gathered}x_1=F(x_1,x_2,x_3),\qquad x_2=F(x_2,x_1,x_2),\\x_3=F(x_3,x_2,x_1).\end{gathered}\tag{67}$$

Now, we show that, for any (x_1,x_2,x_3), $(y_1,y_2,y_3)\in X^3$, there exists $(z_1,z_2,z_3)\in X^3$ which is comparable to both of them. Indeed, denote

$$\begin{gathered}z_1=\max\{x_1,y_1\},\qquad z_2=\min\{x_2,y_2\},\\z_3=\max\{x_3,y_3\}.\end{gathered}\tag{68}$$

Then we have $x_1\leq z_1$, $x_2\geq z_2$, $x_3\leq z_3$ and $y_1\leq z_1$, $y_2\geq z_2$, $y_3\leq z_3$. This implies that (z_1,z_2,z_3) is comparable with (x_1,x_2,x_3) and with (y_1,y_2,y_3), so

$$(x_1,x_2,x_3)\leq(z_1,z_2,z_3),\qquad (y_1,y_2,y_3)\leq(z_1,z_2,z_3).\tag{69}$$

Now, by Theorem 10, it follows that F has a unique triple fixed point which is in fact the solution the (52). □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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