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Research Article

Hybrid Extragradient-Like Viscosity Methods for Generalized Mixed Equilibrium Problems, Variational Inclusions, and Optimization Problems

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We introduce and analyze a new hybrid extragradient-like viscosity iterative algorithm for finding a common solution of a generalized mixed equilibrium problem, a finite family of variational inclusions for maximal monotone and inverse strongly monotone mappings, and a fixed point problem of infinitely many nonexpansive mappings in a real Hilbert space. Under some mild conditions, we prove the strong convergence of the sequence generated by the proposed algorithm to a common solution of these three problems which also solves an optimization problem.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, C be a nonempty closed convex subset of H, and P_C be the metric projection of H onto C. Let $S : C \to H$ be a nonlinear mapping on C. We denote by $\mathrm{Fix}(S)$ the set of fixed points of S and by $\mathbf{R}$ the set of all real numbers. A mapping V is called strongly positive on H if there exists a constant $\overline{\gamma} \in (0, 1]$ such that

$$\langle Vx, x \rangle \geq \overline{\gamma} \|x\|^2, \quad \forall x \in H. \tag{1}$$

A mapping $S : C \to H$ is called L-Lipschitz continuous if there exists a constant $L \geq 0$ such that

$$\|Sx - Sy\| \leq L \|x - y\|, \quad \forall x, y \in C. \tag{2}$$

In particular, if $L = 1$ then S is called a nonexpansive mapping; if $L \in [0, 1)$ then A is called a contraction.

Let $A : C \to H$ be a nonlinear mapping on C. Recall that the classical variational inequality problem (VIP) is to find a point $x \in C$ such that

$$\langle Ax, y - x \rangle \geq 0, \quad \forall y \in C. \tag{3}$$

The solution set of VIP (3) is denoted by $\mathrm{VI}(C, A)$. The VIP (3) was first discussed by Lions [1] and has been extensively studied since then. See, for example, [2–5].

In 1976, Korpelevič [6] proposed an iterative algorithm for solving the VIP (3) in Euclidean space $\mathbf{R}^n$:

$$\begin{aligned} y_n &= P_C (x_n - \tau A x_n), \\ x_{n+1} &= P_C (x_n - \tau A y_n), \quad \forall n \geq 0, \end{aligned} \tag{4}$$

with $\tau > 0$ as a given number, which is known as the extragradient method. The literature on the VIP is vast and Korpelevich's extragradient method has received great attention given by many authors, see, for example, [7–23] and the references therein. Let $\varphi : C \to \mathbf{R}$ be a real-valued function, $A : H \to H$ be a nonlinear mapping, and $\Theta : C \times C \to \mathbf{R}$ be a bifunction. In 2008, Peng and Yao [14] introduced the following generalized mixed equilibrium problem (GMEP) of finding $x \in C$ such that

$$\Theta (x, y) + \varphi (y) - \varphi (x) + \langle Ax, y - x \rangle \geq 0, \quad \forall y \in C. \tag{5}$$

We denote the set of solutions of GMEP (5) by $\mathrm{GMEP}(\Theta, \varphi, A)$. The GMEP (5) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, Nash equilibrium problems in noncooperative games, and others. The GMEP is further considered and studied; see, for example, [13, 16, 24–28].

We present some special cases of GMEP (5) as follows.

If $\varphi = 0$, then GMEP (5) reduces to the generalized equilibrium problem (GEP) which is to find $x \in C$ such that

$$\Theta(x, y) + \langle Ax, y - x\rangle \geq 0, \quad \forall y \in C. \tag{6}$$

This problem was introduced and studied by S. Takahashi and W. Takahashi [29]. The set of solutions of GEP is denoted by $\mathrm{GEP}(\Theta, A)$.

If $A = 0$, then GMEP (5) reduces to the mixed equilibrium problem (MEP) which is to find $x \in C$ such that

$$\Theta(x, y) + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{7}$$

It is considered and studied in [30, 31]. The set of solutions of MEP is denoted by $\mathrm{MEP}(\Theta, \varphi)$.

If $\varphi = 0$, $A = 0$, then GMEP (5) reduces to the equilibrium problem (EP) which is to find $x \in C$ such that

$$\Theta(x, y) \geq 0, \quad \forall y \in C. \tag{8}$$

This was considered and studied in [32, 33]. The set of solutions of EP is denoted by $\mathrm{EP}(\Theta)$. It is worth mentioning that the EP is a unified model of several problems, namely, variational inequality problems, optimization problems, saddle point problems, complementarity problems, fixed point problems, Nash equilibrium problems, and so forth.

For the bifunction $\Theta : C \times C \to \mathbf{R}$ and real-valued function $\varphi : C \to \mathbf{R}$ in the GMEP (5), as in [14], we assume that Θ is a bifunction satisfying conditions (H1)–(H4) and φ is a lower semicontinuous and convex function with restriction (H5), where

(H1) $\Theta(x, x) = 0$ for all $x \in C$;

(H2) Θ is monotone, that is; $\Theta(x, y) + \Theta(y, x) \leq 0$ for any $x, y \in C$;

(H3) Θ is upper-hemicontinuous, that is; for each $x, y, z \in C$,

$$\limsup_{t \to 0^+} \Theta(tz + (1-t)x, y) \leq \Theta(x, y); \tag{9}$$

(H4) $\Theta(x, \cdot)$ is convex and lower semicontinuous for each $x \in C$;

(H5) for each $x \in H$ and $r > 0$, there exists a bounded subset $D_x \subset C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,

$$\Theta(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0. \tag{10}$$

A differentiable function $K : H \to \mathbf{R}$ is called

(i) convex, if

$$K(y) - K(x) \geq \langle K'(x), y - x\rangle, \quad \forall x, y \in H, \tag{11}$$

where K' is the Frechet derivative of K at x;

(ii) strongly convex, if there exists a constant $\sigma > 0$ such that

$$K(y) - K(x) - \langle K'(x), y - x\rangle \geq \frac{\sigma}{2}\|x - y\|^2, \quad \forall x, y \in H. \tag{12}$$

It is easy to see that if $K : H \to \mathbf{R}$ is a differentiable strongly convex function with constant $\sigma > 0$, then $K' : H \to H$ is strongly monotone with constant $\sigma > 0$.

Given a positive number $r > 0$, let $S_r^{(\Theta,\varphi)} : H \to C$ be the solution set of the auxiliary mixed equilibrium problem; that is, for each $x \in H$,

$$S_r^{(\Theta,\varphi)}(x) := \Big\{ y \in C : \Theta(y, z) + \varphi(z) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), z - y\rangle \geq 0, \ \forall z \in C \Big\}. \tag{13}$$

In particular, whenever $K(x) = (1/2)\|x\|^2$, $\forall x \in H$, $S_r^{(\Theta,\varphi)}$ is rewritten as $T_r^{(\Theta,\varphi)}$.

Let $\Theta_1, \Theta_2 : C \times C \to \mathbf{R}$ be two bifunctions and $B_1, B_2 : C \to H$ be two nonlinear mappings. Consider the following system of generalized equilibrium problems (SGEP): find $(x^*, y^*) \in C \times C$ such that

$$\begin{gathered} \Theta_1(x^*, x) + \langle B_1 y^*, x - x^*\rangle + \frac{1}{\mu_1}\langle x^* - y^*, x - x^*\rangle \geq 0, \quad \forall x \in C, \\ \Theta_2(y^*, y) + \langle B_2 x^*, y - y^*\rangle + \frac{1}{\mu_2}\langle y^* - x^*, y - y^*\rangle \geq 0, \quad \forall y \in C, \end{gathered} \tag{14}$$

where $\mu_1 > 0$ and $\mu_2 > 0$ are two constants. It is introduced and studied in [15], that the SGEP reduces to a system of variational inequalities whenever $\Theta_1 \equiv \Theta_2 \equiv 0$. It is worth mentioning that the system of variational inequalities is a tool to solve the Nash equilibrium problem for noncooperative games.

In 2010, Ceng and Yao [15] transformed the SGEP into a fixed point problem in the following way.

Proposition CY (see [15]). *Let $\Theta_1, \Theta_2 : C \times C \to \mathbf{R}$ be two bifunctions satisfying conditions (H1)–(H4) and let $B_i : C \to H$ be $\tilde{\beta}_i$-inverse-strongly monotone for $i = 1, 2$. Let $\mu_i \in (0, 2\tilde{\beta}_i)$ for $i = 1, 2$. Then, $(x^*, y^*) \in C \times C$ is a solution of SGEP if and only if x^* is a fixed point of the mapping $G : C \to C$ defined by $G = T_{\mu_1}^{\Theta_1}(I - \mu_1 B_1)T_{\mu_2}^{\Theta_2}(I - \mu_2 B_2)$, where $y^* = T_{\mu_2}^{\Theta_2}(I - \mu_2 B_2)x^*$. Here, one denotes the fixed point set of G by $\mathrm{SGEP}(G)$.*

Let $\{T_n\}_{n=1}^{\infty}$ be an infinite family of nonexpansive mappings on H and $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of nonnegative numbers in $[0,1]$. For any $n \geq 1$, define a self-mapping W_n on H as follows:

$$\begin{aligned}
U_{n,n+1} &= I, \\
U_{n,n} &= \lambda_n T_n U_{n,n+1} + (1-\lambda_n) I, \\
U_{n,n-1} &= \lambda_{n-1} T_{n-1} U_{n,n} + (1-\lambda_{n-1}) I, \\
&\vdots \\
U_{n,k} &= \lambda_k T_k U_{n,k+1} + (1-\lambda_k) I, \\
U_{n,k-1} &= \lambda_{k-1} T_{k-1} U_{n,k} + (1-\lambda_{k-1}) I, \\
&\vdots \\
U_{n,2} &= \lambda_2 T_2 U_{n,3} + (1-\lambda_2) I, \\
W_n &= U_{n,1} = \lambda_1 T_1 U_{n,2} + (1-\lambda_1) I.
\end{aligned} \tag{15}$$

Such a mapping W_n is called the W-mapping generated by $T_n, T_{n-1}, \ldots, T_1$ and $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$.

In 2011, for the case where $C = H$, let $f : H \to H$ be a contraction, $K : H \to \mathrm{R}$ be differentiable and strongly convex, $\{\alpha_n\}$, $\{\beta_n\} \subset (0,1)$ and $x_0, u \in H$ be given. Yao et al. [25] proposed the hybrid iterative algorithm for finding a common element of the set $\mathrm{MEP}(\Theta,\varphi)$ and the fixed set $\bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i)$ of an infinite family of nonexpansive mappings $\{T_i\}_{i=1}^{\infty}$ on H as follows:

$$\begin{aligned}
&\Theta(y_n, z) + \varphi(z) - \varphi(y_n) \\
&\quad + \frac{1}{r}\left\langle K'(y_n) - K'(x_n), z - y_n \right\rangle \geq 0, \quad z \in H, \\
&x_{n+1} = \alpha_n (u + \gamma f(x_n)) + \beta_n x_n \\
&\quad + ((1-\beta_n) I - \alpha_n (I + \mu V)) W_n y_n, \quad \forall n \geq 1.
\end{aligned} \tag{16}$$

They proved the strong convergence of the sequence generated by the hybrid iterative algorithm (16) to a point $x^* \in \bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i) \cap \mathrm{MEP}(\Theta,\varphi)$ under some appropriate conditions. This point x^* also solves the following optimization problem:

$$\min_{x \in \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \cap \mathrm{MEP}(\Theta,\varphi)} \frac{\mu}{2}\langle Vx, x\rangle + \frac{1}{2}\|x-u\|^2 - h(x), \tag{OP0}$$

where $h : H \to \mathbf{R}$ is the potential function of γf.

Let $f : H \to H$ be a contraction and V be a strongly positive bounded linear operator on H. Assume that $\varphi : H \to \mathbf{R}$ is a lower semicontinuous and convex functional, that $\Theta, \Theta_1, \Theta_2 : H \times H \to \mathbf{R}$ satisfy conditions (H1)–(H4), and that $A, B_1, B_2 : H \to H$ are inverse-strongly monotone. Very recently, Ceng et al. [16] introduced the following hybrid extragradient-like iterative algorithm:

$$\begin{aligned}
z_n &= S_{r_n}^{(\Theta,\varphi)}(x_n - r_n A x_n), \\
y_n &= T_{\mu_1}^{\Theta_1}(I - \mu_1 B_1) T_{\mu_2}^{\Theta_2}(I - \mu_2 B_2) z_n, \\
x_{n+1} &= \alpha_n (u + \gamma f(x_n)) + \beta_n x_n \\
&\quad + ((1-\beta_n) I - \alpha_n (I + \mu V)) W_n y_n, \quad \forall n \geq 0,
\end{aligned} \tag{17}$$

for finding a common solution of GMEP (5), SGEP (14), and the fixed point problem of an infinite family of nonexpansive mappings $\{T_i\}_{i=1}^{\infty}$ on H, where $\{r_n\} \subset (0,\infty)$, $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$, and $x_0, u \in H$ are given. The authors proved the strong convergence of the sequence generated by the hybrid iterative algorithm (17) to a point $x^* \in \bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i) \cap \mathrm{GMEP}(\Theta,\varphi,A) \cap \mathrm{SGEP}(G)$ under some suitable conditions. This point x^* also solves the following optimization problem:

$$\begin{aligned}
&\min_{x \in \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \cap \mathrm{GMEP}(\Theta,\varphi,A) \cap \mathrm{SGEP}(G)} \frac{\mu}{2}\langle Vx, x\rangle \\
&\quad + \frac{1}{2}\|x-u\|^2 - h(x),
\end{aligned} \tag{OP1}$$

where $h : H \to \mathbf{R}$ is the potential function of γf.

On the other hand, let B be a single-valued mapping of C into H and R be a set-valued mapping with $D(R) = C$. Consider the following variational inclusion: find a point $x \in C$ such that

$$0 \in Bx + Rx. \tag{18}$$

We denote by $I(B,R)$ the solution set of the variational inclusion (18). In particular, if $B = R = 0$, then $I(B,R) = C$. If $B = 0$, then problem (18) becomes the inclusion problem introduced by Rockafellar [34]. It is known that problem (18) provides a convenient framework for the unified study of optimal solutions in many optimization related areas including mathematical programming, complementarity problems, variational inequalities, optimal control, mathematical economics, equilibria and game theory, etc. Let a set-valued mapping $R : D(R) \subset H \to 2^H$ be maximal monotone. We define the resolvent operator $J_{R,\lambda} : H \to \overline{D(R)}$ associated with R and λ as follows:

$$J_{R,\lambda} = (I + \lambda R)^{-1}, \quad \forall x \in H, \tag{19}$$

where λ is a positive number.

In 1998, Huang [35] studied problem (18) in the case where R is maximal monotone and B is strongly monotone and Lipschitz continuous with $D(R) = C = H$. Subsequently, Zeng et al. [36] further studied problem (18) in the case which is more general than Huang's [35]. Moreover, the authors [36] obtained the same strong convergence conclusion as in Huang's result [35]. In addition, the authors also gave the geometric convergence rate estimate for approximate solutions. Also, various types of iterative algorithms for solving variational inclusions have been further studied and developed; for more details, refer to [21, 26, 37, 38] and the references therein.

Motivated and inspired by the above facts, we, in this paper, introduce and analyze a new iterative algorithm by a hybrid extragradient-like viscosity method for finding a common element of the set of solutions of a generalized mixed equilibrium problem, the set of solutions of a finite family of variational inclusions for maximal monotone and inverse strong monotone mappings, and the set of fixed points of a countable family of nonexpansive mappings in a real Hilbert space. Under some appropriate conditions, we prove the strong convergence of the sequence generated by the proposed algorithm to a common solution of these three problems. Such a solution also solves an optimization problem. Several special cases are also discussed. The results presented in this paper are the supplement, extension, improvement, and generalization of the previously known results in this area.

2. Preliminaries

Throughout this paper, we assume that H is a real Hilbert space whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x and $x_n \to x$ to indicate that the sequence $\{x_n\}$ converges strongly to x. Moreover, we use $\omega_w(x_n)$ to denote the weak ω-limit set of the sequence $\{x_n\}$; that is,

$$\omega_w(x_n) := \{x \in H : x_{n_i} \rightharpoonup x \text{ for some subsequence } \{x_{n_i}\} \text{ of } \{x_n\}\}. \quad (20)$$

Recall that a mapping $A : C \to H$ is called

(i) monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C; \quad (21)$$

(ii) η-strongly monotone if there exists a constant $\eta > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \eta\|x - y\|^2, \quad \forall x, y \in C; \quad (22)$$

(iii) ζ-inverse-strongly monotone if there exists a constant $\zeta > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \zeta\|Ax - Ay\|^2, \quad \forall x, y \in C. \quad (23)$$

It is easy to see that the projection P_C is 1-ism. Inverse strongly monotone (also referred to as cocoercive) operators have been applied widely in solving practical problems in various fields.

The metric (or nearest point) projection from H onto C is the mapping $P_C : H \to C$ which assigns to each point $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \inf_{y\in C}\|x - y\| =: d(x, C). \quad (24)$$

Some important properties of projections are gathered in the following proposition.

Proposition 1. *For given $x \in H$ and $z \in C$:*

(i) $z = P_C x \Leftrightarrow \langle x - z, y - z\rangle \leq 0$, $\forall y \in C$;

(ii) $z = P_C x \Leftrightarrow \|x - z\|^2 \leq \|x - y\|^2 - \|y - z\|^2$, $\forall y \in C$;

(iii) $\langle P_C x - P_C y, x - y\rangle \geq \| P_C x - P_C y\|^2$, $\forall y \in H$. *(This implies that P_C is nonexpansive and monotone.)*

By using the technique of [31], we can readily obtain the following elementary result.

Proposition 2 (see [16, Lemma 1 and Proposition 1]). *Let C be a nonempty closed convex subset of a real Hilbert space H and let $\varphi : C \to \mathbf{R}$ be a lower semicontinuous and convex function. Let $\Theta : C \times C \to \mathbf{R}$ be a bifunction satisfying the conditions (H1)–(H4). Assume that*

(i) *$K : H \to \mathbf{R}$ is strongly convex with constant $\sigma > 0$ and the function $x \mapsto \langle y - x, K'(x)\rangle$ is weakly upper semicontinuous for each $y \in H$;*

(ii) *for each $x \in H$ and $r > 0$, there exists a bounded subset $D_x \subset C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,*

$$\Theta(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle K'(z) - K'(x), y_x - z\rangle < 0. \quad (25)$$

Then the following hold:

(a) for each $x \in H$, $S_r^{(\Theta,\varphi)}(x) \neq \emptyset$;

(b) $S_r^{(\Theta,\varphi)}$ is single-valued;

(c) $S_r^{(\Theta,\varphi)}$ is nonexpansive if K' is Lipschitz continuous with constant $\nu > 0$ and

$$\langle K'(x_1) - K'(x_2), u_1 - u_2\rangle \geq \langle K'(u_1) - K'(u_2), u_1 - u_2\rangle, \quad \forall (x_1, x_2) \in H \times H, \quad (26)$$

where $u_i = S_r^{(\Theta,\varphi)}(x_i)$ for $i = 1, 2$;

(d) for all $s, t > 0$ and $x \in H$

$$\begin{aligned}&\langle K'(S_s^{(\Theta,\varphi)}x) - K'(S_t^{(\Theta,\varphi)}x), S_s^{(\Theta,\varphi)}x - S_t^{(\Theta,\varphi)}x\rangle \\ &\leq \frac{s-t}{s}\langle K'(S_s^{(\Theta,\varphi)}x) - K'(x), S_s^{(\Theta,\varphi)}x - S_t^{(\Theta,\varphi)}x\rangle;\end{aligned} \quad (27)$$

(e) $\mathrm{Fix}(S_r^{(\Theta,\varphi)}) = \mathrm{MEP}(\Theta, \varphi)$;

(f) $\mathrm{MEP}(\Theta, \varphi)$ is closed and convex.

In particular, whenever $\Theta : C\times C \to \mathbf{R}$ is a bifunction satisfying the conditions (H1)–(H4) and $K(x) = (1/2)\|x\|^2, \forall x \in H$, then, that is, for any $x, y \in H$,

$$\|S_r^{(\Theta,\varphi)}x - S_r^{(\Theta,\varphi)}y\|^2 \leq \langle S_r^{(\Theta,\varphi)}x - S_r^{(\Theta,\varphi)}y, x - y\rangle \quad (28)$$

($S_r^{(\Theta,\varphi)}$ is firmly nonexpansive) and

$$\left\|S_s^{(\Theta,\varphi)}x - S_t^{(\Theta,\varphi)}x\right\| \le \frac{|s-t|}{s}\left\|S_s^{(\Theta,\varphi)}x - x\right\|, \quad \forall s,t>0,\ x\in H. \tag{29}$$

In this case, $S_r^{(\Theta,\varphi)}$ is rewritten as $T_r^{(\Theta,\varphi)}$. If, in addition, $\varphi \equiv 0$, then $T_r^{(\Theta,\varphi)}$ is rewritten as T_r^{Θ} (see [15, Lemma 2.1] for more details).

Remark 3. Suppose $K : H \to \mathbf{R}$ is strongly convex with constant $\sigma > 0$ and $K' : H \to H$ is Lipschitz continuous with constant $\nu > 0$. Then $K' : H \to H$ is σ-strongly monotone and ν-Lipschitz continuous with positive constants $\sigma, \nu > 0$. Utilizing Proposition 2(d) we obtain that for all $s,t > 0$ and $x \in H$

$$\begin{aligned}\sigma&\left\|S_s^{(\Theta,\varphi)}x - S_t^{(\Theta,\varphi)}x\right\|^2\\ &\le \left\langle K'\left(S_s^{(\Theta,\varphi)}x\right) - K'\left(S_t^{(\Theta,\varphi)}x\right), S_s^{(\Theta,\varphi)}x - S_t^{(\Theta,\varphi)}x\right\rangle\\ &\le \frac{s-t}{s}\left\langle K'\left(S_s^{(\Theta,\varphi)}x\right) - K'(x), S_s^{(\Theta,\varphi)}x - S_t^{(\Theta,\varphi)}x\right\rangle\\ &\le \frac{|s-t|}{s}\left\|K'\left(S_s^{(\Theta,\varphi)}x\right) - K'(x)\right\|\left\|S_s^{(\Theta,\varphi)}x - S_t^{(\Theta,\varphi)}x\right\|\\ &\le \frac{|s-t|}{s}\nu\left\|S_s^{(\Theta,\varphi)}x - x\right\|\left\|S_s^{(\Theta,\varphi)}x - S_t^{(\Theta,\varphi)}x\right\|,\end{aligned} \tag{30}$$

which immediately implies that

$$\left\|S_s^{(\Theta,\varphi)}x - S_t^{(\Theta,\varphi)}x\right\| \le \frac{|s-t|}{s}\cdot\frac{\nu}{\sigma}\left\|S_s^{(\Theta,\varphi)}x - x\right\|. \tag{31}$$

We need some facts and tools in a real Hilbert space H which are listed as lemmas below.

Lemma 4. *Let X be a real inner product space. Then there holds the following inequality:*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall x,y\in X. \tag{32}$$

Lemma 5. *Let H be a real Hilbert space. Then the following hold:*

(a) $\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle$ *for all* $x,y \in H$;

(b) $\|\lambda x + \mu y\|^2 = \lambda\|x\|^2 + \mu\|y\|^2 - \lambda\mu\|x-y\|^2$ *for all* $x, y \in H$ *and* $\lambda,\mu \in [0,1]$ *with* $\lambda+\mu = 1$;

(c) *if* $\{x_n\}$ *is a sequence in H such that* $x_n \rightharpoonup x$, *it follows that*

$$\limsup_{n\to\infty}\|x_n - y\|^2 = \limsup_{n\to\infty}\|x_n - x\|^2 + \|x-y\|^2, \quad \forall y\in H. \tag{33}$$

We have the following crucial lemmas concerning the W-mappings defined by (15).

Lemma 6 (see [39, Lemma 3.2]). *Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on H such that $\bigcap_{n=1}^{\infty}\mathrm{Fix}(T_n) \neq \emptyset$ and let $\{\lambda_n\}$ be a sequence in $(0,b]$ for some $b \in (0,1)$. Then, for every $x \in H$ and $k \ge 1$ the limit $\lim_{n\to\infty}U_{n,k}x$ exists, where $U_{n,k}$ is defined by* (15).

Remark 7 (see [40, Remark 3.1]). It can be known from Lemma 6 that if D is a nonempty bounded subset of H, then for $\epsilon > 0$ there exists $n_0 \ge k$ such that for all $n > n_0$

$$\sup_{x\in D}\|U_{n,k}x - U_k x\| \le \epsilon. \tag{34}$$

Remark 8 (see [40, Remark 3.2]). Utilizing Lemma 6, we define a mapping $W : H \to H$ as follows:

$$Wx = \lim_{n\to\infty}W_n x = \lim_{n\to\infty}U_{n,1}x, \quad \forall x\in H. \tag{35}$$

Such a W is called the W-mapping generated by $T_1, T_2, \ldots$ and $\lambda_1, \lambda_2, \ldots$. Since W_n is nonexpansive, $W : H \to H$ is also nonexpansive. Indeed, observe that for each $x, y \in H$

$$\|Wx - Wy\| = \lim_{n\to\infty}\|W_n x - W_n y\| \le \|x-y\|. \tag{36}$$

If $\{x_n\}$ is a bounded sequence in H, then we put $D = \{x_n : n \ge 1\}$. Hence, it is clear from Remark 3 that for an arbitrary $\epsilon > 0$ there exists $N_0 \ge 1$ such that for all $n > N_0$

$$\begin{aligned}&\|W_n x_n - W x_n\|\\ &\quad = \|U_{n,1}x_n - U_1 x_n\| \le \sup_{x\in D}\|U_{n,1}x - U_1 x\| \le \epsilon.\end{aligned} \tag{37}$$

This implies that

$$\lim_{n\to\infty}\|W_n x_n - W x_n\| = 0. \tag{38}$$

Lemma 9 (see [39, Lemma 3.3]). *Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on H such that $\bigcap_{n=1}^{\infty}\mathrm{Fix}(T_n) \neq \emptyset$ and let $\{\lambda_n\}$ be a sequence in $(0,b]$ for some $b \in (0,1)$. Then,* $\mathrm{Fix}(W) = \bigcap_{n=1}^{\infty}\mathrm{Fix}(T_n)$.

Lemma 10 (see [41, Theorem 10.4 (Demiclosedness Principle)]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be nonexpansive. Then $I - T$ is demiclosed on C. That is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I-T)x_n\}$ strongly converges to some y, it follows that $(I-T)x = y$. Here I is the identity operator of H.*

Lemma 11. *Let $A : C \to H$ be a monotone mapping. In the context of the variational inequality problem the characterization of the projection (see Proposition 1(i)) implies*

$$u \in VI(C,A) \Longleftrightarrow u = P_C(u - \lambda A u), \quad \forall \lambda > 0. \tag{39}$$

The following lemma can be easily proven, and therefore, we omit the proof.

Lemma 12. *Let $f : H \to H$ be an l-Lipschitzian mapping with constant $l \ge 0$ and $F : H \to H$ be a κ-Lipschitzian and*

η-strongly monotone operator with positive constants $\kappa, \eta > 0$. Then for $0 \le \gamma l < \mu\eta$,

$$\langle (\mu F - \gamma f) x - (\mu F - \gamma f) y, x - y \rangle \ge (\mu\eta - \gamma l) \|x - y\|^2, \quad \forall x, y \in H. \tag{40}$$

That is, $\mu F - \gamma f$ is strongly monotone with constant $\mu\eta - \gamma l$.

Let C be a nonempty closed convex subset of a real Hilbert space H. We introduce some notations. Let λ be a number in $(0, 1]$ and let $\mu > 0$. Associating with a nonexpansive mapping $T : C \to C$, we define the mapping $T^\lambda : C \to H$ by

$$T^\lambda x := Tx - \lambda\mu F(Tx), \quad \forall x \in C, \tag{41}$$

where $F : H \to H$ is an operator such that, for some positive constants $\kappa, \eta > 0$, F is κ-Lipschitzian and η-strongly monotone on H; that is, F satisfies the conditions:

$$\begin{aligned} \|Fx - Fy\| &\le \kappa \|x - y\|, \\ \langle Fx - Fy, x - y \rangle &\ge \eta \|x - y\|^2 \end{aligned} \tag{42}$$

for all $x, y \in H$.

Lemma 13 (see [42, Lemma 3.1]). *T^λ is a contraction provided $0 < \mu < (2\eta/\kappa^2)$; that is,*

$$\left\|T^\lambda x - T^\lambda y\right\| \le (1 - \lambda\tau) \|x - y\|, \quad \forall x, y \in C, \tag{43}$$

where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \in (0, 1]$.

Recall that a set-valued mapping $T : D(T) \subset H \to 2^H$ is called monotone if for all $x, y \in D(T)$, $f \in Tx$, and $g \in Ty$ imply

$$\langle f - g, x - y \rangle \ge 0. \tag{44}$$

A set-valued mapping T is called maximal monotone if T is monotone and $(I + \lambda T)D(T) = H$ for each $\lambda > 0$, where I is the identity mapping of H. We denote by $G(T)$ the graph of T. It is known that a monotone mapping T is maximal if and only if, for $(x, f) \in H \times H$, $\langle f - g, x - y \rangle \ge 0$ for every $(y, g) \in G(T)$ implies that $f \in Tx$. Next we provide an example to illustrate the concept of maximal monotone mapping.

Let $A : C \to H$ be a monotone, k-Lipschitz-continuous mapping and let $N_C v$ be the normal cone to C at $v \in C$; that is,

$$N_C v = \{w \in H : \langle v - u, w \rangle \ge 0, \ \forall u \in C\}. \tag{45}$$

Define

$$Tv = \begin{cases} Av + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases} \tag{46}$$

Then, T is maximal monotone and $0 \in Tv$ if and only if $v \in \mathrm{VI}(C, A)$; see [34].

Assume that $R : D(R) \subset H \to 2^H$ is a maximal monotone mapping. Let $\lambda > 0$. In terms of Huang [35] (see also [36]), there holds the following property for the resolvent operator $J_{R,\lambda} : H \to \overline{D(R)}$.

Lemma 14. *$J_{R,\lambda}$ is single-valued and firmly nonexpansive; that is,*

$$\langle J_{R,\lambda} x - J_{R,\lambda} y, x - y \rangle \ge \|J_{R,\lambda} x - J_{R,\lambda} y\|^2, \quad \forall x, y \in H. \tag{47}$$

Consequently, $J_{R,\lambda}$ is nonexpansive and monotone.

Lemma 15 (see [21]). *Let R be a maximal monotone mapping with $D(R) = C$. Then for any given $\lambda > 0$, $u \in C$ is a solution of problem (17) if and only if $u \in C$ satisfies*

$$u = J_{R,\lambda}(u - \lambda B u). \tag{48}$$

Lemma 16 (see [36]). *Let R be a maximal monotone mapping with $D(R) = C$ and let $B : C \to H$ be a strongly monotone, continuous, and single-valued mapping. Then for each $z \in H$, the equation $z \in (B + \lambda R)x$ has a unique solution x_λ for $\lambda > 0$.*

Lemma 17 (see [21]). *Let R be a maximal monotone mapping with $D(R) = C$ and $B : C \to H$ be a monotone, continuous, and single-valued mapping. Then $(I + \lambda(R + B))C = H$ for each $\lambda > 0$. In this case, $R + B$ is maximal monotone.*

Lemma 18 (see [30]). *Let C be a nonempty closed convex subset of a real Hilbert space H, and $g : C \to \mathbf{R} \cup +\infty$ be a proper lower semicontinuous differentiable convex function. If x^* is a solution the minimization problem*

$$g(x^*) = \inf_{x \in C} g(x), \tag{49}$$

then,

$$\left\langle g'(x), x - x^* \right\rangle \ge 0, \quad \forall x \in C. \tag{50}$$

In particular, if x^* solves (OP), then

$$\langle u + (\gamma f - (I + \mu V)) x^*, x - x^* \rangle \le 0. \tag{51}$$

Lemma 19 (see [43]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \le (1 - \alpha_n) s_n + \alpha_n \beta_n + \gamma_n, \quad \forall n \ge 1, \tag{52}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are three real sequences satisfying the conditions:

(i) $\{\alpha_n\} \subset [0, 1]$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\limsup_{n \to \infty} \beta_n \le 0$;

(iii) $\gamma_n \ge 0$ $(\forall n \ge 1)$, $\sum_{n=1}^{\infty} \gamma_n < \infty$.

Then $\lim_{n \to \infty} s_n = 0$.

3. Main Results

We introduce and analyze a new iterative algorithm by hybrid extragradient-like viscosity method for finding a common element of the set of solutions of a generalized mixed equilibrium problem, the set of solutions of a finite family of variational inclusions for maximal monotone and inverse strong monotone mappings, and the set of fixed points of a countable family of nonexpansive mappings in a real Hilbert space. Under appropriate conditions imposed on the parameter sequences we will prove a strong convergence of the proposed algorithm.

Theorem 20. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let N be an integer. Let Θ be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying (H1)–(H4) and $\varphi : C \to \mathbf{R}$ be a lower semicontinuous and convex functional. Let $R_i : C \to 2^H$ be a maximal monotone mapping and let $A : H \to H$ and $B_i : C \to H$ be ζ-inverse strongly monotone and η_i-inverse strongly monotone, respectively, where $i \in \{1, 2, \ldots, N\}$. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on H and $\{\lambda_n\}$ be a sequence in $(0, b]$ for some $b \in (0, 1)$. Let $F : H \to H$ be a κ-Lipschitzian and η-strongly monotone operator with positive constants $\kappa, \eta > 0$. Let $0 < \mu < (2\eta/\kappa^2)$ and $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Let $f : H \to H$ be an l-Lipschitzian mapping with $0 \leq \gamma l < \tau$. Let W_n be the W-mapping defined by (15) and V be a $\overline{\gamma}$-strongly positive bounded linear operator with $\gamma l < (1 + \mu)\overline{\gamma}$. Assume that $\Omega := \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \cap GMEP(\Theta, \varphi, A) \cap \bigcap_{i=1}^{N} I(B_i, R_i)$ is nonempty. Suppose $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\sigma_n\}$ are three sequences in $(0, 1)$. Assume that*

(i) *$K : H \to \mathbf{R}$ is strongly convex with constant $\sigma > 0$ and its derivative K' is Lipschitz continuous with constant $\nu > 0$ such that the function $x \mapsto \langle y - x, K'(x)\rangle$ is weakly upper semicontinuous for each $y \in H$;*

(ii) *for each $x \in H$, there exist a bounded subset $D_x \subset C$ and $z_x \in C$ such that for any $y \notin D_x$,*

$$\Theta(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\left\langle K'(y) - K'(x), z_x - y\right\rangle < 0; \tag{53}$$

(iii) *$\lim_{n\to\infty}\sigma_n = \lim_{n\to\infty}(\alpha_n/\sigma_n) = 0$, $\limsup_{n\to\infty}(\alpha_{n+1}/\alpha_n) < \infty$, $\sum_{n=1}^{\infty}\sigma_n = \infty$ and*

$$0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1; \tag{54}$$

(iv) *$\mu_i \in (0, 2\eta_i)$, $i \in \{1, 2, \ldots, N\}$, and $\{r_n\} \subset [0, 2\zeta]$ satisfies*

$$0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < 2\zeta; \tag{55}$$

(v) *$\sum_{n=1}^{\infty}(|\beta_{n+1} - \beta_n| + |\sigma_{n+1} - \sigma_n| + |r_{n+1} - r_n|) < \infty$.*

Given $x_1 \in H$ arbitrarily, then the sequence $\{x_n\}$ generated iteratively by

$$\begin{aligned} u_n &= S_{r_n}^{(\Theta,\varphi)}(I - r_n A)x_n, \\ z_n &= J_{R_N,\mu_N}(I - \mu_N B_N) J_{R_{N-1},\mu_{N-1}}(I - \mu_{N-1}B_{N-1}) \cdots J_{R_1,\mu_1}(I - \mu_1 B_1) u_n, \\ y_n &= \alpha_n(u + \gamma f(x_n)) + \beta_n x_n + ((1 - \beta_n) I - \alpha_n(I + \mu V)) W_n z_n, \\ x_{n+1} &= \sigma_n \gamma f(y_n) + (I - \sigma_n \mu F) W_n y_n, \quad \forall n \geq 1, \end{aligned} \tag{56}$$

converges strongly to $x^* \in \Omega$ provided $S_r^{(\Theta,\varphi)}$ is firmly nonexpansive, where $x^* = P_\Omega(I - (\mu F - \gamma f))x^*$ is a unique solution of the VIP:

$$\langle(\gamma f - \mu F)x^*, y - x^*\rangle \leq 0, \quad \forall y \in \Omega. \tag{57}$$

Proof. Since $\lim_{n\to\infty}\alpha_n = 0$ and $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$, we may assume, without loss of generality, that $\alpha_n \leq (1 - \beta_n)(1 + \mu\|V\|)^{-1}$. Since V is a $\overline{\gamma}$-strongly positive bounded linear operator on H, we know that

$$\|V\| = \sup\{\langle Vu, u\rangle : u \in H, \|u\| = 1\}. \tag{58}$$

Observe that

$$\begin{aligned} \langle((1 - \beta_n) I - \alpha_n(I + \mu V))u, u\rangle &= 1 - \beta_n - \alpha_n - \alpha_n\mu\langle Vu, u\rangle \\ &\geq 1 - \beta_n - \alpha_n - \alpha_n\mu\|V\| \\ &\geq 0; \end{aligned} \tag{59}$$

that is, $(1 - \beta_n)I - \alpha_n(I + \mu V)$ is positive. It follows that

$$\begin{aligned} &\|(1 - \beta_n) I - \alpha_n(I + \mu V)\| \\ &\quad = \sup\{\langle((1 - \beta_n) I - \alpha_n(I + \mu V))u, u\rangle : u \in H, \|u\| = 1\} \\ &\quad = \sup\{1 - \beta_n - \alpha_n - \alpha_n\mu\langle Vu, u\rangle : u \in H, \|u\| = 1\} \\ &\quad \leq 1 - \beta_n - \alpha_n - \alpha_n\mu\overline{\gamma}. \end{aligned} \tag{60}$$

Put

$$\Lambda^i = J_{R_i,\mu_i}(I - \mu_i B_i) J_{R_{i-1},\mu_{i-1}}(I - \mu_{i-1}B_{i-1}) \cdots J_{R_1,\mu_1}(I - \mu_1 B_1) \tag{61}$$

for all $i \in \{1, 2, \ldots, N\}$ and $\Lambda^0 = I$, where I is the identity mapping on H. Then we have that $z_n = \Lambda^N u_n$.

We divide the rest of the proof into several steps.

Step 1. We show that $\{x_n\}$ is bounded. Indeed, take $p \in \Omega$ arbitrarily. Since $p = S_{r_n}^{(\Theta,\varphi)}(p - r_n A p)$, A is ζ-inverse strongly monotone and $0 \le r_n \le 2\zeta$, we have, for any $n \ge 1$,

$$\begin{aligned}
&\|u_n - p\|^2 \\
&= \left\|S_{r_n}^{(\Theta,\varphi)}(I - r_n A)x_n - S_{r_n}^{(\Theta,\varphi)}(I - r_n A)p\right\|^2 \\
&\le \|(I - r_n A)\, x_n - (I - r_n A)\, p\|^2 \\
&= \|(x_n - p) - r_n\,(Ax_n - Ap)\|^2 \\
&= \|x_n - p\|^2 - 2r_n \langle x_n - p, Ax_n - Ap\rangle + r_n^2 \|Ax_n - Ap\|^2 \\
&\le \|x_n - p\|^2 - 2r_n\zeta \|Ax_n - Ap\|^2 + r_n^2 \|Ax_n - Ap\|^2 \\
&= \|x_n - p\|^2 + r_n\,(r_n - 2\zeta)\,\|Ax_n - Ap\|^2 \\
&\le \|x_n - p\|^2.
\end{aligned} \tag{62}$$

Since $p = J_{R_i,\mu_i}(I - \mu_i B_i)p$, $\Lambda^i p = p$, and B_i is η_i-inverse strongly monotone, where $\mu_i \in (0, 2\eta_i)$, $i \in \{1, 2, \ldots, N\}$, by Lemma 14 we deduce that for each $n \ge 1$

$$\begin{aligned}
&\|z_n - p\|^2 \\
&= \left\|J_{R_N,\mu_N}\,(I - \mu_N B_N)\,\Lambda^{N-1}u_n - J_{R_N,\mu_N}\,(I - \mu_N B_N)\,\Lambda^{N-1}p\right\|^2 \\
&\le \left\|(I - \mu_N B_N)\,\Lambda^{N-1}u_n - (I - \mu_N B_N)\,\Lambda^{N-1}p\right\|^2 \\
&= \left\|\left(\Lambda^{N-1}u_n - \Lambda^{N-1}p\right) - \mu_N\left(B_N\Lambda^{N-1}u_n - B_N\Lambda^{N-1}p\right)\right\|^2 \\
&\le \left\|\Lambda^{N-1}u_n - \Lambda^{N-1}p\right\|^2 + \mu_N\,(\mu_N - 2\eta_N) \\
&\quad \times \left\|B_N\Lambda^{N-1}u_n - B_N\Lambda^{N-1}p\right\|^2 \\
&\le \left\|\Lambda^{N-1}u_n - \Lambda^{N-1}p\right\|^2 \\
&\vdots \\
&\le \left\|\Lambda^{0}u_n - \Lambda^{0}p\right\|^2 \\
&= \|u_n - p\|^2.
\end{aligned} \tag{63}$$

Combining (62) and (63), we have

$$\|z_n - p\| \le \|x_n - p\|. \tag{64}$$

Set $\overline{V} = I + \mu V$. Then from (56) and (64), we obtain

$$\begin{aligned}
&\|y_n - p\| \\
&= \left\|\alpha_n\,(u + \gamma f\,(x_n)) + \beta_n x_n + \left((1 - \beta_n)\,I - \alpha_n\overline{V}\right)W_n z_n - p\right\| \\
&= \left\|\alpha_n u + \alpha_n\left(\gamma f\,(x_n) - \overline{V}p\right) + \beta_n\,(x_n - p)\right. \\
&\qquad \left. + \left((1 - \beta_n)\,I - \alpha_n\overline{V}\right)(W_n z_n - p)\right\| \\
&\le (1 - \beta_n - \alpha_n - \alpha_n\mu\overline{\gamma})\,\|z_n - p\| + \beta_n\,\|x_n - p\| \\
&\quad + \alpha_n\,\|u\| + \alpha_n\left\|\gamma f\,(x_n) - \overline{V}p\right\| \\
&\le (1 - \beta_n - \alpha_n - \alpha_n\mu\overline{\gamma})\,\|x_n - p\| + \beta_n\,\|x_n - p\| \\
&\quad + \alpha_n\,\|u\| + \alpha_n\left\|\gamma f\,(x_n) - \overline{V}p\right\| \\
&\le (1 - \alpha_n - \alpha_n\mu\overline{\gamma})\,\|x_n - p\| + \alpha_n\,\|u\| \\
&\quad + \alpha_n\left(\gamma\,\|f\,(x_n) - f\,(p)\| + \left\|\gamma f\,(p) - \overline{V}p\right\|\right) \\
&\le (1 - \alpha_n - \alpha_n\mu\overline{\gamma})\,\|x_n - p\| + \alpha_n\,\|u\| \\
&\quad + \alpha_n\left(\gamma l\,\|x_n - p\| + \left\|\gamma f\,(p) - \overline{V}p\right\|\right) \\
&\le [1 - ((1 + \mu)\,\overline{\gamma} - \gamma l)\,\alpha_n]\,\|x_n - p\| \\
&\quad + \alpha_n\left(\left\|\gamma f\,(p) - \overline{V}p\right\| + \|u\|\right) \\
&= [1 - ((1 + \mu)\,\overline{\gamma} - \gamma l)\,\alpha_n]\,\|x_n - p\| \\
&\quad + ((1 + \mu)\,\overline{\gamma} - \gamma l)\,\alpha_n \frac{\left\|\gamma f\,(p) - \overline{V}p\right\| + \|u\|}{(1 + \mu)\,\overline{\gamma} - \gamma l}.
\end{aligned} \tag{65}$$

Therefore, by Lemma 13 we have

$$\begin{aligned}
&\|x_{n+1} - p\| \\
&= \|\sigma_n\gamma f\,(y_n) + (I - \sigma_n\mu F)\,W_n y_n - p\| \\
&= \|\sigma_n\gamma\,(f\,(y_n) - f\,(p)) + (I - \sigma_n\mu F)\,W_n y_n \\
&\qquad - (I - \sigma_n\mu F)\,W_n p + \sigma_n\,(\gamma f\,(p) - \mu F p)\| \\
&\le \sigma_n\gamma\,\|f\,(y_n) - f\,(p)\| \\
&\quad + \|(I - \sigma_n\mu F)\,W_n y_n - (I - \sigma_n\mu F)\,W_n p\| \\
&\quad + \sigma_n\,\|\gamma f\,(p) - \mu F p\| \\
&\le \sigma_n\gamma l\,\|y_n - p\| + (1 - \sigma_n\tau)\,\|y_n - p\| \\
&\quad + \sigma_n\,\|\gamma f\,(p) - \mu F p\| \\
&= (1 - \sigma_n\,(\tau - \gamma l))\,\|y_n - p\| + \sigma_n\,\|\gamma f\,(p) - \mu F p\| \\
&\le (1 - \sigma_n\,(\tau - \gamma l)) \\
&\quad \times \left[\vphantom{\frac{a}{b}}(1 - ((1 + \mu)\,\overline{\gamma} - \gamma l)\,\alpha_n)\,\|x_n - p\|\right. \\
&\qquad \left. + ((1 + \mu)\,\overline{\gamma} - \gamma l)\,\alpha_n \frac{\left\|\gamma f\,(p) - \overline{V}p\right\| + \|u\|}{(1 + \mu)\,\overline{\gamma} - \gamma l}\right] \\
&\quad + \sigma_n\,\|\gamma f\,(p) - \mu F p\|
\end{aligned}$$

$$\le (1-\sigma_n(\tau-\gamma l))\max\left\{\|x_n-p\|,\frac{\|\gamma f(p)-\overline{V}p\|+\|u\|}{(1+\mu)\overline{\gamma}-\gamma l}\right\}$$
$$+\sigma_n\|\gamma f(p)-\mu Fp\|$$
$$=(1-\sigma_n(\tau-\gamma l))\max\left\{\|x_n-p\|,\frac{\|\gamma f(p)-\overline{V}p\|+\|u\|}{(1+\mu)\overline{\gamma}-\gamma l}\right\}$$
$$+\sigma_n(\tau-\gamma l)\frac{\|\gamma f(p)-\mu Fp\|}{\tau-\gamma l}$$
$$\le\max\left\{\|x_n-p\|,\frac{\|\gamma f(p)-\overline{V}p\|+\|u\|}{(1+\mu)\overline{\gamma}-\gamma l},\frac{\|\gamma f(p)-\mu Fp\|}{\tau-\gamma l}\right\}. \tag{66}$$

By induction, we get

$$\|x_n-p\|\le\max\left\{\|x_0-p\|,\frac{\|\gamma f(p)-\overline{V}p\|+\|u\|}{(1+\mu)\overline{\gamma}-\gamma l},\frac{\|\gamma f(p)-\mu Fp\|}{\tau-\gamma l}\right\}. \tag{67}$$

Therefore, $\{x_n\}$ is bounded and so are the sequences $\{u_n\}$, $\{z_n\}$, $\{y_n\}$, $\{f(x_n)\}$, $\{f(y_n)\}$, $\{W_nx_n\}$, and $\{W_nz_n\}$.

Step 2. We show that $\|x_{n+1}-x_n\|\to 0$ as $n\to\infty$.

Indeed, define

$$y_n=\beta_nx_n+(1-\beta_n)w_n,\quad \forall n\ge 1. \tag{68}$$

Then from the definition of w_n, we obtain

$$\begin{aligned}
&w_{n+1}-w_n\\
&=\frac{y_{n+1}-\beta_{n+1}x_{n+1}}{1-\beta_{n+1}}-\frac{y_n-\beta_nx_n}{1-\beta_n}\\
&=\frac{\alpha_{n+1}(u+\gamma f(x_{n+1}))+((1-\beta_{n+1})I-\alpha_{n+1}\overline{V})W_{n+1}z_{n+1}}{1-\beta_{n+1}}\\
&\quad-\frac{\alpha_n(u+\gamma f(x_n))+((1-\beta_n)I-\alpha_n\overline{V})W_nz_n}{1-\beta_n}\\
&=\frac{\alpha_{n+1}}{1-\beta_{n+1}}(u+\gamma f(x_{n+1}))\\
&\quad-\frac{\alpha_n}{1-\beta_n}(u+\gamma f(x_n))+W_{n+1}z_{n+1}-W_nz_n\\
&\quad+\frac{\alpha_n}{1-\beta_n}\overline{V}W_nz_n-\frac{\alpha_{n+1}}{1-\beta_{n+1}}\overline{V}W_{n+1}z_{n+1}\\
&=\frac{\alpha_{n+1}}{1-\beta_{n+1}}\left[u+\gamma f(x_{n+1})-\overline{V}W_{n+1}z_{n+1}\right]\\
&\quad+\frac{\alpha_n}{1-\beta_n}\left[\overline{V}W_nz_n-u-\gamma f(x_n)\right]\\
&\quad+W_{n+1}z_{n+1}-W_{n+1}z_n+W_{n+1}z_n-W_nz_n.
\end{aligned} \tag{69}$$

It follows that

$$\begin{aligned}
&\|w_{n+1}-w_n\|\\
&\le\frac{\alpha_{n+1}}{1-\beta_{n+1}}\left(\|u\|+\|\gamma f(x_{n+1})\|+\|\overline{V}W_{n+1}z_{n+1}\|\right)\\
&\quad+\frac{\alpha_n}{1-\beta_n}\left(\|\overline{V}W_nz_n\|+\|u\|+\|\gamma f(x_n)\|\right)\\
&\quad+\|W_{n+1}z_{n+1}-W_{n+1}z_n\|+\|W_{n+1}z_n-W_nz_n\|\\
&\le\frac{\alpha_{n+1}}{1-\beta_{n+1}}\left(\|u\|+\|\gamma f(x_{n+1})\|+\|\overline{V}W_{n+1}z_{n+1}\|\right)\\
&\quad+\frac{\alpha_n}{1-\beta_n}\left(\|\overline{V}W_nz_n\|+\|u\|+\|\gamma f(x_n)\|\right)\\
&\quad+\|W_{n+1}z_n-W_nz_n\|+\|z_{n+1}-z_n\|.
\end{aligned} \tag{70}$$

From (15), since W_n, T_n, and $U_{n,i}$ are all nonexpansive, we have

$$\begin{aligned}
\|W_{n+1}z_n-W_nz_n\|&=\|\lambda_1T_1U_{n+1,2}z_n-\lambda_1T_1U_{n,2}z_n\|\\
&\le\lambda_1\|U_{n+1,2}z_n-U_{n,2}z_n\|\\
&=\lambda_1\|\lambda_2T_2U_{n+1,3}z_n-\lambda_2T_2U_{n,3}z_n\|\\
&\le\lambda_1\lambda_2\|U_{n+1,3}z_n-U_{n,3}z_n\|\\
&\ \ \vdots\\
&\le\lambda_1\lambda_2\cdots\lambda_n\|U_{n+1,n+1}z_n-U_{n,n+1}z_n\|\\
&\le M\prod_{i=1}^{n}\lambda_i,
\end{aligned} \tag{71}$$

$$\begin{aligned}
\|W_{n+1}y_n-W_ny_n\|&=\|\lambda_1T_1U_{n+1,2}y_n-\lambda_1T_1U_{n,2}y_n\|\\
&\le\lambda_1\|U_{n+1,2}y_n-U_{n,2}y_n\|\\
&=\lambda_1\|\lambda_2T_2U_{n+1,3}y_n-\lambda_2T_2U_{n,3}y_n\|\\
&\le\lambda_1\lambda_2\|U_{n+1,3}y_n-U_{n,3}y_n\|\\
&\ \ \vdots\\
&\le\lambda_1\lambda_2\cdots\lambda_n\|U_{n+1,n+1}y_n-U_{n,n+1}y_n\|\\
&\le M\prod_{i=1}^{n}\lambda_i,
\end{aligned} \tag{72}$$

where M is a constant such that

$$\begin{aligned}
&\sup_{n\ge1}\{\|U_{n+1,n+1}z_n\|+\|U_{n,n+1}z_n\|\}\le M,\\
&\sup_{n\ge1}\{\|U_{n+1,n+1}y_n\|+\|U_{n,n+1}y_n\|\}\le M.
\end{aligned} \tag{73}$$

On the other hand, we estimate $\|z_{n+1}-z_n\|$. Taking into consideration that $0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1$

and $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < 2\zeta$, we may assume, without loss of generality, that $\{r_n\} \subset [c,d] \subset (0,2\zeta)$ and $\{\beta_n\} \subset [e,f] \subset (0,1)$. Utilizing Remark 3 and Lemma 14, we have

$$\begin{aligned}
&\|z_{n+1}-z_n\|^2\\
&= \big\|J_{R_N,\mu_N}(I-\mu_N B_N)\Lambda^{N-1}u_{n+1}\\
&\qquad - J_{R_N,\mu_N}(I-\mu_N B_N)\Lambda^{N-1}u_n\big\|^2\\
&\leq \big\|(I-\mu_N B_N)\Lambda^{N-1}u_{n+1}-(I-\mu_N B_N)\Lambda^{N-1}u_n\big\|^2\\
&= \big\|\left(\Lambda^{N-1}u_{n+1}-\Lambda^{N-1}u_n\right)\\
&\qquad -\mu_N\left(B_N\Lambda^{N-1}u_{n+1}-B_N\Lambda^{N-1}u_n\right)\big\|^2\\
&\leq \big\|\Lambda^{N-1}u_{n+1}-\Lambda^{N-1}u_n\big\|^2+\mu_N(\mu_N-2\eta_N)\\
&\quad\times\big\|B_N\Lambda^{N-1}u_{n+1}-B_N\Lambda^{N-1}u_n\big\|^2\\
&\leq \big\|\Lambda^{N-1}u_{n+1}-\Lambda^{N-1}u_n\big\|^2\\
&\vdots\\
&\leq \big\|\Lambda^{0}u_{n+1}-\Lambda^{0}u_n\big\|^2\\
&= \|u_{n+1}-u_n\|^2,\\
&\|(I-r_{n+1}A)x_{n+1}-(I-r_nA)x_n\|\\
&= \|x_{n+1}-x_n-r_{n+1}(Ax_{n+1}-Ax_n)+(r_n-r_{n+1})Ax_n\|\\
&\leq \|x_{n+1}-x_n-r_{n+1}(Ax_{n+1}-Ax_n)\|+|r_{n+1}-r_n|\,\|Ax_n\|\\
&\leq \|x_{n+1}-x_n\|+|r_{n+1}-r_n|\,\|Ax_n\|,
\end{aligned}\tag{74}$$

$$\begin{aligned}
&\|u_{n+1}-u_n\|\\
&= \big\|S^{(\Theta,\varphi)}_{r_{n+1}}(I-r_{n+1}A)x_{n+1}-S^{(\Theta,\varphi)}_{r_n}(I-r_nA)x_n\big\|\\
&= \big\|S^{(\Theta,\varphi)}_{r_{n+1}}(I-r_{n+1}A)x_{n+1}-S^{(\Theta,\varphi)}_{r_{n+1}}(I-r_nA)x_n\\
&\qquad +S^{(\Theta,\varphi)}_{r_{n+1}}(I-r_nA)x_n-S^{(\Theta,\varphi)}_{r_n}(I-r_nA)x_n\big\|\\
&\leq \big\|S^{(\Theta,\varphi)}_{r_{n+1}}(I-r_{n+1}A)x_{n+1}-S^{(\Theta,\varphi)}_{r_{n+1}}(I-r_nA)x_n\big\|\\
&\quad +\big\|S^{(\Theta,\varphi)}_{r_{n+1}}(I-r_nA)x_n-S^{(\Theta,\varphi)}_{r_n}(I-r_nA)x_n\big\|\\
&\leq \|(I-r_{n+1}A)x_{n+1}-(I-r_nA)x_n\|\\
&\quad +\big\|S^{(\Theta,\varphi)}_{r_{n+1}}(I-r_nA)x_n-S^{(\Theta,\varphi)}_{r_n}(I-r_nA)x_n\big\|\\
&\leq \|x_{n+1}-x_n\|+|r_{n+1}-r_n|\,\|Ax_n\|\\
&\quad +\big\|S^{(\Theta,\varphi)}_{r_{n+1}}(I-r_nA)x_n-S^{(\Theta,\varphi)}_{r_n}(I-r_nA)x_n\big\|\\
&\leq \|x_{n+1}-x_n\|+|r_{n+1}-r_n|\,\|Ax_n\|\\
&\quad +\frac{|r_{n+1}-r_n|}{r_{n+1}}\cdot\frac{\nu}{\sigma}\\
&\quad\times\big\|S^{(\Theta,\varphi)}_{r_{n+1}}(I-r_nA)x_n-(I-r_nA)x_n\big\|\\
&\leq \|x_{n+1}-x_n\|+|r_{n+1}-r_n|\\
&\quad\times\Big(\|Ax_n\|\\
&\qquad +\frac{\nu}{c\sigma}\big\|S^{(\Theta,\varphi)}_{r_{n+1}}(I-r_nA)x_n-(I-r_nA)x_n\big\|\Big)\\
&\leq \|x_{n+1}-x_n\|+|r_{n+1}-r_n|\,M_1,
\end{aligned}\tag{75}$$

where $\sup_{n\geq1}\{\|Ax_n\|+(\nu/c\sigma)\|S^{(\Theta,\varphi)}_{r_{n+1}}(I-r_nA)x_n-(I-r_nA)x_n\|\}\leq M_1$ for some $M_1>0$.

Note that

$$\begin{aligned}
y_{n+1}-y_n &= \beta_n(x_{n+1}-x_n)+(\beta_{n+1}-\beta_n)(x_{n+1}-w_{n+1})\\
&\quad +(1-\beta_n)(w_{n+1}-w_n),\\
&x_{n+2}-x_{n+1}\\
&= \sigma_n\gamma\left(f(y_{n+1})-f(y_n)\right)+(\sigma_{n+1}-\sigma_n)\\
&\quad\times\left(\gamma f(y_{n+1})-\mu FW_{n+1}y_{n+1}\right)\\
&\quad +(I-\sigma_n\mu F)W_{n+1}y_{n+1}-(I-\sigma_n\mu F)W_ny_n.
\end{aligned}\tag{76}$$

Hence, from (70), (71), (74), and (75) it follows that

$$\begin{aligned}
&\|y_{n+1}-y_n\|\\
&\leq \beta_n\|x_{n+1}-x_n\|+|\beta_{n+1}-\beta_n|\,\|x_{n+1}-w_{n+1}\|\\
&\quad +(1-\beta_n)\|w_{n+1}-w_n\|\\
&\leq \beta_n\|x_{n+1}-x_n\|+|\beta_{n+1}-\beta_n|\,\|x_{n+1}-w_{n+1}\|\\
&\quad +(1-\beta_n)\Big[\frac{\alpha_{n+1}}{1-\beta_{n+1}}\Big(\|u\|+\|\gamma f(x_{n+1})\|\\
&\qquad\qquad +\big\|\overline{V}W_{n+1}z_{n+1}\big\|\Big)\\
&\qquad +\frac{\alpha_n}{1-\beta_n}\Big(\big\|\overline{V}W_nz_n\big\|+\|u\|+\|\gamma f(x_n)\|\Big)\\
&\qquad +\|W_{n+1}z_n-W_nz_n\|+\|z_{n+1}-z_n\|\Big]\\
&\leq \beta_n\|x_{n+1}-x_n\|+|\beta_{n+1}-\beta_n|\,\|x_{n+1}-w_{n+1}\|\\
&\quad +(1-\beta_n)\Big[\frac{\alpha_{n+1}}{1-\beta_{n+1}}\Big(\|u\|+\|\gamma f(x_{n+1})\|
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad + \left\| \overline{V} W_{n+1} z_{n+1} \right\| \Big) \\
&\qquad + \frac{\alpha_n}{1-\beta_n} \left(\left\| \overline{V} W_n z_n \right\| + \|u\| + \|\gamma f(x_n)\| \right) \\
&\qquad + M \prod_{i=1}^{n} \lambda_i + \|z_{n+1} - z_n\| \Big] \\
&\leq \beta_n \|x_{n+1} - x_n\| + |\beta_{n+1} - \beta_n| \|x_{n+1} - w_{n+1}\| \\
&\quad + (1-\beta_n) \Big[\frac{\alpha_{n+1}}{1-\beta_{n+1}} \Big(\|u\| + \|\gamma f(x_{n+1})\| \\
&\qquad\qquad + \left\| \overline{V} W_{n+1} z_{n+1} \right\| \Big) \\
&\qquad + \frac{\alpha_n}{1-\beta_n} \left(\left\| \overline{V} W_n z_n \right\| + \|u\| + \|\gamma f(x_n)\| \right) \\
&\qquad + M \prod_{i=1}^{n} \lambda_i + \|u_{n+1} - u_n\| \Big] \\
&\leq \beta_n \|x_{n+1} - x_n\| + |\beta_{n+1} - \beta_n| \|x_{n+1} - w_{n+1}\| \\
&\quad + (1-\beta_n) \Big[\frac{\alpha_{n+1}}{1-\beta_{n+1}} \Big(\|u\| + \|\gamma f(x_{n+1})\| \\
&\qquad\qquad + \left\| \overline{V} W_{n+1} z_{n+1} \right\| \Big) \\
&\qquad + \frac{\alpha_n}{1-\beta_n} \left(\left\| \overline{V} W_n z_n \right\| + \|u\| + \|\gamma f(x_n)\| \right) \\
&\qquad + M \prod_{i=1}^{n} \lambda_i + \|x_{n+1} - x_n\| \\
&\qquad + |r_{n+1} - r_n| M_1 \Big] \\
&\leq \|x_{n+1} - x_n\| + |\beta_{n+1} - \beta_n| M_2 \\
&\quad + M_2 (\alpha_{n+1} + \alpha_n) + M_2 \prod_{i=1}^{n} \lambda_i + |r_{n+1} - r_n| M_2,
\end{aligned} \tag{77}
$$

where $\sup_{n\geq 1}\{(1/(1-f))(\|\overline{V}W_n z_n\| + \|u\| + \|\gamma f(x_n)\|) + \|x_n - w_n\| + M_1 + M\} \leq M_2$ for some $M_2 > 0$. So, utilizing Lemma 13 we obtain from (72) and (77) that

$$
\begin{aligned}
&\|x_{n+2} - x_{n+1}\| \\
&\leq \sigma_n \gamma \|f(y_{n+1}) - f(y_n)\| + |\sigma_{n+1} - \sigma_n| \\
&\quad \times \|\gamma f(y_{n+1}) - \mu F W_{n+1} y_{n+1}\| \\
&\quad + \|(I - \sigma_n \mu F) W_{n+1} y_{n+1} - (I - \sigma_n \mu F) W_n y_n\| \\
&\leq \sigma_n \gamma l \|y_{n+1} - y_n\| + |\sigma_{n+1} - \sigma_n| \\
&\quad \times \|\gamma f(y_{n+1}) - \mu F W_{n+1} y_{n+1}\| \\
&\quad + (1 - \sigma_n \tau) \|W_{n+1} y_{n+1} - W_n y_n\| \\
&\leq \sigma_n \gamma l \|y_{n+1} - y_n\| + |\sigma_{n+1} - \sigma_n| \\
&\quad \times \|\gamma f(y_{n+1}) - \mu F W_{n+1} y_{n+1}\| \\
&\quad + (1 - \sigma_n \tau) (\|W_{n+1} y_{n+1} - W_{n+1} y_n\| \\
&\qquad\qquad + \|W_{n+1} y_n - W_n y_n\|) \\
&\leq \sigma_n \gamma l \|y_{n+1} - y_n\| + |\sigma_{n+1} - \sigma_n| \\
&\quad \times \|\gamma f(y_{n+1}) - \mu F W_{n+1} y_{n+1}\| \\
&\quad + (1 - \sigma_n \tau) \left(\|y_{n+1} - y_n\| + M \prod_{i=1}^{n} \lambda_i \right) \\
&\leq (1 - \sigma_n (\tau - \gamma l)) \|y_{n+1} - y_n\| + |\sigma_{n+1} - \sigma_n| \\
&\quad \times \|\gamma f(y_{n+1}) - \mu F W_{n+1} y_{n+1}\| + M \prod_{i=1}^{n} \lambda_i \\
&\leq (1 - \sigma_n (\tau - \gamma l)) \\
&\quad \times \Big[\|x_{n+1} - x_n\| \\
&\qquad + |\beta_{n+1} - \beta_n| M_2 + M_2 (\alpha_{n+1} + \alpha_n) \\
&\qquad + M_2 \prod_{i=1}^{n} \lambda_i \\
&\qquad + |r_{n+1} - r_n| M_2 \Big] + |\sigma_{n+1} - \sigma_n| \\
&\quad \times \|\gamma f(y_{n+1}) - \mu F W_{n+1} y_{n+1}\| + M \prod_{i=1}^{n} \lambda_i \\
&\leq (1 - \sigma_n (\tau - \gamma l)) \|x_{n+1} - x_n\| \\
&\quad + |\beta_{n+1} - \beta_n| M_3 + M_3 (\alpha_{n+1} + \alpha_n) + M_3 \prod_{i=1}^{n} \lambda_i \\
&\quad + |r_{n+1} - r_n| M_3 + |\sigma_{n+1} - \sigma_n| M_3 \\
&\quad + M_3 \prod_{i=1}^{n} \lambda_i \\
&= (1 - \sigma_n (\tau - \gamma l)) \|x_{n+1} - x_n\| \\
&\quad + (|\beta_{n+1} - \beta_n| + |r_{n+1} - r_n| + |\sigma_{n+1} - \sigma_n|) M_3 \\
&\quad + 2M_3 \prod_{i=1}^{n} \lambda_i + M_3 (\alpha_{n+1} + \alpha_n) \\
&\leq (1 - \sigma_n (\tau - \gamma l)) \|x_{n+1} - x_n\| \\
&\quad + (|\beta_{n+1} - \beta_n| + |r_{n+1} - r_n| + |\sigma_{n+1} - \sigma_n|) M_3 \\
&\quad + 2M_3 b^n \\
&\quad + \sigma_n (\tau - \gamma l) \frac{M_3}{\tau - \gamma l} \cdot \frac{\alpha_n}{\sigma_n} \left(\frac{\alpha_{n+1}}{\alpha_n} + 1 \right),
\end{aligned} \tag{78}
$$

where $\sup_{n\geq1}\{\|\gamma f(y_n) - \mu F W_n y_n\| + M_2 + M\} \leq M_3$ for some $M_3 > 0$. Applying Lemma 19 to (78), we deduce from conditions (iii) and (v) that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{79}$$

Step 3. $\| x_n - y_n \| \to 0$ as $n \to \infty$. Indeed, utilizing Lemmas 4 and 5(b) we obtain from (56) and (64) that

$$\begin{aligned}
&\|y_n - p\|^2 \\
&= \|\alpha_n ((u + \gamma f(x_n)) - (I + \mu V) W_n z_n) \\
&\qquad + \beta_n (x_n - p) + (1 - \beta_n)(W_n z_n - p)\|^2 \\
&\leq \|\beta_n (x_n - p) + (1 - \beta_n)(W_n z_n - p)\|^2 \\
&\quad + 2\alpha_n \langle (u + \gamma f(x_n)) - (I + \mu V) W_n z_n, y_n - p\rangle \\
&= \beta_n \|x_n - p\|^2 + (1 - \beta_n)\|W_n z_n - p\|^2 \\
&\quad - \beta_n (1 - \beta_n)\|x_n - W_n z_n\|^2 \\
&\quad + 2\alpha_n \|(u + \gamma f(x_n)) - (I + \mu V) W_n z_n\| \|y_n - p\| \\
&\leq \beta_n \|x_n - p\|^2 + (1 - \beta_n)\|z_n - p\|^2 \\
&\quad - \beta_n (1 - \beta_n)\|x_n - W_n z_n\|^2 \\
&\quad + 2\alpha_n \|(u + \gamma f(x_n)) - (I + \mu V) W_n z_n\| \|y_n - p\| \\
&\leq \beta_n \|x_n - p\|^2 + (1 - \beta_n)\|x_n - p\|^2 \\
&\quad - \beta_n (1 - \beta_n)\|x_n - W_n z_n\|^2 \\
&\quad + 2\alpha_n \|(u + \gamma f(x_n)) - (I + \mu V) W_n z_n\| \|y_n - p\| \\
&= \|x_n - p\|^2 - \beta_n (1 - \beta_n)\|x_n - W_n z_n\|^2 \\
&\quad + 2\alpha_n \|(u + \gamma f(x_n)) - (I + \mu V) W_n z_n\| \|y_n - p\|.
\end{aligned} \tag{80}$$

Utilizing Lemmas 4 and 13, we conclude from (56) and (80) that

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&= \|\sigma_n \gamma (f(y_n) - f(p)) + (I - \sigma_n \mu F) W_n y_n \\
&\qquad - (I - \sigma_n \mu F) W_n p + \sigma_n (\gamma f(p) - \mu F p)\|^2 \\
&\leq \|\sigma_n \gamma (f(y_n) - f(p)) + (I - \sigma_n \mu F) W_n y_n \\
&\qquad - (I - \sigma_n \mu F) W_n p\|^2 \\
&\quad + 2\sigma_n \langle \gamma f(p) - \mu F p, x_{n+1} - p\rangle \\
&\leq [\sigma_n \gamma l \|y_n - p\| + (1 - \sigma_n \tau)\|y_n - p\|]^2 \\
&\quad + 2\sigma_n \langle \gamma f(p) - \mu F p, x_{n+1} - p\rangle \\
&= (1 - \sigma_n (\tau - \gamma l))^2 \|y_n - p\|^2 \\
&\quad + 2\sigma_n \|\gamma f(p) - \mu F p\| \|x_{n+1} - p\| \\
&\leq \|y_n - p\|^2 + 2\sigma_n \|\gamma f(p) - \mu F p\| \|x_{n+1} - p\| \\
&\leq \|x_n - p\|^2 - \beta_n (1 - \beta_n)\|x_n - W_n z_n\|^2 \\
&\quad + 2\alpha_n \|(u + \gamma f(x_n)) - (I + \mu V) W_n z_n\| \|y_n - p\| \\
&\quad + 2\sigma_n \|\gamma f(p) - \mu F p\| \|x_{n+1} - p\|,
\end{aligned} \tag{81}$$

which leads to

$$\begin{aligned}
&e(1 - f)\|x_n - W_n z_n\|^2 \\
&\leq \beta_n (1 - \beta_n)\|x_n - W_n z_n\|^2 \\
&\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\quad + 2\alpha_n \|(u + \gamma f(x_n)) - (I + \mu V) W_n z_n\| \|y_n - p\| \\
&\quad + 2\sigma_n \|\gamma f(p) - \mu F p\| \|x_{n+1} - p\| \\
&\leq \|x_n - x_{n+1}\| (\|x_n - p\| + \|x_{n+1} - p\|) \\
&\quad + 2\alpha_n \|(u + \gamma f(x_n)) - (I + \mu V) W_n z_n\| \|y_n - p\| \\
&\quad + 2\sigma_n \|\gamma f(p) - \mu F p\| \|x_{n+1} - p\|.
\end{aligned} \tag{82}$$

Since $\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0$, $\lim_{n\to\infty}\alpha_n = 0$, and $\lim_{n\to\infty}\sigma_n = 0$, we deduce from the boundedness of $\{x_n\}$, $\{y_n\}$, $\{f(x_n)\}$ and $\{W_n z_n\}$ that

$$\lim_{n\to\infty} \|x_n - W_n z_n\| = 0. \tag{83}$$

Note that

$$\begin{aligned}
&\|y_n - x_n\| \\
&= \|\alpha_n ((u + \gamma f(x_n)) - (I + \mu V) W_n z_n) \\
&\qquad + (1 - \beta_n)(W_n z_n - x_n)\| \\
&\leq \alpha_n \|(u + \gamma f(x_n)) - (I + \mu V) W_n z_n\| \\
&\quad + (1 - \beta_n)\|W_n z_n - x_n\| \\
&\leq \alpha_n \|(u + \gamma f(x_n)) - (I + \mu V) W_n z_n\| \\
&\quad + \|W_n z_n - x_n\|.
\end{aligned} \tag{84}$$

So, it follows from (83) and $\lim_{n\to\infty}\alpha_n = 0$ that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{85}$$

Step 4. $\|x_n - u_n\| \to 0$, $\|u_n - z_n\| \to 0$ and $\|z_n - Wz_n\| \to 0$ as $n \to \infty$. Indeed, for $p \in \Omega$, we find that

$$\begin{aligned}\|u_n - p\|^2 &= \left\|S_{r_n}^{(\Theta,\varphi)}(I - r_n A)x_n - S_{r_n}^{(\Theta,\varphi)}(I - r_n A)p\right\|^2 \\ &\le \|(I - r_n A)x_n - (I - r_n A)p\|^2 \\ &= \|x_n - p - r_n(Ax_n - Ap)\|^2 \\ &\le \|x_n - p\|^2 + r_n(r_n - 2\zeta)\|Ax_n - Ap\|^2.\end{aligned} \tag{86}$$

From (56), (63), and (86), we obtain

$$\begin{aligned}\|y_n - p\|^2 &= \|\alpha_n((u + \gamma f(x_n)) - (I + \mu V)W_n z_n) \\ &\quad + \beta_n(x_n - p) + (1 - \beta_n)(W_n z_n - p)\|^2 \\ &\le \|\beta_n(x_n - p) + (1 - \beta_n)(W_n z_n - p)\|^2 \\ &\quad + 2\alpha_n\langle (u + \gamma f(x_n)) - (I + \mu V)W_n z_n, y_n - p\rangle \\ &\le \beta_n\|x_n - p\|^2 + (1 - \beta_n)\|W_n z_n - p\|^2 \\ &\quad + 2\alpha_n\|(u + \gamma f(x_n)) - (I + \mu V)W_n z_n\|\|y_n - p\| \\ &\le \beta_n\|x_n - p\|^2 + (1 - \beta_n)\|z_n - p\|^2 \\ &\quad + 2\alpha_n\|(u + \gamma f(x_n)) - (I + \mu V)W_n z_n\|\|y_n - p\| \\ &\le \beta_n\|x_n - p\|^2 + (1 - \beta_n)\|u_n - p\|^2 \\ &\quad + 2\alpha_n\|(u + \gamma f(x_n)) - (I + \mu V)W_n z_n\|\|y_n - p\| \\ &\le \beta_n\|x_n - p\|^2 + (1 - \beta_n) \\ &\quad \times \left[\|x_n - p\|^2 + r_n(r_n - 2\zeta)\|Ax_n - Ap\|^2\right] \\ &\quad + 2\alpha_n\|(u + \gamma f(x_n)) - (I + \mu V)W_n z_n\| \\ &\quad \times \|y_n - p\| \\ &= \|x_n - p\|^2 + (1 - \beta_n)r_n(r_n - 2\zeta)\|Ax_n - Ap\|^2 \\ &\quad + 2\alpha_n\|(u + \gamma f(x_n)) - (I + \mu V)W_n z_n\|\|y_n - p\|,\end{aligned} \tag{87}$$

which immediately implies that

$$\begin{aligned}&(1 - f)c(2\zeta - d)\|Ax_n - Ap\|^2 \\ &\le (1 - \beta_n)r_n(2\zeta - r_n)\|Ax_n - Ap\|^2 \\ &\le \|x_n - p\|^2 - \|y_n - p\|^2 \\ &\quad + 2\alpha_n\|(u + \gamma f(x_n)) - (I + \mu V)W_n z_n\|\|y_n - p\| \\ &\le \|x_n - y_n\|(\|x_n - p\| + \|y_n - p\|) \\ &\quad + 2\alpha_n\|(u + \gamma f(x_n)) - (I + \mu V)W_n z_n\|\|y_n - p\|.\end{aligned} \tag{88}$$

From $\lim_{n\to\infty}\alpha_n = 0$ and $\lim_{n\to\infty}\|x_n - y_n\| = 0$, we have

$$\lim_{n\to\infty}\|Ax_n - Ap\| = 0. \tag{89}$$

Furthermore, from the firm nonexpansivity of $S_{r_n}^{(\Theta,\varphi)}$, we have

$$\begin{aligned}\|u_n - p\|^2 &= \left\|S_{r_n}^{(\Theta,\varphi)}(I - r_n A)x_n - S_{r_n}^{(\Theta,\varphi)}(I - r_n A)p\right\|^2 \\ &\le \langle (I - r_n A)x_n - (I - r_n A)p, u_n - p\rangle \\ &= \frac{1}{2}\left[\|(I - r_n A)x_n - (I - r_n A)p\|^2 + \|u_n - p\|^2\right. \\ &\qquad \left. - \|(I - r_n A)x_n - (I - r_n A)p - (u_n - p)\|^2\right] \\ &\le \frac{1}{2}\left[\|x_n - p\|^2 + \|u_n - p\|^2\right. \\ &\qquad \left. - \|x_n - u_n - r_n(Ax_n - Ap)\|^2\right] \\ &= \frac{1}{2}\left[\|x_n - p\|^2 + \|u_n - p\|^2 - \|x_n - u_n\|^2\right. \\ &\qquad \left. + 2r_n\langle Ax_n - Ap, x_n - u_n\rangle - r_n^2\|Ax_n - Ap\|^2\right],\end{aligned} \tag{90}$$

which implies that

$$\begin{aligned}\|u_n - p\|^2 \le{}& \|x_n - p\|^2 - \|x_n - u_n\|^2 \\ &+ 2r_n\|Ax_n - Ap\|\|x_n - u_n\|.\end{aligned} \tag{91}$$

From (87) and (91), we have

$$\begin{aligned}\|y_n - p\|^2 &\le \beta_n\|x_n - p\|^2 + (1 - \beta_n)\|u_n - p\|^2 \\ &\quad + 2\alpha_n\|(u + \gamma f(x_n)) - (I + \mu V)W_n z_n\|\|y_n - p\| \\ &\le \beta_n\|x_n - p\|^2 \\ &\quad + (1 - \beta_n)\left[\|x_n - p\|^2\right. \\ &\qquad \left. - \|x_n - u_n\|^2 + 2r_n\|Ax_n - Ap\|\|x_n - u_n\|\right] \\ &\quad + 2\alpha_n\|(u + \gamma f(x_n)) - (I + \mu V)W_n z_n\|\|y_n - p\| \\ &\le \|x_n - p\|^2 - (1 - \beta_n)\|x_n - u_n\|^2 \\ &\quad + 2r_n\|Ax_n - Ap\|\|x_n - u_n\| \\ &\quad + 2\alpha_n\|(u + \gamma f(x_n)) - (I + \mu V)W_n z_n\|\|y_n - p\|.\end{aligned} \tag{92}$$

It follows that

$$\begin{aligned}
&(1-f)\|x_n-u_n\|^2\\
&\le(1-\beta_n)\|x_n-u_n\|^2\\
&\le\|x_n-p\|^2-\|y_n-p\|^2\\
&\quad+2r_n\|Ax_n-Ap\|\|x_n-u_n\|\\
&\quad+2\alpha_n\|(u+\gamma f(x_n))-(I+\mu V)W_nz_n\|\|y_n-p\|\\
&\le\|x_n-y_n\|(\|x_n-p\|+\|y_n-p\|)\\
&\quad+2d\|Ax_n-Ap\|\|x_n-u_n\|\\
&\quad+2\alpha_n\|(u+\gamma f(x_n))-(I+\mu V)W_nz_n\|\|y_n-p\|.
\end{aligned}\tag{93}$$

So, from (85), (89), and $\lim_{n\to\infty}\alpha_n=0$, we obtain

$$\lim_{n\to\infty}\|x_n-u_n\|=0.\tag{94}$$

Next we show that $\lim_{n\to\infty}\|A_i\Lambda^iu_n-A_ip\|=0$, $i=1,2,\ldots,N$. Observe that

$$\begin{aligned}
&\|\Lambda^iu_n-p\|^2\\
&=\|J_{R_i,\mu_i}(I-\mu_iB_i)\Lambda^{i-1}u_n-J_{R_i,\mu_i}(I-\mu_iB_i)p\|^2\\
&\le\|(I-\mu_iB_i)\Lambda^{i-1}u_n-(I-\mu_iB_i)p\|^2\\
&\le\|\Lambda^{i-1}u_n-p\|^2+\mu_i(\mu_i-2\eta_i)\\
&\quad\times\|B_i\Lambda^{i-1}u_n-B_ip\|^2\\
&\le\|u_n-p\|^2+\mu_i(\mu_i-2\eta_i)\|B_i\Lambda^{i-1}u_n-B_ip\|^2\\
&\le\|x_n-p\|^2+\mu_i(\mu_i-2\eta_i)\|B_i\Lambda^{i-1}u_n-B_ip\|^2.
\end{aligned}\tag{95}$$

Combining (87) and (95), we have

$$\begin{aligned}
&\|y_n-p\|^2\\
&\le\beta_n\|x_n-p\|^2+(1-\beta_n)\|z_n-p\|^2\\
&\quad+2\alpha_n\|(u+\gamma f(x_n))-(I+\mu V)W_nz_n\|\|y_n-p\|\\
&\le\beta_n\|x_n-p\|^2+(1-\beta_n)\|\Lambda^iu_n-p\|^2\\
&\quad+2\alpha_n\|(u+\gamma f(x_n))-(I+\mu V)W_nz_n\|\|y_n-p\|\\
&\le\beta_n\|x_n-p\|^2+(1-\beta_n)\\
&\quad\times\left[\|x_n-p\|^2+\mu_i(\mu_i-2\eta_i)\|B_i\Lambda^{i-1}u_n-B_ip\|^2\right]\\
&\quad+2\alpha_n\|(u+\gamma f(x_n))-(I+\mu V)W_nz_n\|\|y_n-p\|\\
&=\|x_n-p\|^2+(1-\beta_n)\mu_i(\mu_i-2\eta_i)\\
&\quad\times\|B_i\Lambda^{i-1}u_n-B_ip\|^2\\
&\quad+2\alpha_n\|(u+\gamma f(x_n))-(I+\mu V)W_nz_n\|\|y_n-p\|,
\end{aligned}\tag{96}$$

which leads to

$$\begin{aligned}
&(1-f)\mu_i(2\eta_i-\mu_i)\|B_i\Lambda^{i-1}u_n-B_ip\|^2\\
&\le(1-\beta_n)\mu_i(2\eta_i-\mu_i)\|B_i\Lambda^{i-1}u_n-B_ip\|^2\\
&\le\|x_n-p\|^2-\|y_n-p\|^2\\
&\quad+2\alpha_n\|(u+\gamma f(x_n))-(I+\mu V)W_nz_n\|\|y_n-p\|\\
&\le\|x_n-y_n\|(\|x_n-p\|+\|y_n-p\|)\\
&\quad+2\alpha_n\|(u+\gamma f(x_n))-(I+\mu V)W_nz_n\|\|y_n-p\|.
\end{aligned}\tag{97}$$

Since $\mu_i\in(0,2\eta_i)$, $i=1,2,\ldots,N$ and $\lim_{n\to\infty}\alpha_n=0$, we obtain from (85) that

$$\lim_{n\to\infty}\|B_i\Lambda^{i-1}u_n-B_ip\|=0,\quad\forall i\in\{1,2,\ldots,N\}.\tag{98}$$

By Lemmas 5(a) and (14), we obtain

$$\begin{aligned}
&\|\Lambda^iu_n-p\|^2\\
&=\|J_{R_i,\mu_i}(I-\mu_iB_i)\Lambda^{i-1}u_n-J_{R_i,\mu_i}(I-\mu_iB_i)p\|^2\\
&\le\left\langle(I-\mu_iB_i)\Lambda^{i-1}u_n-(I-\mu_iB_i)p,\Lambda^iu_n-p\right\rangle\\
&=\frac{1}{2}\Big(\|(I-\mu_iB_i)\Lambda^{i-1}u_n-(I-\mu_iB_i)p\|^2\\
&\qquad+\|\Lambda^iu_n-p\|^2-\|(I-\mu_iB_i)\Lambda^{i-1}u_n\\
&\qquad\quad-(I-\mu_iB_i)p-(\Lambda^iu_n-p)\|^2\Big)\\
&\le\frac{1}{2}\Big(\|\Lambda^{i-1}u_n-p\|^2+\|\Lambda^iu_n-p\|^2\\
&\qquad-\|\Lambda^{i-1}u_n-\Lambda^iu_n-\mu_i(B_i\Lambda^{i-1}u_n-B_ip)\|^2\Big)\\
&\le\frac{1}{2}\Big(\|u_n-p\|^2+\|\Lambda^iu_n-p\|^2\\
&\qquad-\|\Lambda^{i-1}u_n-\Lambda^iu_n-\mu_i(B_i\Lambda^{i-1}u_n-B_ip)\|^2\Big)\\
&\le\frac{1}{2}\Big(\|x_n-p\|^2+\|\Lambda^iu_n-p\|^2\\
&\qquad-\|\Lambda^{i-1}u_n-\Lambda^iu_n-\mu_i(B_i\Lambda^{i-1}u_n-B_ip)\|^2\Big),
\end{aligned}\tag{99}$$

which implies

$$\begin{aligned} &\left\|\Lambda^i u_n - p\right\|^2 \\ &\leq \left\|x_n - p\right\|^2 \\ &\quad - \left\|\Lambda^{i-1} u_n - \Lambda^i u_n - \mu_i \left(B_i \Lambda^{i-1} u_n - B_i p\right)\right\|^2 \\ &= \left\|x_n - p\right\|^2 - \left\|\Lambda^{i-1} u_n - \Lambda^i u_n\right\|^2 \\ &\quad - \mu_i^2 \left\|B_i \Lambda^{i-1} u_n - B_i p\right\|^2 \\ &\quad + 2\mu_i \left\langle \Lambda^{i-1} u_n - \Lambda^i u_n, B_i \Lambda^{i-1} u_n - B_i p\right\rangle \\ &\leq \left\|x_n - p\right\|^2 - \left\|\Lambda^{i-1} u_n - \Lambda^i u_n\right\|^2 \\ &\quad + 2\mu_i \left\|\Lambda^{i-1} u_n - \Lambda^i u_n\right\| \left\|B_i \Lambda^{i-1} u_n - B_i p\right\|. \end{aligned} \tag{100}$$

Combining (87) and (100), we have

$$\begin{aligned} &\left\|y_n - p\right\|^2 \\ &\leq \beta_n \left\|x_n - p\right\|^2 + (1 - \beta_n)\left\|z_n - p\right\|^2 \\ &\quad + 2\alpha_n \left\|(u + \gamma f(x_n)) - (I + \mu V) W_n z_n\right\| \left\|y_n - p\right\| \\ &\leq \beta_n \left\|x_n - p\right\|^2 + (1 - \beta_n)\left\|\Lambda^i u_n - p\right\|^2 \\ &\quad + 2\alpha_n \left\|(u + \gamma f(x_n)) - (I + \mu V) W_n z_n\right\| \left\|y_n - p\right\| \\ &\leq \beta_n \left\|x_n - p\right\|^2 + (1 - \beta_n) \\ &\quad \times \Big[\left\|x_n - p\right\|^2 - \left\|\Lambda^{i-1} u_n - \Lambda^i u_n\right\|^2 \\ &\qquad + 2\mu_i \left\|\Lambda^{i-1} u_n - \Lambda^i u_n\right\| \left\|B_i \Lambda^{i-1} u_n - B_i p\right\|\Big] \\ &\quad + 2\alpha_n \left\|(u + \gamma f(x_n)) - (I + \mu V) W_n z_n\right\| \left\|y_n - p\right\| \\ &\leq \left\|x_n - p\right\|^2 - (1 - \beta_n)\left\|\Lambda^{i-1} u_n - \Lambda^i u_n\right\|^2 \\ &\quad + 2\mu_i \left\|\Lambda^{i-1} u_n - \Lambda^i u_n\right\| \left\|B_i \Lambda^{i-1} u_n - B_i p\right\| \\ &\quad + 2\alpha_n \left\|(u + \gamma f(x_n)) - (I + \mu V) W_n z_n\right\| \left\|y_n - p\right\|, \end{aligned} \tag{101}$$

which yields

$$\begin{aligned} &(1 - f)\left\|\Lambda^{i-1} u_n - \Lambda^i u_n\right\|^2 \\ &\leq (1 - \beta_n)\left\|\Lambda^{i-1} u_n - \Lambda^i u_n\right\|^2 \\ &\leq \left\|x_n - p\right\|^2 - \left\|y_n - p\right\|^2 \\ &\quad + 2\mu_i \left\|\Lambda^{i-1} u_n - \Lambda^i u_n\right\| \left\|B_i \Lambda^{i-1} u_n - B_i p\right\| \\ &\quad + 2\alpha_n \left\|(u + \gamma f(x_n)) - (I + \mu V) W_n z_n\right\| \\ &\quad \times \left\|y_n - p\right\| \\ &\leq \left\|x_n - y_n\left(\left\|x_n - p\right\| + \left\|y_n - p\right\|\right) + 2\mu_i\right\| \\ &\quad \times \left\|\Lambda^{i-1} u_n - \Lambda^i u_n\right\| \left\|B_i \Lambda^{i-1} u_n - B_i p\right\| \\ &\quad + 2\alpha_n \left\|(u + \gamma f(x_n)) - (I + \mu V) W_n z_n\right\| \\ &\quad \times \left\|y_n - p\right\|. \end{aligned} \tag{102}$$

From (85), (98), and $\lim_{n\to\infty}\alpha_n = 0$, we get

$$\lim_{n\to\infty}\left\|\Lambda^{i-1} u_n - \Lambda^i u_n\right\| = 0, \quad \forall i \in \{1, 2, \ldots, N\}. \tag{103}$$

From (103) we get

$$\begin{aligned} &\left\|u_n - z_n\right\| \\ &\quad = \left\|\Lambda^0 u_n - \Lambda^N u_n\right\| \\ &\quad \leq \left\|\Lambda^0 u_n - \Lambda^1 u_n\right\| + \left\|\Lambda^1 u_n - \Lambda^2 u_n\right\| \\ &\qquad + \cdots + \left\|\Lambda^{N-1} u_n - \Lambda^N u_n\right\| \\ &\quad \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{104}$$

By (94) and (104), we have

$$\begin{aligned} \left\|x_n - z_n\right\| &\leq \left\|x_n - u_n\right\| + \left\|u_n - z_n\right\| \\ &\longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \end{aligned} \tag{105}$$

together with (85), yields

$$\begin{aligned} \left\|y_n - z_n\right\| &\leq \left\|y_n - x_n\right\| + \left\|x_n - z_n\right\| \\ &\longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{106}$$

Note that

$$\left\|z_n - W_n z_n\right\| \leq \left\|y_n - z_n\right\| + \left\|x_n - W_n z_n\right\|. \tag{107}$$

Hence from (83) and (106) we have

$$\lim_{n\to\infty}\left\|z_n - W_n z_n\right\| = 0. \tag{108}$$

Also, observe that

$$\left\|z_n - W z_n\right\| \leq \left\|z_n - W_n z_n\right\| + \left\|W_n z_n - W z_n\right\|. \tag{109}$$

From (108), Remark 8, and the boundedness of $\{z_n\}$ we immediately get

$$\lim_{n\to\infty}\left\|z_n - W z_n\right\| = 0. \tag{110}$$

Step 5. We show that $\limsup_{n\to\infty}\langle(\gamma f - \mu F)x^*, x_n - x^*\rangle \leq 0$, where $x^* \in \Omega$ is a unique solution of the VIP:

$$\left\langle(\gamma f - \mu F)x^*, y - x^*\right\rangle \leq 0, \quad \forall y \in \Omega. \tag{111}$$

Indeed, it is clear that $\mu F - \gamma f : H \to H$ is $(\mu\kappa + \gamma l)$-Lipschitzian. Note that

$$\begin{aligned} &\mu\eta \geq \tau \\ &\Longleftrightarrow \mu\eta \geq 1 - \sqrt{1 - \mu\left(2\eta - \mu\kappa^2\right)} \\ &\Longleftrightarrow \sqrt{1 - \mu\left(2\eta - \mu\kappa^2\right)} \geq 1 - \mu\eta \\ &\Longleftrightarrow 1 - 2\mu\eta + \mu^2\kappa^2 \geq 1 - 2\mu\eta + \mu^2\eta^2 \\ &\Longleftrightarrow \kappa^2 \geq \eta^2 \\ &\Longleftrightarrow \kappa \geq \eta. \end{aligned} \tag{112}$$

It is clear that

$$\langle (\mu F - \gamma f) x - (\mu F - \gamma f) y, x - y \rangle \geq (\mu\eta - \gamma l) \|x - y\|^2 \quad \forall x, y \in H. \tag{113}$$

Hence by Lemma 12 we deduce from $0 \leq \gamma l < \tau \leq \mu\eta$ that $\mu F - \gamma f$ is $(\mu\eta - \gamma l)$-strongly monotone. In the meantime, it is easy to see that $\mu F - \gamma f$ is $(\mu\kappa + \gamma l)$-Lipschitzian with the constant $\mu\kappa + \gamma l > 0$. Thus, there exists a unique solution x^* in Ω to the VIP (111). Equivalently, $P_\Omega(I - (\mu F - \gamma f))$ has a unique fixed point $x^* \in \Omega$; that is, $x^* = P_\Omega(I - (\mu F - \gamma f))x^*$.

First, we observe that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\begin{aligned} &\limsup_{n\to\infty} \langle (\gamma f - \mu F) x^*, x_n - x^* \rangle \\ &\quad = \lim_{i\to\infty} \left\langle (\gamma f - \mu F) x^*, x_{n_i} - x^* \right\rangle. \end{aligned} \tag{114}$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to some w. Without loss of generality, we may assume that $x_{n_i} \rightharpoonup w$. From (94) and (103)–(105), we have that $u_{n_i} \rightharpoonup w$, $\Lambda^m u_{n_i} \rightharpoonup w$ and $z_{n_i} \rightharpoonup w$, where $m \in \{1, 2, \ldots, N\}$. By (110) we have that $\|Wz_n - z_n\| \to 0$ as $n \to \infty$. Then, by Lemma 10 we obtain $w \in \mathrm{Fix}(W) = \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$ (due to Lemma 9). Next, we prove that $w \in \bigcap_{m=1}^{N} I(B_m, R_m)$. As a matter of fact, since B_m is η_m-inverse strongly monotone, B_m is a monotone and Lipschitz continuous mapping. It follows from Lemma 17 that $R_m + B_m$ is maximal monotone. Let $(v, g) \in G(R_m + B_m)$; that is, $g - B_m v \in R_m v$. Again, since $\Lambda^m u_n = J_{R_m,\mu_m}(I - \mu_m B_m)\Lambda^{m-1} u_n$, $n \geq 1$, $m \in \{1, 2, \ldots, N\}$, we have

$$\Lambda^{m-1} u_n - \mu_m B_m \Lambda^{m-1} u_n \in \left(I + \mu_m R_m\right) \Lambda^m u_n; \tag{115}$$

that is,

$$\frac{1}{\mu_m} \left(\Lambda^{m-1} u_n - \Lambda^m u_n - \mu_m B_m \Lambda^{m-1} u_n\right) \in R_m \Lambda^m u_n. \tag{116}$$

In terms of the monotonicity of R_m, we get

$$\begin{aligned} &\Big\langle v - \Lambda^m u_n, g - B_m v \\ &\quad - \frac{1}{\mu_m} \left(\Lambda^{m-1} u_n - \Lambda^m u_n - \mu_m B_m \Lambda^{m-1} u_n\right) \Big\rangle \geq 0, \end{aligned} \tag{117}$$

and hence

$$\begin{aligned} &\langle v - \Lambda^m u_n, g \rangle \\ &\geq \Big\langle v - \Lambda^m u_n, B_m v \\ &\qquad + \frac{1}{\mu_m} \left(\Lambda^{m-1} u_n - \Lambda^m u_n - \mu_m B_m \Lambda^{m-1} u_n\right) \Big\rangle \\ &= \Big\langle v - \Lambda^m u_n, B_m v - B_m \Lambda^m u_n + B_m \Lambda^m u_n \\ &\qquad - B_m \Lambda^{m-1} u_n + \frac{1}{\mu_m} \left(\Lambda^{m-1} u_n - \Lambda^m u_n\right) \Big\rangle \\ &\geq \left\langle v - \Lambda^m u_n, B_m \Lambda^m u_n - B_m \Lambda^{m-1} u_n \right\rangle \\ &\qquad + \left\langle v - \Lambda^m u_n, \frac{1}{\mu_m} \left(\Lambda^{m-1} u_n - \Lambda^m u_n\right) \right\rangle. \end{aligned} \tag{118}$$

In particular,

$$\begin{aligned} \left\langle v - \Lambda^m u_{n_i}, g \right\rangle &\geq \left\langle v - \Lambda^m u_{n_i}, B_m \Lambda^m u_{n_i} - B_m \Lambda^{m-1} u_{n_i} \right\rangle \\ &\quad + \left\langle v - \Lambda^m u_{n_i}, \frac{1}{\mu_m} \left(\Lambda^{m-1} u_{n_i} - \Lambda^m u_{n_i}\right) \right\rangle. \end{aligned} \tag{119}$$

Since $\|\Lambda^m u_n - \Lambda^{m-1} u_n\| \to 0$ (due to (103)) and $\| B_m \Lambda^m u_n - B_m \Lambda^{m-1} u_n \| \to 0$ (due to the Lipschitz continuity of B_m), we conclude from $\Lambda^m u_{n_i} \rightharpoonup w$ and $\mu_m \in (0, 2\eta_m)$, $m \in \{1, 2, \ldots, N\}$ that

$$\lim_{i\to\infty} \left\langle v - \Lambda^m u_{n_i}, g \right\rangle = \langle v - w, g \rangle \geq 0. \tag{120}$$

It follows from the maximal monotonicity of $B_m + R_m$ that $0 \in (R_m + B_m)w$; that is, $w \in I(B_m, R_m)$. Therefore, $w \in \bigcap_{m=1}^{N} I(B_m, R_m)$.

Next, we show that $w \in \mathrm{GMEP}(\Theta, \varphi, A)$. In fact, from $z_n = S_{r_n}^{(\Theta,\varphi)}(I - r_n A)x_n$, we know that

$$\begin{aligned} &\Theta\left(u_n, y\right) + \varphi(y) - \varphi\left(u_n\right) + \langle A x_n, y - u_n \rangle \\ &\quad + \frac{1}{r_n} \left\langle K'\left(u_n\right) - K'\left(x_n\right), y - u_n \right\rangle \geq 0, \quad \forall y \in C. \end{aligned} \tag{121}$$

From (H2) it follows that

$$\begin{aligned} &\varphi(y) - \varphi\left(u_n\right) + \langle A x_n, y - u_n \rangle \\ &\quad + \frac{1}{r_n} \left\langle K'\left(u_n\right) - K'\left(x_n\right), y - u_n \right\rangle \\ &\geq \Theta\left(y, u_n\right), \quad \forall y \in C. \end{aligned} \tag{122}$$

Replacing n by n_i, we have

$$\varphi(y)-\varphi(u_{n_i})+\langle Ax_{n_i},y-u_{n_i}\rangle + \left\langle \frac{K'(u_{n_i})-K'(x_{n_i})}{r_{n_i}}, y-u_{n_i}\right\rangle \geq \Theta(y,u_{n_i}),\quad \forall y\in C. \tag{123}$$

Put $u_t = ty+(1-t)w$ for all $t\in(0,1]$ and $y\in C$. Then, from (123) we have

$$\begin{aligned}\langle u_t-u_{n_i},Au_t\rangle &\geq \langle u_t-u_{n_i},Au_t\rangle-\varphi(u_t)+\varphi(u_{n_i})-\langle u_t-u_{n_i},Ax_{n_i}\rangle\\ &\quad-\left\langle \frac{K'(u_{n_i})-K'(x_{n_i})}{r_{n_i}},u_t-u_{n_i}\right\rangle+\Theta(u_t,u_{n_i})\\ &\geq \langle u_t-u_{n_i},Au_t-Au_{n_i}\rangle+\langle u_t-u_{n_i},Au_{n_i}-Ax_{n_i}\rangle\\ &\quad-\varphi(u_t)+\varphi(u_{n_i})\\ &\quad-\left\langle \frac{K'(u_{n_i})-K'(x_{n_i})}{r_{n_i}},u_t-u_{n_i}\right\rangle+\Theta(u_t,u_{n_i}).\end{aligned}\tag{124}$$

Since $\|u_{n_i}-x_{n_i}\|\to 0$ as $i\to\infty$, we deduce from the Lipschitz continuity of A and K' that $\|Au_{n_i}-Ax_{n_i}\|\to 0$ and $\|K'(u_{n_i})-K'(x_{n_i})\|\to 0$ as $i\to\infty$. Further, from the monotonicity of A, we have $\langle u_t-u_{n_i},Au_t-Au_{n_i}\rangle\geq 0$. So, from (H4), the weakly lower semicontinuity of φ, $((K'(u_{n_i})-K'(x_{n_i}))/r_{n_i})\to 0$ and $u_{n_i}\rightharpoonup w$, we have

$$\langle u_t-w,Au_t\rangle\geq-\varphi(u_t)+\varphi(w)+\Theta(u_t,w),\quad \text{as } i\longrightarrow\infty. \tag{125}$$

From (H1), (H4), and (125) we also have

$$\begin{aligned}0&=\Theta(u_t,u_t)+\varphi(u_t)-\varphi(u_t)\\ &\leq t\Theta(u_t,y)+(1-t)\Theta(u_t,w)+t\varphi(y)\\ &\quad+(1-t)\varphi(w)-\varphi(u_t)\\ &=t[\Theta(u_t,y)+\varphi(y)-\varphi(u_t)]\\ &\quad+(1-t)[\Theta(u_t,w)+\varphi(w)-\varphi(w)-\varphi(u_t)]\\ &\leq t[\Theta(u_t,y)+\varphi(y)-\varphi(u_t)]+(1-t)\langle u_t-w,Au_t\rangle\\ &=t[\Theta(u_t,y)+\varphi(y)-\varphi(u_t)]+(1-t)t\langle y-w,Au_t\rangle,\end{aligned}\tag{126}$$

and hence

$$0\leq\Theta(u_t,y)+\varphi(y)-\varphi(u_t)+(1-t)\langle y-w,Au_t\rangle. \tag{127}$$

Letting $t\to 0$, we have, for each $y\in C$,

$$0\leq\Theta(w,y)+\varphi(y)-\varphi(w)+\langle Aw,y-w\rangle. \tag{128}$$

This implies that $w\in \mathrm{GMEP}(\Theta,\varphi,A)$. Therefore, $w\in\bigcap_{n=1}^{\infty}\mathrm{Fix}(T_n)\cap\mathrm{GMEP}(\Theta,\varphi,A)\cap\bigcap_{i=1}^{N}I(B_i,R_i):=\Omega$. This shows that $\omega_w(x_n)\subset\Omega$. Consequently, from (111) and (114) we have

$$\begin{aligned}&\limsup_{n\to\infty}\langle(\gamma f-\mu F)x^*,x_n-x^*\rangle\\ &\quad=\langle(\gamma f-\mu F)x^*,w-x^*\rangle\leq 0.\end{aligned}\tag{129}$$

Step 6. Finally, we show that $x_n\to x^*\in\Omega$, where $x^*=P_\Omega(I-(\mu F-\gamma f))x^*$.

Indeed, in terms of Lemma 4 we have

$$\begin{aligned}\|y_n-x^*\|^2 &=\left\|\alpha_n(u+\gamma f(x_n)-\overline{V}x^*)+\beta_n(x_n-x^*)\right.\\ &\quad\left.+((1-\beta_n)I-\alpha_n\overline{V})(W_nz_n-x^*)\right\|^2\\ &\leq\left\|\beta_n(x_n-x^*)+((1-\beta_n)I-\alpha_n\overline{V})(W_nz_n-x^*)\right\|^2\\ &\quad+2\alpha_n\langle u+\gamma f(x_n)-\overline{V}x^*,y_n-x^*\rangle\\ &\leq\left[\left\|((1-\beta_n)I-\alpha_n\overline{V})(W_nz_n-x^*)\right\|\right.\\ &\quad\left.+\beta_n\|x_n-x^*\|\right]^2+2\alpha_n\left\|u+\gamma f(x_n)-\overline{V}x^*\right\|\|y_n-x^*\|\\ &\leq[(1-\beta_n-\alpha_n(1+\mu)\overline{\gamma})\|W_nz_n-x^*\|+\beta_n\|x_n-x^*\|]^2\\ &\quad+2\alpha_n\left\|u+\gamma f(x_n)-\overline{V}x^*\right\|\|y_n-x^*\|\\ &\leq[(1-\beta_n-\alpha_n(1+\mu)\overline{\gamma})\|z_n-x^*\|+\beta_n\|x_n-x^*\|]^2\\ &\quad+2\alpha_n\left\|u+\gamma f(x_n)-\overline{V}x^*\right\|\|y_n-x^*\|\\ &\leq[(1-\beta_n-\alpha_n(1+\mu)\overline{\gamma})\|x_n-x^*\|+\beta_n\|x_n-x^*\|]^2\\ &\quad+2\alpha_n\left\|u+\gamma f(x_n)-\overline{V}x^*\right\|\|y_n-x^*\|\\ &=(1-\alpha_n(1+\mu)\overline{\gamma})^2\|x_n-x^*\|^2\\ &\quad+2\alpha_n\left\|u+\gamma f(x_n)-\overline{V}x^*\right\|\|y_n-x^*\|\\ &\leq\|x_n-x^*\|^2+2\alpha_n\left\|u+\gamma f(x_n)-\overline{V}x^*\right\|\|y_n-x^*\|.\end{aligned}\tag{130}$$

Utilizing Lemmas 4 and 13, we conclude from (130) that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &=\|\sigma_n\gamma(f(y_n)-f(x^*))+(I-\sigma_n\mu F)W_ny_n\\ &\quad-(I-\sigma_n\mu F)W_nx^*+\sigma_n(\gamma f(x^*)-\mu Fx^*)\|^2\\ &\leq\|\sigma_n\gamma(f(y_n)-f(x^*))+(I-\sigma_n\mu F)W_ny_n\\ &\quad-(I-\sigma_n\mu F)W_nx^*\|^2\\ &\quad+2\sigma_n\langle\gamma f(x^*)-\mu Fx^*,x_{n+1}-x^*\rangle\end{aligned}$$

$$
\begin{aligned}
&\leq [\sigma_n \gamma \| f(y_n) - f(x^*) \| \\
&\qquad + \| (I - \sigma_n \mu F) W_n y_n - (I - \sigma_n \mu F) W_n x^* \|]^2 \\
&\quad + 2\sigma_n \langle \gamma f(x^*) - \mu F x^*, x_{n+1} - x^* \rangle \\
&\leq [\sigma_n \gamma l \| y_n - x^* \| + (1 - \sigma_n \tau) \| y_n - x^* \|]^2 \\
&\quad + 2\sigma_n \langle \gamma f(x^*) - \mu F x^*, x_{n+1} - x^* \rangle \\
&= (1 - \sigma_n (\tau - \gamma l))^2 \| y_n - x^* \|^2 \\
&\quad + 2\sigma_n \langle \gamma f(x^*) - \mu F x^*, x_{n+1} - x^* \rangle \\
&\leq (1 - \sigma_n (\tau - \gamma l)) \\
&\quad \times \left[\| x_n - x^* \|^2 + 2\alpha_n \| u + \gamma f(x_n) - \overline{V} x^* \| \| y_n - x^* \| \right] \\
&\quad + 2\sigma_n \langle \gamma f(x^*) - \mu F x^*, x_{n+1} - x^* \rangle \\
&\leq (1 - \sigma_n (\tau - \gamma l)) \| x_n - x^* \|^2 \\
&\quad + 2\alpha_n \| u + \gamma f(x_n) - \overline{V} x^* \| \| y_n - x^* \| \\
&\quad + 2\sigma_n \langle \gamma f(x^*) - \mu F x^*, x_{n+1} - x^* \rangle \\
&= (1 - \sigma_n (\tau - \gamma l)) \| x_n - x^* \|^2 \\
&\quad + \sigma_n (\tau - \gamma l) \cdot \frac{2}{\tau - \gamma l} \\
&\qquad \times \left[\frac{\alpha_n}{\sigma_n} \| u + \gamma f(x_n) - \overline{V} x^* \| \right. \\
&\qquad\quad \left. \times \| y_n - x^* \| + \langle \gamma f(x^*) - \mu F x^*, x_{n+1} - x^* \rangle \right].
\end{aligned}
\tag{131}
$$

Note that $0 \leq \gamma l < \tau$. Hence, $\sum_{n=1}^{\infty} \sigma_n = \infty$ leads to $\sum_{n=1}^{\infty} \sigma_n(\tau - \gamma l) = \infty$. In addition, since $\lim_{n\to\infty}\sigma_n = 0$ and $\lim_{n\to\infty}(\alpha_n/\sigma_n) = 0$, we get from (129)

$$
\begin{aligned}
\limsup_{n\to\infty} \frac{2}{\tau - \gamma l} & \left[\frac{\alpha_n}{\sigma_n} \| u + \gamma f(x_n) - \overline{V} x^* \| \| y_n - x^* \| \right. \\
& \left. + \langle \gamma f(x^*) - \mu F x^*, x_{n+1} - x^* \rangle \right] \leq 0.
\end{aligned}
\tag{132}
$$

Applying Lemma 19 to (131), we have that $x_n \to x^*$ as $n \to \infty$. This completes the proof. □

Corollary 21. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let N be an integer. Let Θ be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying (H1)–(H4) and $\varphi : C \to \mathbf{R}$ be a lower semicontinuous and convex functional. Let $R_i : C \to 2^H$ be a maximal monotone mapping and let $A : H \to H$ and $B_i : C \to H$ be ζ-inverse strongly monotone and η_i-inverse strongly monotone, respectively, where $i \in \{1, 2, \ldots, N\}$. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on H and $\{\lambda_n\}$ be a sequence in $(0, b]$ for some $b \in (0, 1)$. Let V be a $\overline{\gamma}$-strongly positive bounded linear operator. Given $u \in H$ and $\mu > 0$, let $Fx = ((1/\mu)I + V)x - (1/\mu)u$. Suppose $0 < \mu((1/\mu) + \|V\|)^2 < 2((1/\mu) + \overline{\gamma})$ and $\tau = 1 - \sqrt{1 - \mu(2((1/\mu) + \overline{\gamma}) - \mu((1/\mu) + \|V\|)^2)}$. Let $f : H \to H$ be an l-Lipschitzian mapping with $0 \leq \gamma l < \min\{\tau, (1 + \mu)\overline{\gamma}\}$. Let W_n be the W-mapping defined by (15). Assume that $\Omega := \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \cap GMEP(\Theta, \varphi, A) \cap \bigcap_{i=1}^{N} I(B_i, R_i)$ is nonempty. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\sigma_n\}$ be three sequences in $(0, 1)$. Assume that the conditions (i)–(v) of Theorem 20 hold. If $S_r^{(\Theta,\varphi)}$ is firmly nonexpansive, then for a given arbitrary $x_1 \in H$, the sequence $\{x_n\}$ generated iteratively by*

$$
\begin{aligned}
u_n &= S_{r_n}^{(\Theta,\varphi)} (I - r_n A) x_n, \\
z_n &= J_{R_N,\mu_N} (I - \mu_N B_N) J_{R_{N-1},\mu_{N-1}} (I - \mu_{N-1} B_{N-1}) \\
&\quad \cdots J_{R_1,\mu_1} (I - \mu_1 B_1) u_n, \\
y_n &= \alpha_n (u + \gamma f(x_n)) + \beta_n x_n \\
&\quad + ((1 - \beta_n) I - \alpha_n (I + \mu V)) W_n z_n, \\
x_{n+1} &= \sigma_n (u + \gamma f(y_n)) + (I - \sigma_n (I + \mu V)) W_n y_n, \\
&\qquad \forall n \geq 1,
\end{aligned}
\tag{133}
$$

converges strongly to $x^* \in \Omega$, which solves the following optimization problem:

$$
\min_{x\in\Omega} \frac{\mu}{2} \langle Vx, x \rangle + \frac{1}{2} \| x - u \|^2 - h(x), \tag{OP2}
$$

where $h : H \to \mathbf{R}$ is the potential function of γf.

Proof. Given $u \in H$ and $\mu > 0$, let $Fx = ((1/\mu)I + V)x - (1/\mu)u$. Then $F : H \to H$ is a κ-Lipschitzian and η-strongly monotone operator with positive constants $\kappa = (1/\mu) + \|V\|$ and $\eta = (1/\mu) + \overline{\gamma}$. Suppose $0 < \mu < (2\eta/\kappa^2) = (2((1/\mu) + \overline{\gamma}))/(((1/\mu) + \|V\|)^2)$ and $\tau = 1 - \sqrt{1 - \mu(2((1/\mu) + \overline{\gamma}) - \mu((1/\mu) + \|V\|)^2)}$. In this case, it is easy from (56) to see that

$$
\begin{aligned}
x_{n+1} &= \sigma_n \gamma f(y_n) + (I - \sigma_n \mu F) W_n y_n \\
&= \sigma_n \gamma f(y_n) + (I - \sigma_n (I + \mu V)) W_n y_n + \sigma_n u \\
&= \sigma_n (u + \gamma f(y_n)) + (I - \sigma_n (I + \mu V)) W_n y_n.
\end{aligned}
\tag{134}
$$

Then, for $0 \leq \gamma l < \min\{\tau, (1 + \mu)\overline{\gamma}\}$, all conditions of Theorem 20 are satisfied. Therefore, utilizing Theorem 20 we infer that $\{x_n\}$ converges strongly to $x^* \in \Omega$, where $x^* = P_\Omega(I - (\mu F - \gamma f))x^*$ is a unique solution of the VIP:

$$
\langle (\gamma f - \mu F) x^*, y - x^* \rangle \leq 0, \quad \forall y \in \Omega. \tag{135}
$$

Utilizing Lemma 18, we know that x^* solves the following optimization problem:

$$
\min_{x\in\Omega} \frac{\mu}{2} \langle Vx, x \rangle + \frac{1}{2} \| x - u \|^2 - h(x), \tag{OP2$'$}
$$

where $h : H \to \mathbf{R}$ is the potential function of γf. □

Corollary 22. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let Θ be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying (H1)–(H4) and $\varphi : C \to \mathbf{R}$ be a lower semicontinuous and convex functional. Let $R_i : C \to 2^H$ be a maximal monotone mapping and let $A : H \to H$ and $B_i : C \to H$ be ζ-inverse strongly monotone and η_i-inverse strongly monotone, respectively, for $i = 1, 2$. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on H and $\{\lambda_n\}$ be a sequence in $(0, b]$ for some $b \in (0, 1)$. Let $F : H \to H$ be a κ-Lipschitzian and η-strongly monotone operator with positive constants $\kappa, \eta > 0$. Let $0 < \mu < (2\eta/\kappa^2)$ and $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Let $f : H \to H$ be an l-Lipschitzian mapping with $0 \leq \gamma l < \tau$. Let W_n be the W-mapping defined by (15) and V be a $\overline{\gamma}$-strongly positive bounded linear operator with $\gamma l < (1 + \mu)\overline{\gamma}$. Assume that $\Omega := \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \cap GMEP(\Theta, \varphi, A) \cap I(B_2, R_2) \cap I(B_1, R_1)$ is nonempty. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\sigma_n\}$ be three sequences in $(0, 1)$. Assume that the conditions (i)–(v) of Theorem 20 hold. If $S_r^{(\Theta,\varphi)}$ is firmly nonexpansive, then for a given arbitrary $x_1 \in H$, the sequence $\{x_n\}$ generated iteratively by*

$$\begin{aligned} u_n &= S_{r_n}^{(\Theta,\varphi)} (I - r_n A) x_n, \\ z_n &= J_{R_2,\mu_2} (I - \mu_2 B_2) J_{R_1,\mu_1} (I - \mu_1 B_1) u_n, \\ y_n &= \alpha_n (u + \gamma f (x_n)) + \beta_n x_n \\ &\quad + ((1 - \beta_n) I - \alpha_n (I + \mu V)) W_n z_n, \\ x_{n+1} &= \sigma_n \gamma f (y_n) + (I - \sigma_n \mu F) W_n y_n, \quad \forall n \geq 1, \end{aligned} \tag{136}$$

converges strongly to $x^* \in \Omega$, where $x^* = P_\Omega(I - (\mu F - \gamma f))x^*$ is a unique solution of the VIP:

$$\langle (\gamma f - \mu F) x^*, y - x^* \rangle \leq 0, \quad \forall y \in \Omega. \tag{137}$$

Corollary 23. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let N be an integer. Let Θ be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying (H1)–(H4) and $\varphi : C \to \mathbf{R}$ be a lower semicontinuous and convex functional. Let $R_i : C \to 2^H$ be a maximal monotone mapping and let $B_i : C \to H$ be η_i-inverse strongly monotone, respectively, where $i \in \{1, 2, \ldots, N\}$. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on H and $\{\lambda_n\}$ be a sequence in $(0, b]$ for some $b \in (0, 1)$. Let $F : H \to H$ be a κ-Lipschitzian and η-strongly monotone operator with positive constants $\kappa, \eta > 0$. Let $0 < \mu < (2\eta/\kappa^2)$ and $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Let $f : H \to H$ be an l-Lipschitzian mapping with $0 \leq \gamma l < \tau$. Let W_n be the W-mapping defined by (15) and V be a $\overline{\gamma}$-strongly positive bounded linear operator with $\gamma l < (1 + \mu)\overline{\gamma}$. Assume that $\Omega := \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \cap MEP(\Theta, \varphi) \cap \bigcap_{i=1}^{N} I(B_i, R_i)$ is nonempty. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\sigma_n\}$ be three sequences in $(0, 1)$. Assume that the conditions (i)–(v) of Theorem 20 hold, where $\{r_n\}$ is a bounded sequence such that $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < 2\zeta$ for some $\zeta > 0$. If $S_r^{(\Theta,\varphi)}$ is firmly nonexpansive, then for a given arbitrary $x_1 \in H$, the sequence $\{x_n\}$ generated iteratively by*

$$\begin{aligned} &\Theta (u_n, y) + \varphi (y) - \varphi (u_n) \\ &\quad + \frac{1}{r_n} \langle k' (u_n) - k' (x_n), y - u_n \rangle \geq 0, \quad \forall y \in C, \\ &z_n = J_{R_N,\mu_N} (I - \mu_N B_N) J_{R_{N-1},\mu_{N-1}} (I - \mu_{N-1} B_{N-1}) \\ &\quad \cdots J_{R_1,\mu_1} (I - \mu_1 B_1) u_n, \\ &y_n = \alpha_n (u + \gamma f (x_n)) + \beta_n x_n \\ &\quad + ((1 - \beta_n) I - \alpha_n (I + \mu V)) W_n z_n, \\ &x_{n+1} = \sigma_n \gamma f (y_n) + (I - \sigma_n \mu F) W_n y_n, \quad \forall n \geq 1, \end{aligned} \tag{138}$$

converges strongly to $x^* \in \Omega$, where $x^* = P_\Omega(I - (\mu F - \gamma f))x^*$ is a unique solution of the VIP:

$$\langle (\gamma f - \mu F) x^*, y - x^* \rangle \leq 0, \quad \forall y \in \Omega. \tag{139}$$

Proof. In Theorem 20, for all $n \geq 1$, $u_n = S_{r_n}^{(\Theta,\varphi)}(I - r_n A)x_n$ is equivalent to

$$\begin{aligned} &\Theta (u_n, y) + \varphi (y) - \varphi (u_n) + \langle A x_n, y - u_n \rangle \\ &\quad + \frac{1}{r_n} \langle k' (u_n) - k' (x_n), y - u_n \rangle \geq 0, \quad \forall y \in C. \end{aligned} \tag{140}$$

Put $A \equiv 0$. Then it follows that

$$\begin{aligned} &\Theta (u_n, y) + \varphi (y) - \varphi (u_n) \\ &\quad + \frac{1}{r_n} \langle k' (u_n) - k' (x_n), y - u_n \rangle \geq 0, \quad \forall y \in C. \end{aligned} \tag{141}$$

Observe that for all $\zeta \in (0, \infty)$

$$\langle Ax - Ay, x - y \rangle \geq \zeta \|Ax - Ay\|^2, \quad \forall x, y \in H. \tag{142}$$

So, whenever $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < 2\zeta$ for some $\zeta \in (0, \infty)$, we obtain the desired result by using Theorem 20. □

Let $T : H \to H$ be a k-strictly pseudocontractive mapping. For recent convergence result for strictly pseudocontractive mappings, we refer to [44]. Putting $A = I - T$, we know that for all $x, y \in H$

$$\|(I - A) x - (I - A) y\|^2 \leq \|x - y\|^2 + k\|Ax - Ay\|^2. \tag{143}$$

Note that

$$\begin{aligned} &\|(I - A) x - (I - A) y\|^2 \\ &\quad = \|x - y\|^2 + \|Ax - Ay\|^2 - 2 \langle Ax - Ay, x - y \rangle . \end{aligned} \tag{144}$$

Hence, we have for all $x, y \in H$

$$\langle Ax - Ay, x - y \rangle \geq \frac{1 - k}{2} \|Ax - Ay\|^2. \tag{145}$$

Consequently, if $T : H \to H$ is a k-strictly pseudocontractive mapping, then the mapping $A = I - T$ is $(1 - k)/2$-inverse strongly monotone.

Corollary 24. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let N be an integer. Let Θ be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying (H1)–(H4) and $\varphi : C \to \mathbf{R}$ be a lower semicontinuous and convex functional. Let $R_i : C \to 2^H$ be a maximal monotone mapping and let $T : H \to H$ and $B_i : C \to H$ be a k-strictly pseudocontractive mapping and an η_i-inverse strongly monotone mapping, respectively, where $i \in \{1, 2, \ldots, N\}$. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on H and $\{\lambda_n\}$ be a sequence in $(0, b]$ for some $b \in (0, 1)$. Let $F : H \to H$ be a κ-Lipschitzian and η-strongly monotone operator with positive constants $\kappa, \eta > 0$. Let $0 < \mu < (2\eta/\kappa^2)$ and $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Let $f : H \to H$ be an l-Lipschitzian mapping with $0 \le \gamma l < \tau$. Let W_n be the W-mapping defined by (15) and V be a $\overline{\gamma}$-strongly positive bounded linear operator with $\gamma l < (1 + \mu)\overline{\gamma}$. Assume that $\Omega := \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \cap GMEP(\Theta, \varphi, A) \cap \bigcap_{i=1}^{N} I(B_i, R_i)$ is nonempty, where $A = I - T$. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\sigma_n\}$ be three sequences in $(0, 1)$. Assume that the conditions (i)–(v) of Theorem 20 hold where $\zeta = ((1-k)/2)$. If $S_r^{(\Theta,\varphi)}$ is firmly nonexpansive, then for a given arbitrary $x_1 \in H$, the sequence $\{x_n\}$ generated iteratively by*

$$\begin{aligned} u_n &= S_{r_n}^{(\Theta,\varphi)} \left((1 - r_n) x_n + r_n T x_n \right), \\ z_n &= J_{R_N,\mu_N} \left(I - \mu_N B_N \right) J_{R_{N-1},\mu_{N-1}} \left(I - \mu_{N-1} B_{N-1} \right) \\ &\quad \cdots J_{R_1,\mu_1} \left(I - \mu_1 B_1 \right) u_n, \\ y_n &= \alpha_n \left(u + \gamma f \left(x_n \right) \right) + \beta_n x_n \\ &\quad + \left(\left(1 - \beta_n \right) I - \alpha_n \left(I + \mu V \right) \right) W_n z_n, \\ x_{n+1} &= \sigma_n \gamma f \left(y_n \right) + \left(I - \sigma_n \mu F \right) W_n y_n, \quad \forall n \ge 1, \end{aligned} \tag{146}$$

converges strongly to $x^* \in \Omega$, where $x^* = P_\Omega(I - (\mu F - \gamma f))x^*$ is a unique solution of the VIP:

$$\left\langle \left(\gamma f - \mu F \right) x^*, y - x^* \right\rangle \le 0, \quad \forall y \in \Omega. \tag{147}$$

Proof. Since T is a k-strictly pseudocontractive mapping, the mapping $A = I - T$ is $(1 - k)/2$-inverse strongly monotone. In this case, put $\zeta = (1 - k)/2$. Moreover, we obtain that

$$\begin{aligned} u_n &= S_{r_n}^{(\Theta,\varphi)} \left(I - r_n A \right) x_n \\ &= S_{r_n}^{(\Theta,\varphi)} \left(x_n - r_n \left(I - T \right) x_n \right) \\ &= S_{r_n}^{(\Theta,\varphi)} \left(\left(1 - r_n \right) x_n + r_n T x_n \right). \end{aligned} \tag{148}$$

So, from Theorem 20, we obtain the desired result. □

Corollary 25. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let N be an integer. Let Θ be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying (H1)–(H4) and $\varphi : C \to \mathbf{R}$ be a lower semicontinuous and convex functional. Let $R_i : C \to 2^H$ be a maximal monotone mapping and let $A : H \to H$ and $B_i : C \to H$ be ζ-inverse strongly monotone and η_i-inverse strongly monotone, respectively, where $i \in \{1, 2, \ldots, N\}$. Let $F : H \to H$ be a κ-Lipschitzian and η-strongly monotone operator with positive constants $\kappa, \eta > 0$. Let $0 < \mu < (2\eta/\kappa^2)$ and $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Let $f : H \to H$ be an l-Lipschitzian mapping with $0 \le \gamma l < \tau$. Let V be a $\overline{\gamma}$-strongly positive bounded linear operator with $\gamma l < (1 + \mu)\overline{\gamma}$. Assume that $\Omega := GMEP(\Theta, \varphi, A) \cap \bigcap_{i=1}^{N} I(B_i, R_i)$ is nonempty. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\sigma_n\}$ be three sequences in $(0, 1)$. Assume that conditions (i)–(v) of Theorem 20 hold. Given $x_1 \in H$ arbitrarily, let the sequence $\{x_n\}$ be generated iteratively by*

$$\begin{aligned} u_n &= S_{r_n}^{(\Theta,\varphi)} \left(I - r_n A \right) x_n, \\ z_n &= J_{R_N,\mu_N} \left(I - \mu_N B_N \right) J_{R_{N-1},\mu_{N-1}} \left(I - \mu_{N-1} B_{N-1} \right) \\ &\quad \cdots J_{R_1,\mu_1} \left(I - \mu_1 B_1 \right) u_n, \\ y_n &= \alpha_n \left(u + \gamma f \left(x_n \right) \right) + \beta_n x_n \\ &\quad + \left(\left(1 - \beta_n \right) I - \alpha_n \left(I + \mu V \right) \right) z_n, \\ x_{n+1} &= \sigma_n \gamma f \left(y_n \right) + \left(I - \sigma_n \mu F \right) y_n, \quad \forall n \ge 1. \end{aligned} \tag{149}$$

If $S_r^{(\Theta,\varphi)}$ is firmly nonexpansive, then $\{x_n\}$ converges strongly to $x^* \in \Omega$, where $x^* = P_\Omega(I - (\mu F - \gamma f))x^*$ is a unique solution of the VIP:

$$\left\langle \left(\gamma f - \mu F \right) x^*, y - x^* \right\rangle \le 0, \quad \forall y \in \Omega. \tag{150}$$

Proof. Put $T_n x = x$ for all integers $n \ge 1$ and all $x \in H$. Then, the desired result follows from Theorem 20. □

Corollary 26. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let N be an integer. Let $R_i : C \to 2^H$ be a maximal monotone mapping and let $B_i : C \to H$ be η_i-inverse strongly monotone, where $i \in \{1, 2, \ldots, N\}$. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on H and $\{\lambda_n\}$ be a sequence in $(0, b]$ for some $b \in (0, 1)$. Let $F : H \to H$ be a κ-Lipschitzian and η-strongly monotone operator with positive constants $\kappa, \eta > 0$. Let $0 < \mu < (2\eta/\kappa^2)$ and $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Let $f : H \to H$ be an l-Lipschitzian mapping with $0 \le \gamma l < \tau$. Let W_n be the W-mapping defined by (15) and V be a $\overline{\gamma}$-strongly positive bounded linear operator with $\gamma l < (1+\mu)\overline{\gamma}$. Assume that $\Omega := \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \cap \bigcap_{i=1}^{N} I(B_i, R_i)$ is nonempty. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\sigma_n\}$ be three sequences in $(0, 1)$. Assume that,*

(i) $\lim_{n\to\infty} \sigma_n = \lim_{n\to\infty}(\alpha_n/\sigma_n) = 0$, $\limsup_{n\to\infty} (\alpha_{n+1}/\alpha_n) < \infty$, $\sum_{n=1}^{\infty} \sigma_n = \infty$ *and*

$$0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1; \tag{151}$$

(ii) $\mu_i \in (0, 2\eta_i)$, $i \in \{1, 2, \ldots, N\}$;

(iii) $\sum_{n=1}^{\infty}(|\beta_{n+1} - \beta_n| + |\sigma_{n+1} - \sigma_n|) < \infty$.

For a given arbitrary $x_1 \in H$, let the sequence $\{x_n\}$ be generated iteratively by

$$\begin{aligned} z_n &= J_{R_N,\mu_N}\left(I-\mu_N B_N\right) J_{R_{N-1},\mu_{N-1}}\left(I-\mu_{N-1}B_{N-1}\right) \\ &\quad \cdots J_{R_1,\mu_1}\left(I-\mu_1 B_1\right) x_n, \\ y_n &= \alpha_n\left(u+\gamma f\left(x_n\right)\right)+\beta_n x_n \\ &\quad +\left(\left(1-\beta_n\right) I-\alpha_n\left(I+\mu V\right)\right) W_n z_n, \\ x_{n+1} &= \sigma_n \gamma f\left(y_n\right)+\left(I-\sigma_n \mu F\right) W_n y_n, \quad \forall n \geq 1. \end{aligned} \tag{152}$$

Then the sequence $\{x_n\}$ converges strongly to $x^* \in \Omega$, where $x^* = P_\Omega(I-(\mu F-\gamma f))x^*$ is a unique solution of the VIP:

$$\left\langle\left(\gamma f-\mu F\right) x^*, y-x^*\right\rangle \leq 0, \quad \forall y \in \Omega. \tag{153}$$

Proof. Put $\Theta(x,y) = 0$, $\varphi(x) = 0$ for all $x, y \in C$, $Ax = 0$ for all $x \in H$ and $r_n = 1$. Take $K(x) = (1/2)\,\|x\|^2$ for all $x \in H$. Then we get $u_n = x_n$ in Theorem 20 and the conclusion follows. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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