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Research Article

Global Analysis of Almost Periodic Solution of a Discrete Multispecies Mutualism System

Hui Zhang, Bin Jing, Yingqi Li, and Xiaofeng Fang

Mathematics and OR Section, Xi'an Research Institute of High-Tech, Hongqing Town, Xi'an, Shaanxi 710025, China

Correspondence should be addressed to Hui Zhang; zh53054958@163.com

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This paper discusses a discrete multispecies Lotka-Volterra mutualism system. We first obtain the permanence of the system. Assuming that the coefficients in the system are almost periodic sequences, we obtain the sufficient conditions for the existence of a unique almost periodic solution which is globally attractive. In particular, for the discrete two-species Lotka-Volterra mutualism system, the sufficient conditions for the existence of a unique uniformly asymptotically stable almost periodic solution are obtained. An example together with numerical simulation indicates the feasibility of the main result.

1. Introduction

In this paper, we consider a discrete multispecies Lotka-Volterra mutualism system:

$$x_i(k+1) = x_i(k)\exp\left\{a_i(k) - b_i(k)x_i(k) + \sum_{j=1, j\neq i}^{n} c_{ij}(k)\frac{x_j(k)}{d_{ij}+x_j(k)}\right\}, \quad i = 1, 2, \ldots, n, \tag{1}$$

where $x_i(k)$ stand for the densities of species x_i at the kth generation, $a_i(k)$ represent the natural growth rates of species x_i at the kth generation, $b_i(k)$ are the intraspecific effects of the kth generation of species x_i on own population, $c_{ij}(k)$ measure the interspecific mutualism effects of the kth generation of species x_j on species x_i $(i, j = 1, 2, \ldots, n, i \neq j)$, and $d_{ij}(\geq 1)$ are positive control constants.

A number of scholars have studied the difference system (see [1–8] and the references cited therein) since the discrete time models governed by the difference equation are more appropriate than the continuous ones when the populations have short life expectancy, nonoverlapping generations in the real world.

Recently, as far as the multispecies Lotka-Volterra ecosystem is concerned, Wendi and Zhengyi [1] proposed the following Lotka-Volterra model:

$$x_i(k+1) = x_i(k)\exp\left[r_i(k) - \sum_{j=1}^{n} a_{ij}(k)x_j(k)\right], \quad i = 1, 2, \ldots, n. \tag{2}$$

By constructing a suitable Lyapunov function and using the finite covering theorem of mathematic analysis, they obtained a set of sufficient conditions which ensure the system to be globally asymptotically stable.

Chen [3] studied the dynamic behavior of the discrete $n+m$-species Lotka-Volterra competition predator-prey systems:

$$x_i(k+1) = x_i(k)\exp\left[b_i(k) - \sum_{l=1}^{n} a_{il}(k)x_l(k) - \sum_{l=1}^{m} c_{il}(k)y_l(k)\right],$$

$$y_j(k+1) = y_j(k)\exp\left[-r_j(k) + \sum_{l=1}^{n} d_{jl}(k)\, x_l(k) \times \sum_{l=1}^{m} e_{jl}(k)\, y_l(k)\right], \tag{3}$$

where $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. Sufficient conditions which ensure the permanence and the global stability of the systems are obtained; for periodic case, sufficient conditions which ensure the existence of a globally stable positive periodic solution of the systems are obtained.

At the same time, a few scholars have investigated the mutualism system (see [2, 8–13] in detail). However, to the best of the authors' knowledge, still no scholar has done works on discrete multispecies mutualism system. So we propose the discrete multispecies Lotka-Volterra mutualism system (1).

Notice that the investigation of almost periodic solutions for difference equations is one of the most important topics in the qualitative theory of difference equations due to the applications in biology, ecology, neural network, and so forth (see [7, 14–21] and the references cited therein), and little work has been done previously on an almost periodic version which is corresponding to system (1). Then, we will further investigate the existence of a unique almost periodic solution of system (1) which is globally attractive.

Denote as Z and Z^+ the set of integers and the set of nonnegative integers, respectively. For any bounded sequence $g(n)$ defined on Z, define $g^u = \sup_{n\in Z} g(n)$, $g^l = \inf_{n\in Z} g(n)$.

Throughout this paper, we assume the following.

(H1) $a_i(k)$, $b_i(k)$, and $c_{ij}(k)$ are bounded nonnegative almost periodic sequences such that

$$0 < a_i^l \le a_i(k) \le a_i^u, \qquad 0 < b_i^l \le b_i(k) \le b_i^u,$$
$$0 < c_{ij}^l \le c_{ij}(k) \le c_{ij}^u. \tag{4}$$

From the point of view of biology, in the sequel, we assume that $\mathbf{x}(0) = (x_1(0), x_2(0), \ldots, x_n(0)) > \mathbf{0}$. Then, it is easy to see that, for given $\mathbf{x}(0) > \mathbf{0}$, the system (1) has a positive sequence solution $\mathbf{x}(k) = (x_1(k), x_2(k), \ldots, x_n(k))$ $(k \in Z^+)$ passing through $\mathbf{x}(0)$.

The remaining part of this paper is organized as follows. In Section 2, we will introduce some definitions and several useful lemmas. In Section 3, by applying the theory of difference inequality, we present the permanence results for system (1). In Section 4, we establish the sufficient conditions for the existence of a unique globally attractive almost periodic solution of system (1). In particular, for the discrete two-species Lotka-Volterra mutualism system, the sufficient conditions for the existence of a unique uniformly asymptotically stable almost periodic solution are obtained. The main result is illustrated by an example with a numerical simulation in the last section.

2. Preliminaries

Firstly, we give the definitions of the terminologies involved.

Definition 1 (see [22]). A sequence $x : Z \to R$ is called an almost periodic sequence if the ε-translation set of x

$$E\{\varepsilon, x\} = \{\tau \in Z : |x(n+\tau) - x(n)| < \varepsilon, \forall n \in Z\} \tag{5}$$

is a relatively dense set in Z for all $\varepsilon > 0$; that is, for any given $\varepsilon > 0$, there exists an integer $l(\varepsilon) > 0$ such that each interval of length $l(\varepsilon)$ contains an integer $\tau \in E\{\varepsilon, x\}$ with

$$|x(n+\tau) - x(n)| < \varepsilon, \quad \forall n \in Z. \tag{6}$$

τ is called an ε-translation number of $x(n)$.

Definition 2 (see [23]). A sequence $x : Z^+ \to R$ is called an asymptotically almost periodic sequence if

$$x(n) = p(n) + q(n), \quad \forall n \in Z^+, \tag{7}$$

where $p(n)$ is an almost periodic sequence and $\lim_{n\to\infty} q(n) = 0$.

Definition 3 (see [24]). A solution $(x_1(k), x_2(k), \ldots, x_n(k))$ of system (1) is said to be globally attractive if, for any other solution $(x_1^*(k), x_2^*(k), \ldots, x_n^*(k))$ of system (1), one has

$$\lim_{k\to+\infty} \left(x_i^*(k) - x_i(k)\right) = 0, \quad i = 1, 2, \ldots, n. \tag{8}$$

Now, we present some results which will play an important role in the proof of the main result.

Lemma 4 (see [25]). *If $\{x(n)\}$ is an almost periodic sequence, then $\{x(n)\}$ is bounded.*

Lemma 5 (see [26]). *$\{x(n)\}$ is an almost periodic sequence if and only if, for any sequence $m_i \subset Z$, there exists a subsequence $\{m_{i_k}\} \subset \{m_i\}$ such that the sequence $\{x(n+m_{i_k})\}$ converges uniformly for all $n \in Z$ as $k \to \infty$. Furthermore, the limit sequence is also an almost periodic sequence.*

Lemma 6 (see [23]). *$\{x(n)\}$ is an asymptotically almost periodic sequence if and only if, for any sequence $m_i \subset Z$ satisfying $m_i > 0$ and $m_i \to \infty$ as $i \to \infty$, there exists a subsequence $\{m_{i_k}\} \subset \{m_i\}$ such that the sequence $\{x(n+m_{i_k})\}$ converges uniformly for all $n \in Z^+$ as $k \to \infty$.*

Lemma 7 (see [25]). *Suppose that $\{p_1(n)\}$ and $\{p_2(n)\}$ are almost periodic real sequences. Then, $\{p_1(n) + p_2(n)\}$ and $\{p_1(n)p_2(n)\}$ are almost periodic; $1/p_1(n)$ is also almost periodic provided that $p_1(n) \neq 0$ for all $n \in Z$. Moreover, if $\varepsilon > 0$ is an arbitrary real number, then there exists a relatively dense set that is ε-almost periodic common to $\{p_1(n)\}$ and $\{p_2(n)\}$.*

Lemma 8 (see [3, 11, 27]). *Assume that $\{x(n)\}$ satisfies $x(n) > 0$ and*

$$x(n+1) \le x(n)\exp\{a(n) - b(n)x(n)\}, \tag{9}$$

for $n \in N$, where $a(n)$ and $b(n)$ are nonnegative sequences bounded above and below by positive constants. Then,

$$\limsup_{n\to+\infty} x(n) \le \frac{1}{b^l}\exp\{a^u - 1\}. \tag{10}$$

Lemma 9 (see [3, 11, 27]). *Assume that $\{x(n)\}$ satisfies*

$$\begin{gathered} x(n+1) \ge x(n)\exp\{a(n) - b(n)x(n)\}, \quad n \ge N_0, \\ \limsup_{n\to+\infty} x(n) \le x^*, \end{gathered} \tag{11}$$

and $x(N_0) > 0$, where $a(n)$ and $b(n)$ are nonnegative sequences bounded above and below by positive constants and $N_0 \in N$. Then,

$$\liminf_{n\to+\infty} x(n) \ge \min\left\{\frac{a^l}{b^u}\exp\{a^l - b^u x^*\}, \frac{a^l}{b^u}\right\}. \tag{12}$$

Consider the following almost periodic difference system:

$$x(n+1) = f(n, x(n)), \quad n \in Z^+, \tag{13}$$

where $f : Z^+ \times S_B \to R^K$, $S_B = \{x \in R^k : \|x\| < B\}$, and $f(n,x)$ is almost periodic in n uniformly for $x \in S_B$ and is continuous in x. The product system of (13) is the following system:

$$x(n+1) = f(n, x(n)), \qquad y(n+1) = f(n, y(n)), \tag{14}$$

and Zhang [28] obtained the following lemma.

Lemma 10 (see [28]). *Suppose that there exists a Lyapunov function $V(n,x,y)$ defined for $n \in Z^+$, $\|x\| < B$, and $\|y\| < B$ satisfying the following conditions:*

(i) *$a(\|x-y\|) \le V(n,x,y) \le b(\|x-y\|)$, where $a, b \in K$ with $K = \{a \in C(R^+, R^+) : a(0) = 0$ and a is increasing$\}$;*

(ii) *$\|V(n,x_1,y_1) - V(n,x_2,y_2)\| \le L[\|x_1 - x_2\| + \|y_1 - y_2\|]$, where $L > 0$ is a constant;*

(iii) *$\Delta V_{(14)}(n,x,y) \le -\alpha V(n,x,y)$, where $0 < \alpha < 1$ is a constant, and*

$$\Delta V_{(14)}(n,x,y) \equiv V(n+1, f(n,x), f(n,y)) - V(n,x,y). \tag{15}$$

Moreover, if there exists a solution $\varphi(n)$ of (13) such that $\|\varphi(n)\| \le B^* < B$ for $n \in Z^+$, then there exists a unique uniformly asymptotically stable almost periodic solution $p(n)$ of (13) which is bounded by B^*. In particular, if $f(n,x)$ is periodic of period ω, then there exists a unique uniformly asymptotically stable periodic solution of (13) of period ω.

3. Permanence

In this section, we establish a permanence result for system (1), which can be found by Lemmas 8 and 9.

Proposition 11. *Assume that (H1) holds. Then, any positive solution $(x_1(k), x_2(k), \ldots, x_n(k))$ of system (1) satisfies*

$$m_i \le \liminf_{k\to+\infty} x_i(k) \le \limsup_{k\to+\infty} x_i(k) \le M_i, \quad i = 1, 2, \ldots, n, \tag{16}$$

where

$$\begin{gathered} M_i = \frac{1}{b_i^l}\exp\left\{a_i^u + \sum_{j=1, j\ne i}^{n} c_{ij}^u - 1\right\}, \\ m_i = \min\left\{\frac{a_i^l}{b_i^u}\exp\left(a_i^l - b_i^u M_i\right), \frac{a_i^l}{b_i^u}\right\}. \end{gathered} \tag{17}$$

Theorem 12. *Assume that (H1) holds; then system (1) is permanent.*

It should be noticed that, from Proposition 11, we know that the set

$$[m_1, M_1] \times [m_2, M_2] \times \cdots \times [m_n, M_n] \tag{18}$$

is an invariant set of system (1).

The next result tells us that there exist solutions of system (1) totally in the interval of Proposition 11. To be precise, see the following.

Proposition 13. *System (1) has a solution $(x_1(k), x_2(k), \ldots, x_n(k))$ satisfying $m_i \le x_i(k) \le M_i$ for $k \in Z$.*

Proof. By the almost periodicity of $\{a_i(k)\}$, $\{b_i(k)\}$, and $\{c_{ij}(k)\}$, there exists an integer valued sequence $\{\delta_p\}$ with $\delta_p \to \infty$ as $p \to \infty$ such that

$$\begin{gathered} a_i(k+\delta_p) \longrightarrow a_i(k), \\ b_i(k+\delta_p) \longrightarrow b_i(k), \\ c_{ij}(k+\delta_p) \longrightarrow c_{ij}(k) \\ \text{as } p \longrightarrow +\infty. \end{gathered} \tag{19}$$

Let ε be an arbitrary small positive number. It follows from Proposition 11 that there exists a positive integer N_0 such that

$$m_i - \varepsilon \le x_i(k) \le M_i + \varepsilon, \quad k > N_0. \tag{20}$$

Write $x_{ip}(k) = x_i(k+\delta_p)$ for $k \ge N_0 - \delta_p$ and $p = 1, 2, \ldots$. For any positive integer q, it is easy to see that there exists a sequence $\{x_{ip}(k) : p \ge q\}$ such that the sequence $x_p(k)$ has a subsequence, denoted by $\{x_{ip}(k)\}$ again, converging on any finite interval of Z as $p \to \infty$. Thus, we have a sequence $\{y_i(k)\}$ such that

$$x_{ip}(k) \longrightarrow y_i(k) \quad \text{for } k \in Z \text{ as } p \longrightarrow \infty. \tag{21}$$

This, combined with

$$x_i\left(k+1+\delta_p\right) = x_i\left(k+\delta_p\right) \times \exp\left\{a_i\left(k+\delta_p\right) - b_i\left(k+\delta_p\right)x_i\left(k+\delta_p\right) + \sum_{j=1,j\neq i}^{n} c_{ij}\left(k+\delta_p\right)\frac{x_j\left(k+\delta_p\right)}{d_{ij}+x_j\left(k+\delta_p\right)}\right\}, \quad i=1,2,\ldots,n, \tag{22}$$

gives us

$$y_i(k+1) = y_i(k)\exp\left\{a_i(k) - b_i(k)y_i(k) + \sum_{j=1,j\neq i}^{n} c_{ij}(k)\frac{y_j(k)}{d_{ij}+y_j(k)}\right\}, \quad i=1,2,\ldots,n. \tag{23}$$

We can easily see that $\{y_i(k)\}$ is a solution of system (1) and $m_i - \varepsilon \leq y_i(k) \leq M_i + \varepsilon$ for $k \in Z$. Since ε is an arbitrary small positive number, it follows that $m_i \leq y_i(k) \leq M_i$, and hence we complete the proof. □

4. Main Result

The main result of this paper concerns the existence of a globally attractive almost periodic solution of system (1).

Theorem 14. *Assume that (H1) and (H2)*

$$\rho_i = \max\left\{\left|1-b_i^l m_i\right|, \left|1-b_i^u M_i\right|\right\} + \sum_{j=1,j\neq i}^{n} c_{ij}^u M_j < 1, \quad i=1,2,\ldots,n, \tag{24}$$

hold. Then, system (1) *admits a unique almost periodic solution which is globally attractive.*

Proof. It follows from Proposition 13 that there exists a solution $(x_1(k), x_2(k), \ldots, x_n(k))$ of system (1) satisfying $m_i \leq x_i(k) \leq M_i$, $k \in Z^+$. Let $\{\delta_k\}$ be any integer valued sequence such that $\delta_k \to \infty$ as $k \to \infty$. Using the mean value theorem, for $p \neq q$, we get

$$\ln x_i\left(k+\delta_p\right) - \ln x_i\left(k+\delta_q\right) = \frac{1}{\xi_i(k,p,q)}\left[x_i\left(k+\delta_p\right) - x_i\left(k+\delta_q\right)\right], \tag{25}$$

where $\xi_i(k,p,q)$ lies between $x_i(k+\delta_p)$ and $x_i(k+\delta_q)$. Then,

$$\left|x_i\left(k+\delta_p\right) - x_i\left(k+\delta_q\right)\right| \leq M_i\left|\ln x_i\left(k+\delta_p\right) - \ln x_i\left(k+\delta_q\right)\right|, \quad k \in Z^+. \tag{26}$$

For convenience, we introduce $\varphi_i(k,\delta_p,\delta_q)$ through

$$\varphi_i\left(k,\delta_p,\delta_q\right) = \left|\ln x_i\left(k+\delta_p\right) - \ln x_i\left(k+\delta_q\right)\right|, \quad k \in Z^+, \quad \delta_p > 0, \quad \delta_q > 0. \tag{27}$$

Thus,

$$\begin{aligned}
\varphi_i&\left(k+1,\delta_p,\delta_q\right) \\
&= \left|\ln x_i\left(k+1+\delta_p\right) - \ln x_i\left(k+1+\delta_q\right)\right| \\
&= \Bigg|\ln x_i\left(k+\delta_p\right) - \ln x_i\left(k+\delta_q\right) + a_i\left(k+\delta_p\right) \\
&\quad - a_i\left(k+\delta_q\right) - b_i\left(k+\delta_p\right)x_i\left(k+\delta_p\right) \\
&\quad + b_i\left(k+\delta_q\right)x_i\left(k+\delta_q\right) \\
&\quad + \sum_{j=1,j\neq i}^{n} c_{ij}\left(k+\delta_p\right)\frac{x_j\left(k+\delta_p\right)}{d_{ij}+x_j\left(k+\delta_p\right)} \\
&\quad - \sum_{j=1,j\neq i}^{n} c_{ij}\left(k+\delta_q\right)\frac{x_j\left(k+\delta_q\right)}{d_{ij}+x_j\left(k+\delta_q\right)}\Bigg| \\
&\leq \Big|\ln x_i\left(k+\delta_p\right) - \ln x_i\left(k+\delta_q\right) - b_i\left(k+\delta_p\right) \\
&\quad \times \left[x_i\left(k+\delta_p\right) - x_i\left(k+\delta_q\right)\right]\Big| \\
&\quad + \left|a_i\left(k+\delta_p\right) - a_i\left(k+\delta_q\right)\right| \\
&\quad + \left|\left[b_i\left(k+\delta_q\right) - b_i\left(k+\delta_p\right)\right]x_i\left(k+\delta_q\right)\right| \\
&\quad + \sum_{j=1,j\neq i}^{n}\Bigg|c_{ij}\left(k+\delta_p\right)\Bigg[\frac{x_j\left(k+\delta_p\right)}{d_{ij}+x_j\left(k+\delta_p\right)} \\
&\qquad - \frac{x_j\left(k+\delta_q\right)}{d_{ij}+x_j\left(k+\delta_q\right)}\Bigg]\Bigg| \\
&\quad + \sum_{j=1,j\neq i}^{n}\Bigg|\left[c_{ij}\left(k+\delta_p\right) - c_{ij}\left(k+\delta_q\right)\right] \\
&\qquad \times \frac{x_j\left(k+\delta_q\right)}{d_{ij}+x_j\left(k+\delta_q\right)}\Bigg| \\
&\leq \Big|\ln x_i\left(k+\delta_p\right) - \ln x_i\left(k+\delta_q\right) - b_i\left(k+\delta_p\right) \\
&\quad \times \left[x_i\left(k+\delta_p\right) - x_i\left(k+\delta_q\right)\right]\Big| \\
&\quad + \left|a_i\left(k+\delta_p\right) - a_i\left(k+\delta_q\right)\right|
\end{aligned}$$

$$+\left|\left[b_i\left(k+\delta_q\right)-b_i\left(k+\delta_p\right)\right]x_i\left(k+\delta_q\right)\right|$$
$$+\sum_{j=1,j\neq i}^{n}\left|c_{ij}\left(k+\delta_p\right)\left[x_j\left(k+\delta_p\right)-x_j\left(k+\delta_q\right)\right]\right|$$
$$+\sum_{j=1,j\neq i}^{n}\left|\left[c_{ij}\left(k+\delta_p\right)-c_{ij}\left(k+\delta_q\right)\right]x_j\left(k+\delta_q\right)\right|. \quad (28)$$

Let ε_1 be an arbitrary positive number. By the almost periodicity of $\{a_i(k)\}$, $\{b_i(k)\}$, and $\{c_{ij}(k)\}$ and the boundedness of $\{(x_1(k), x_2(k), \ldots, x_n(k))\}$, it follows from Lemmas 5 and 7 that there exists a positive integer $K_1 = K_1(\varepsilon_1)$ such that, for any $\delta_q \geq \delta_p \geq K_1$ and $k \in Z^+$ (if necessary, we can choose subsequences of $\{\delta_p\}$ and $\{\delta_q\}$),

$$\begin{gathered}\left|a_i\left(k+\delta_p\right)-a_i\left(k+\delta_q\right)\right|<\frac{\varepsilon_1}{3},\\ \left|\left[b_i\left(k+\delta_q\right)-b_i\left(k+\delta_p\right)\right]x_i\left(k+\delta_q\right)\right|<\frac{\varepsilon_1}{3},\\ \sum_{j=1,j\neq i}^{n}\left|\left[c_{ij}\left(k+\delta_p\right)-c_{ij}\left(k+\delta_q\right)\right]x_j\left(k+\delta_q\right)\right| <\frac{\varepsilon_1}{3}.\end{gathered} \quad (29)$$

It follows from (25) and (27)–(29) that, for $k \in Z^+$ and $\delta_q \geq \delta_q \geq K_1$,

$$\begin{aligned}\varphi_i\left(k+1,\delta_p,\delta_q\right) &< \left|1-b_i\left(k+\delta_p\right)\xi_i\left(k,\delta_p,\delta_q\right)\right| \times\varphi_i\left(k,\delta_p,\delta_q\right)\\ &\quad+\sum_{j=1,j\neq i}^{n}\left|c_{ij}\left(k+\delta_p\right)\xi_j\left(k,\delta_p,\delta_q\right)\right| \times\varphi_j\left(k,\delta_p,\delta_q\right)+\varepsilon_1\\ &\leq \rho_i\max\left\{\varphi_i\left(k,\delta_p,\delta_q\right)\right\}+\varepsilon_1.\end{aligned} \quad (30)$$

Then,

$$\begin{gathered}\varphi_i\left(k,\delta_p,\delta_q\right)<\rho_i\max\left\{\varphi_i\left(k-1,\delta_p,\delta_q\right)\right\}+\varepsilon_1,\\ \varphi_i\left(k-1,\delta_p,\delta_q\right)<\rho_i\max\left\{\varphi_i\left(k-2,\delta_p,\delta_q\right)\right\}+\varepsilon_1,\\ \vdots\\ \varphi_i\left(1,\delta_p,\delta_q\right)<\rho_i\max\left\{\varphi_i\left(0,\delta_p,\delta_q\right)\right\}+\varepsilon_1.\end{gathered} \quad (31)$$

And we have

$$\varphi_i\left(k,\delta_p,\delta_q\right)<\rho_i^k\max\left\{\varphi_i\left(0,\delta_p,\delta_q\right)\right\}+\frac{1-\rho_i^k}{1-\rho_i}\varepsilon_1, \quad (32)$$

for $k \in Z^+$ and $\delta_q \geq \delta_q \geq K_1$.

Since $\rho_i < 1$, for arbitrary $\varepsilon > 0$, there exists a positive integer $K = K(\varepsilon) > K_1$ such that, for any $\delta_q \geq \delta_p \geq K$,

$$\varphi_i\left(k,\delta_p,\delta_q\right)<\frac{\varepsilon}{\max_{1\leq i\leq n}\{M_i\}} \quad (33)$$

for $k \in Z^+$.

This combined with (26) gives us

$$\begin{gathered}\left|x_i\left(k+\delta_p\right)-x_i\left(k+\delta_q\right)\right|<\varepsilon\\ \text{for } k\in Z^+,\quad \delta_q\geq\delta_q\geq K.\end{gathered} \quad (34)$$

It follows from Lemma 6 that the sequence $\{x_i(k)\}$ $(i = 1, 2, \ldots, n)$ is asymptotically almost periodic. Thus, we can express $\{x_i(k)\}$ as

$$x_i(k)=p_i(k)+q_i(k), \quad (35)$$

where $\{p_i(k)\}$ are almost periodic in $k \in Z$ and $q_i(k) \to 0$ as $k \to \infty$. In the following, we show that $\{p_i(k)\}$ $(i = 1, 2, \ldots, n)$ is an almost periodic solution of system (1).

Define

$$\begin{aligned}f_i(k)&=a_i(k)-b_i(k)\left[p_i(k)+q_i(k)\right]+\sum_{j=1,j\neq i}^{n}c_{ij}(k)\frac{p_j(k)+q_j(k)}{d_{ij}+p_j(k)+q_j(k)},\\ g_i(k)&=a_i(k)-b_i(k)p_i(k)+\sum_{j=1,j\neq i}^{n}c_{ij}(k)\frac{p_j(k)}{d_{ij}+p_j(k)},\quad i=1,2,\ldots,n.\end{aligned} \quad (36)$$

It follows from (1), (35), and the mean value theorem that

$$\begin{aligned}p_i(k+1)+q_i(k+1)&=\left[p_i(k)+q_i(k)\right]\exp\left[f_i(k)\right]\\ &=p_i(k)\left\{\exp\left[f_i(k)\right]-\exp\left[g_i(k)\right]\right\}+p_i(k)\exp\left[g_i(k)\right]+q_i(k)\exp\left[f_i(k)\right]\\ &=-p_i(k)\exp\left[\xi_i(k)\right]\times\left[b_i(k)q_i(k)+\sum_{j=1,j\neq i}^{n}c_{ij}(k)\left(\frac{p_j(k)}{d_{ij}+p_j(k)}-\frac{p_j(k)+q_j(k)}{d_{ij}+p_j(k)+q_j(k)}\right)\right]\\ &\quad+p_i(k)\exp\left[g_i(k)\right]+q_i(k)\exp\left[f_i(k)\right],\end{aligned} \quad (37)$$

where $\xi_i(k) = \theta_i(k) f_i(k) + (1-\theta_i(k)) g_i(k)$ for some $\theta_i(k) \in [0,1]$. Thus,

$$\begin{aligned} p_i(k+1) - p_i(k)\exp[g_i(k)] &= -p_i(k)\exp[\xi_i(k)] \\ &\quad \times \left[b_i(k) q_i(k) - \sum_{j=1, j\neq i}^{n} \frac{d_{ij} c_{ij}(k) q_j(k)}{\left(d_{ij} + p_j(k)\right)\left(d_{ij} + p_j(k) + q_j(k)\right)} \right] \\ &\quad - q_i(k+1) + q_i(k)\exp[f_i(k)]. \end{aligned} \tag{38}$$

Let

$$V_i(k) = p_i(k+1) - p_i(k)\exp[g_i(k)]. \tag{39}$$

By the boundedness of the almost periodic sequences $\{a_i(k)\}$, $\{b_i(k)\}$, $\{c_{ij}(k)\}$, and $\{p_i(k)\}$ and the fact that $q_i(k) \to 0$ as $k \to \infty$, we obtain

$$V_i(k) \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \tag{40}$$

We claim that $V_i(k) \equiv 0$. Otherwise, there exists an integer $k_0 \in Z$ such that $V_i(k_0) \neq 0$. By the almost periodicity of $\{a_i(k)\}$, $\{b_i(k)\}$, $\{c_{ij}(k)\}$, and $\{p_i(k)\}$, there exists an integer valued sequence τ_p such that $\tau_p \to \infty$ as $p \to \infty$ and

$$\begin{aligned} a_i\left(k+\tau_p\right) &\longrightarrow a_i(k), & b_i\left(k+\tau_p\right) &\longrightarrow b_i(k), \\ c_{ij}\left(k+\tau_p\right) &\longrightarrow c_{ij}(k), & p_i\left(k+\tau_p\right) &\longrightarrow p_i(k) \end{aligned} \tag{41}$$

uniformly for all $k \in Z$. Then, we have

$$\begin{aligned} V_i\left(k_0+\tau_p\right) &= p_i\left(k_0+\tau_p+1\right) - p_i\left(k_0+\tau_p\right)\exp\left[g_i\left(k_0+\tau_p\right)\right] \\ &\longrightarrow p_i(k_0+1) - p_i(k_0)\exp[g_i(k_0)] \\ &= V_i(k_0) \end{aligned} \tag{42}$$

as $p \to \infty$, which contradicts the fact that $V_i(k) \to 0$ as $k \to \infty$. This proves the claim. Hence,

$$p_i(k+1) = p_i(k)\exp[g_i(k)]; \tag{43}$$

that is, $\{p_i(k)\}$ is an almost periodic solution of system (1).

Assume that $(x_1(k), x_2(k), \ldots, x_n(k))$ is the solution of system (1) satisfying (H1). Let

$$x_i(k) = p_i(k)\exp(u_i(k)), \quad i = 1, 2, \ldots, n. \tag{44}$$

Then, system (1) is equivalent to

$$\begin{aligned} u_i(k+1) &= u_i(k) - b_i(k) p_i(k)\left[\exp(u_i(k)) - 1\right] \\ &\quad + \sum_{j=1, j\neq i}^{n} c_{ij}(k) \frac{d_{ij} p_j(k)\left[\exp\left(u_j(k)\right) - 1\right]}{\left[d_{ij} + p_j(k)\exp\left(u_j(k)\right)\right]\left[d_{ij} + p_j(k)\right]}, \\ &\qquad i = 1, 2, \ldots, n. \end{aligned} \tag{45}$$

Therefore,

$$\begin{aligned} u_i(k+1) &= u_i(k)\left[1 - b_i(k) p_i(k)\exp(\theta_i(k) u_i(k))\right] \\ &\quad + \sum_{j=1, j\neq i}^{n} c_{ij}(k) \frac{d_{ij} u_j(k) p_j(k)\exp\left(\overline{\theta_j}(k) u_j(k)\right)}{\left[d_{ij} + p_j(k)\exp\left(u_j(k)\right)\right]\left[d_{ij} + p_j(k)\right]}, \\ &\qquad i = 1, 2, \ldots, n, \end{aligned} \tag{46}$$

where $\theta_i(k), \overline{\theta_j}(k) \in [0,1]$. To complete the proof, it suffices to show that

$$\lim_{k\to\infty} u_i(k) = 0, \quad i = 1, 2, \ldots, n. \tag{47}$$

In view of (H2), we can choose $\varepsilon > 0$ such that

$$\begin{aligned} \rho_i^{\varepsilon} = {} & \max\left\{\left|1 - b_i^l(m_i - \varepsilon)\right|, \left|1 - b_i^u(M_i + \varepsilon)\right|\right\} \\ & + \sum_{j=1, j\neq i}^{n} c_{ij}^u\left(M_j + \varepsilon\right) < 1, \quad i = 1, 2, \ldots, n. \end{aligned} \tag{48}$$

Let $\rho = \max\{\rho_i^{\varepsilon}\}$; then $\rho < 1$. According to Proposition 11, there exists a positive integer $k_0 \in Z^+$ such that

$$\begin{gathered} m_i - \varepsilon \le x_i(k) \le M_i + \varepsilon, \\ m_i - \varepsilon \le p_i(k) \le M_i + \varepsilon, \\ i = 1, 2, \ldots, n, \end{gathered} \tag{49}$$

for $k \ge k_0$.

Notice that $\theta_i(k) \in [0,1]$ implies that $p_i(k)\exp(\theta_i(k)u_i(k))$ lies between $p_i(k)$ and $x_i(k)$; $\overline{\theta_j}(k) \in [0,1]$ implies that $p_j(k)\exp(\overline{\theta_j}(k)u_j(k))$ lies between $p_j(k)$ and $x_j(k)$. From (46), we get

$$\begin{aligned} \left|u_i(k+1)\right| \le {} & \max\left\{\left|1 - b_i^l(m_i - \varepsilon)\right|, \left|1 - b_i^u(M_i + \varepsilon)\right|\right\} \\ & \times \left|u_i(k)\right| + \sum_{j=1, j\neq i}^{n} c_{ij}^u\left(M_j^u + \varepsilon\right)\left|u_j(k)\right|, \\ & \qquad i = 1, 2, \ldots, n, \end{aligned} \tag{50}$$

for $k \ge k_0$.

In view of (50), we get

$$\max_{1\le i\le n}\left|u_i(k+1)\right| \le \rho \max_{1\le i\le n}\left|u_i(k)\right|, \quad k \ge k_0. \tag{51}$$

This implies

$$\max_{1\le i\le n}\left|u_i(k)\right| \le \rho^{k-k_0} \max_{1\le i\le n}\left|u_i(k_0)\right|, \quad k \ge k_0. \tag{52}$$

Then, (47) holds, and we can obtain

$$\lim_{k\to+\infty}\left|x_i(k)-p_i(k)\right| = 0, \quad i = 1,2,\ldots,n. \tag{53}$$

Therefore, system (1) admits a unique almost periodic solution which is globally attractive. This ends the proof of Theorem 14. □

In particular, if $n = 2$, we can obtain a discrete two-species Lotka-Volterra mutualism system:

$$\begin{aligned} x_1(n+1) &= x_1(n)\exp\left\{a_1(n) - b_1(n)x_1(n) + c_{12}(n)\frac{x_2(n)}{d_{12}+x_2(n)}\right\}, \\ x_2(n+1) &= x_2(n)\exp\left\{a_2(n) - b_2(n)x_2(n) + c_{21}(n)\frac{x_1(n)}{d_{21}+x_1(n)}\right\}. \end{aligned} \tag{54}$$

In the following, the main result concerns the existence of a uniformly asymptotically stable almost periodic solution of system (54).

From Proposition 11, we denote by Ω the set of all solutions $(x_1(n), x_2(n))$ of system (54) satisfying $m_i \le x_i(n) \le M_i$, $i = 1,2$, for all $n \in Z^+$. According to Lemma 10, we first prove that there is a bounded solution of system (54) and then structure a suitable Lyapunov function for system (54).

Proposition 15. *Assume that (H1) holds. Then, $\Omega \ne \Phi$.*

Proof. By an inductive argument, we have from system (54) that

$$\begin{aligned} x_1(n) &= x_1(0)\exp\sum_{l=0}^{n-1}\left[a_1(l) - b_1(l)x_1(l) + c_{12}(l)\frac{x_2(l)}{d_{12}+x_2(l)}\right], \\ x_2(n) &= x_2(0)\exp\sum_{l=0}^{n-1}\left[a_2(l) - b_2(l)x_1(l) + c_{21}(l)\frac{x_1(l)}{d_{21}+x_1(l)}\right]. \end{aligned} \tag{55}$$

According to Proposition 11, for any solution $(x_1(n), x_2(n))$ of system (54) and an arbitrarily small constant $\varepsilon > 0$, there exists n_0 sufficiently large such that

$$\begin{gathered} m_1 - \varepsilon \le x_1(n) \le M_1 + \varepsilon, \\ m_2 - \varepsilon \le x_2(n) \le M_2 + \varepsilon, \\ \forall n \ge n_0. \end{gathered} \tag{56}$$

Set $\{\tau_k\}$ to be any positive integer sequence such that $\tau_k \to +\infty$ as $k \to +\infty$; we can show that there exists a subsequence of $\{\tau_k\}$ still denoted by $\{\tau_k\}$, such that $x_i(k+\tau_k) \to x_i^*(k)$, $i = 1,2$, uniformly in n on any finite subset C of Z^+ as $k \to +\infty$, where $C = \{a_1, a_2, \ldots, a_m\}$, $a_h \in Z^+ (h = 1,2,\ldots,m)$, and m is a finite number.

As a matter of fact, for any finite subset $C \subset Z^+$, $\tau_k + a_h > n_0$, $h = 1,2,\ldots,m$, when k is large enough. Therefore, $m_i - \varepsilon \le x_i(n+\tau_k) \le M_i + \varepsilon$, $i = 1,2$; that is, $x_i(n+\tau_k)$ are uniformly bounded for k large enough.

Now, for $a_1 \in C$, we can choose a subsequence $\{\tau_k^{(1)}\}$ of $\{\tau_k\}$ such that $\{x_1(a_1+\tau_k^{(1)})\}$ and $\{x_2(a_1+\tau_k^{(1)})\}$ uniformly converge on Z^+ for k large enough.

Analogously, for $a_2 \in C$, we can also choose a subsequence $\{\tau_k^{(2)}\}$ of $\{\tau_k^{(1)}\}$ such that $\{x_1(a_2+\tau_k^{(2)})\}$ and $\{x_2(a_2+\tau_k^{(2)})\}$ uniformly converge on Z^+ for k large enough.

Repeating the above process, for $a_m \in C$, we get a subsequence $\{\tau_k^{(m)}\}$ of $\{\tau_k^{(m-1)}\}$ such that $\{x_1(a_m+\tau_k^{(m)})\}$ and $\{x_2(a_m+\tau_k^{(m)})\}$ uniformly converge on Z^+ for k large enough.

Now, we choose the sequence $\{\tau_k^{(m)}\}$ which is a subsequence of $\{\tau_k\}$ denoted by $\{\tau_k\}$; then, for all $n \in C$, we obtain that $x_i(n+\tau_k) \to x_i^*(n)$, $i = 1,2$, uniformly in $n \in C$ as $k \to +\infty$. Hence, the conclusion is valid by the arbitrariness of C.

Recall the almost periodicity of $\{a_i(n)\}$, $\{b_i(n)\}$, and $\{c_{ij}(n)\}$, for the above sequence $\{\tau_k\}$, $\tau_k \to +\infty$ as $k \to +\infty$; there exists a subsequence denoted by $\{\tau_k\}$ such that

$$\begin{gathered} a_i(n+\tau_k) \longrightarrow a_i(n), \\ b_i(n+\tau_k) \longrightarrow b_i(n), \\ c_{ij}(n+\tau_k) \longrightarrow c_{ij}(n), \\ i,j = 1,2, \quad i \ne j, \end{gathered} \tag{57}$$

as $k \to +\infty$ uniformly on Z^+.

For any $\alpha \in Z^+$, we can assume that $\tau_k + \alpha \ge n_0$ for k large enough. Let $n \in Z^+$, and, by an inductive argument of system (54) from $\tau_k + \alpha$ to $n + \tau_k + \alpha$, we obtain

$$\begin{aligned} & x_1(n+\tau_k+\alpha) \\ &\quad = x_1(\tau_k+\alpha)\exp\sum_{l=\tau_k+\alpha}^{n+\tau_k+\alpha-1}\left[a_1(l) - b_1(l)x_1(l) + c_{12}(l)\frac{x_2(l)}{d_{12}+x_2(l)}\right], \end{aligned}$$

$$x_2(n+\tau_k+\alpha) = x_2(\tau_k+\alpha)\exp\sum_{l=\tau_k+\alpha}^{n+\tau_k+\alpha-1}\left[a_2(l)-b_2(l)x_2(l) + c_{21}(l)\frac{x_1(l)}{d_{21}+x_1(l)}\right]. \quad (58)$$

Thus, it derives that

$$\begin{aligned} x_1(n+\tau_k+\alpha) &= x_1(\tau_k+\alpha) \times\exp\sum_{l=\alpha}^{n+\alpha-1}\left[a_1(l+\tau_k)-b_1(l+\tau_k)x_1(l+\tau_k) + c_{12}(l+\tau_k)\frac{x_2(l+\tau_k)}{d_{12}+x_2(l+\tau_k)}\right],\\ x_2(n+\tau_k+\alpha) &= x_2(\tau_k+\alpha) \times\exp\sum_{l=\alpha}^{n+\alpha-1}\left[a_2(l+\tau_k)-b_2(l+\tau_k)x_2(l+\tau_k) + c_{21}(l+\tau_k)\frac{x_1(l+\tau_k)}{d_{21}+x_1(l+\tau_k)}\right]. \end{aligned} \quad (59)$$

Let $k \to +\infty$; we have

$$\begin{aligned} x_1^*(n+\alpha) &= x_1^*(\alpha)\exp\sum_{l=\alpha}^{n+\alpha-1}\left[a_1(l)-b_1(l)x_1^*(l) + c_{12}(l)\frac{x_2^*(l)}{d_{12}+x_2^*(l)}\right],\\ x_2^*(n+\alpha) &= x_2^*(\alpha)\exp\sum_{l=\alpha}^{n+\alpha-1}\left[a_2(l)-b_2(l)x_2^*(l) + c_{21}(l)\frac{x_1^*(l)}{d_{21}+x_1^*(l)}\right]. \end{aligned} \quad (60)$$

Since α is arbitrary, we know that $(x_1^*(n), x_2^*(n))$ is a solution of system (54) on Z^+, and

$$\begin{aligned} 0 < m_1-\varepsilon \le x_1^*(n) \le M_1+\varepsilon,\\ 0 < m_2-\varepsilon \le x_2^*(n) \le M_2+\varepsilon,\\ \forall n \in Z^+. \end{aligned} \quad (61)$$

Notice that ε is an arbitrarily small positive constant; it follows that

$$\begin{aligned} 0 < m_1 \le x_1^*(n) \le M_1,\\ 0 < m_2 \le x_2^*(n) \le M_2,\\ \forall n \in Z^+. \end{aligned} \quad (62)$$

Thus, $\Omega \ne \Phi$. This completes the proof. □

Theorem 16. *Assume that (H1) holds; furthermore,* $0 < \beta < 1$, *where* $\beta = \min\{\beta_1, \beta_2\}$, *and*

$$\begin{aligned} \beta_i = b_i^l m_i - (b_i^u)^2 M_i^2 - \frac{d_{ij}c_{ij}^u M_j(1+b_i^u M_i)}{(d_{ij}+m_j)^2} - \frac{d_{ji}c_{ji}^u M_i(1+b_j^u M_j + d_{ji}c_{ji}^u M_i)}{(d_{ji}+m_i)^2},\\ i,j=1,2, \quad i\ne j. \end{aligned} \quad (63)$$

Then, there exists a unique uniformly asymptotically stable almost periodic solution of system (54) *which is bounded by* Ω *for all* $n \in Z^+$.

Proof. Denote $p_1(n) = \ln x_1(n)$, $p_2(n) = \ln x_2(n)$. It follows from system (54) that

$$\begin{aligned} p_1(n+1) &= p_1(n)+a_1(n)-b_1(n)e^{p_1(n)} + c_{12}(n)\frac{e^{p_2(n)}}{d_{12}+e^{p_2(n)}},\\ p_2(n+1) &= p_2(n)+a_2(n)-b_2(n)e^{p_2(n)} + c_{21}(n)\frac{e^{p_1(n)}}{d_{21}+e^{p_1(n)}}. \end{aligned} \quad (64)$$

According to Proposition 15, we can see that the system (64) has a bounded solution $(p_1(n), p_2(n))$ satisfying

$$\ln m_i \le \ln p_i(n) \le \ln M_i, \quad i=1,2,\ \forall n \in Z^+. \quad (65)$$

Thus, $|p_i(n)| \le \overline{M_i}$, where $\overline{M_i} = \max\{|\ln m_i|, |\ln M_i|\}$, $i = 1, 2$.

Define the norm $\|(p_1(n), p_2(n))\| = |p_1(n)| + |p_2(n)|$, where $(p_1(n), p_2(n)) \in R^2$. Consider the product system of system (64) as follows:

$$\begin{aligned} p_1(n+1) &= p_1(n)+a_1(n)-b_1(n)e^{p_1(n)} + c_{12}(n)\frac{e^{p_2(n)}}{d_{12}+e^{p_2(n)}},\\ p_2(n+1) &= p_2(n)+a_2(n)-b_2(n)e^{p_2(n)} + c_{21}(n)\frac{e^{p_1(n)}}{d_{21}+e^{p_1(n)}}, \end{aligned}$$

$$q_1(n+1) = q_1(n) + a_1(n) - b_1(n) e^{q_1(n)} + c_{12}(n) \frac{e^{q_2(n)}}{d_{12} + e^{q_2(n)}},$$
$$q_2(n+1) = q_2(n) + a_2(n) - b_2(n) e^{q_2(n)} + c_{21}(n) \frac{e^{q_1(n)}}{d_{21} + e^{q_1(n)}}. \quad (66)$$

We assume that $Q = (p_1(n), p_2(n))$, $W = (q_1(n), q_2(n))$ are any two solutions of system (66) defined on S^*; then, $\|Q\| \le B$, $\|W\| \le B$, where $B = \overline{M_1} + \overline{M_2}$, and $S^* = \{(p_1(n), p_2(n)) \mid \ln m_i \le p_i(n) \le \ln M_i, i = 1, 2, n \in Z^+\}$.

Let us construct a Lyapunov function defined on $Z^+ \times S^* \times S^*$ as follows:

$$V(n, Q, W) = (p_1(n) - q_1(n))^2 + (p_2(n) - q_2(n))^2. \quad (67)$$

It is obvious that the norm $\|Q-W\| = |p_1(n)-q_1(n)|+|p_2(n)-q_2(n)|$ is equivalent to $\|Q - W\|_* = [(p_1(n)-q_1(n))^2+(p_2(n)-q_2(n))^2]^{1/2}$; that is, there are two constants $c_1 > 0$, $c_2 > 0$, such that

$$c_1 \|Q - W\| \le \|Q - W\|_* \le c_2 \|Q - W\|; \quad (68)$$

then,

$$(c_1 \|Q - W\|)^2 \le V(n, Q, W) \le (c_2 \|Q - W\|)^2. \quad (69)$$

Let $\psi, \varphi \in C(R^+, R^+)$, $\psi(x) = c_1^2 x^2$, $\varphi = c_2^2 x^2$; then, condition (i) of Lemma 10 is satisfied.

Moreover, for any $(n, Q, W), (n, \overline{Q}, \overline{W}) \in Z^+ \times S^* \times S^*$, we have

$$\begin{aligned}
&\left|V(n, Q, W) - V\left(n, \overline{Q}, \overline{W}\right)\right| \\
&= \left|(p_1(n) - q_1(n))^2 + (p_2(n) - q_2(n))^2 - (\overline{p_1}(n) - \overline{q_1}(n))^2 - (\overline{p_2}(n) - \overline{q_2}(n))^2\right| \\
&\le [|p_1(n)| + |q_1(n)| + |\overline{p_1}(n)| + |\overline{q_1}(n)|] \times [|p_1(n) - \overline{p_1}(n)| + |q_1(n) - \overline{q_1}(n)|] \\
&\quad + [|p_2(n)| + |q_2(n)| + |\overline{p_2}(n)| + |\overline{q_2}(n)|] \times [|p_2(n) - \overline{p_2}(n)| + |q_2(n) - \overline{q_2}(n)|] \\
&\le L[|p_1(n) - \overline{p_1}(n)| + |p_2(n) - \overline{p_2}(n)| + |q_1(n) - \overline{q_1}(n)| + |q_2(n) - \overline{q_2}(n)|] \\
&= L\left(\left\|Q - \overline{Q}\right\| + \left\|W - \overline{W}\right\|\right),
\end{aligned} \quad (70)$$

where $\overline{Q}(\overline{p_1}(n), \overline{p_2}(n))$, $\overline{W}(\overline{q_1}(n), \overline{q_2}(n))$, and $L = 4\max\{\overline{M_1}, \overline{M_2}\}$. Thus, condition (ii) of Lemma 10 is satisfied.

Finally, calculating the $\Delta V(n)$ of $V(n)$ along the solutions of system (64), we have

$$\begin{aligned}
\Delta V(n) &= V(n+1) - V(n) \\
&= \left[(p_1(n+1) - q_1(n+1))^2 + (p_2(n+1) - q_2(n+1))^2\right] - \left[(p_1(n) - q_1(n))^2 + (p_2(n) - q_2(n))^2\right] \\
&= \sum_{i=1, j\ne i}^{2} \Bigg[-2b_i(n)(p_i(n) - q_i(n))\left(e^{p_i(n)} - e^{q_i(n)}\right) + b_i^2(n)\left(e^{p_i(n)} - e^{q_i(n)}\right)^2 + 2c_{ij}(n) \\
&\qquad \times \left[p_i(n) - q_i(n) - b_i(n)\left(e^{p_i(n)} - e^{q_i(n)}\right)\right] \times \frac{d_{ij}\left(e^{p_j(n)} - e^{q_j(n)}\right)}{\left(d_{ij} + e^{p_j(n)}\right)\left(d_{ij} + e^{q_j(n)}\right)} \\
&\qquad + c_{ij}^2(n) \frac{d_{ij}^2\left(e^{p_j(n)} - e^{q_j(n)}\right)^2}{\left(d_{ij} + e^{p_j(n)}\right)^2\left(d_{ij} + e^{q_j(n)}\right)^2}\Bigg].
\end{aligned} \quad (71)$$

By the mean value theorem, it derives that

$$e^{p_i(n)} - e^{q_i(n)} = \xi_i(n)(p_i(n) - q_i(n)), \quad i = 1, 2, \quad (72)$$

where $\xi_i(n)$ lies between $e^{p_i(n)}$ and $e^{q_i(n)}$. Then,

$$\begin{aligned}
\Delta V(n) = \sum_{i=1, j\ne i}^{2} \Bigg[&-2b_i(n)\xi_i(n)(p_i(n) - q_i(n))^2 + b_i^2(n)\xi_i^2(n)(p_i(n) - q_i(n))^2 \\
&+ \Big(\left(2d_{ij}c_{ij}(n)\xi_j(n) \times (p_i(n) - q_i(n)) \times (p_j(n) - q_j(n)) \times [1 - b_i(n)\xi_i(n)]\right) \\
&\quad \times \left(\left(d_{ij} + e^{p_j(n)}\right) \times \left(d_{ij} + e^{q_j(n)}\right)\right)^{-1}\Big) \\
&+ \frac{d_{ij}^2 c_{ij}^2(n)\xi_j^2(n)(p_j(n) - q_j(n))^2}{\left(d_{ij} + e^{p_j(n)}\right)^2\left(d_{ij} + e^{q_j(n)}\right)^2}\Bigg]
\end{aligned}$$

$$\leq \sum_{i=1,j\neq i}^{2}\Bigg[-2b_i(n)\xi_i(n)(p_i(n)-q_i(n))^2 + b_i^2(n)\xi_i^2(n)(p_i(n)-q_i(n))^2 + \frac{d_{ij}c_{ij}(n)\xi_j(n)[1+b_i(n)\xi_i(n)]}{(d_{ij}+m_j)^2} \times \left[(p_i(n)-q_i(n))^2 + (p_j(n)-q_j(n))^2\right] + \frac{d_{ij}^2c_{ij}^2(n)\xi_j^2(n)}{(d_{ij}+m_j)^2}(p_j(n)-q_j(n))^2\Bigg]$$

$$\leq \sum_{i=1,j\neq i}^{2}\Bigg[\Bigg[-2b_i(n)\xi_i(n)+b_i^2(n)\xi_i^2(n) + \frac{d_{ij}c_{ij}(n)\xi_j(n)[1+b_i(n)\xi_i(n)]}{(d_{ij}+m_j)^2}\Bigg] \times (p_i(n)-q_i(n))^2 + \Big(\big(d_{ij}c_{ij}(n)\xi_j(n) \times \big[1+b_i(n)\xi_i(n) + d_{ij}c_{ij}(n)\xi_j(n)\big]\big) \times (d_{ij}+m_j)^{-2}\Big) \times (p_j(n)-q_j(n))^2\Bigg]$$

$$\leq \sum_{i=1,j\neq i}^{2}\Bigg[\Bigg[-2b_i^l m_i + (b_i^u)^2 M_i^2 + \frac{d_{ij}c_{ij}^u M_j(1+b_i^u M_i)}{(d_{ij}+m_j)^2}\Bigg] \times (p_i(n)-q_i(n))^2 + \frac{d_{ij}c_{ij}^u M_j(1+b_i^u M_i + d_{ij}c_{ij}^u M_j)}{(d_{ij}+m_j)^2} \times (p_j(n)-q_j(n))^2\Bigg]$$

$$\leq -\sum_{i=1}^{2}\beta_i(p_i(n)-q_i(n))^2 \leq -\beta V(n), \tag{73}$$

where $\beta = \min\{\beta_1,\beta_2\}$. By the conditions of Theorem 16, we have $0 < \beta < 1$, and hence condition (iii) of Lemma 10 is satisfied. So, it follows from Lemma 10 that there exists a unique uniformly asymptotically stable almost periodic solution $(p_1^*(n), p_2^*(n))$ of system (64) which is bounded by S^* for all $n \in Z^+$; that is, there exists a unique uniformly asymptotically stable almost periodic solution $(x_1^*(n), x_2^*(n))$ of system (54) which is bounded by Ω for all $n \in Z^+$. This completed the proof. □

5. Numerical Simulations

In this section, we give the following example to check the feasibility of our result.

Example 1. Consider the discrete multispecies Lotka-Volterra mutualism system:

$$\begin{aligned}
x_1(k+1) &= x_1(k)\exp\Big\{1.2 - 0.02\sin(\sqrt{2}k) - (1.05+0.01\sin(\sqrt{3}k))x_1(k) + (0.025+0.002\cos(\sqrt{5}k)) \times \frac{x_2(k)}{1.1+x_2(k)} + (0.02+0.001\cos(\sqrt{2}k)) \times \frac{x_3(k)}{1.2+x_3(k)}\Big\},\\
x_2(k+1) &= x_2(k)\exp\Big\{1.1 - 0.025\cos(\sqrt{3}k) - (1.08+0.015\sin(\sqrt{2}k))x_2(k) + (0.02+0.003\sin(\sqrt{2}k)) \times \frac{x_1(k)}{1.4+x_1(k)} + (0.025+0.002\cos(\sqrt{5}k)) \times \frac{x_3(k)}{1.6+x_3(k)}\Big\},\\
x_3(k+1) &= x_3(k)\exp\Big\{1.15 - 0.03\cos(\sqrt{5}k) - (1.1+0.02\cos(\sqrt{2}n))x_1(k)
\end{aligned}$$

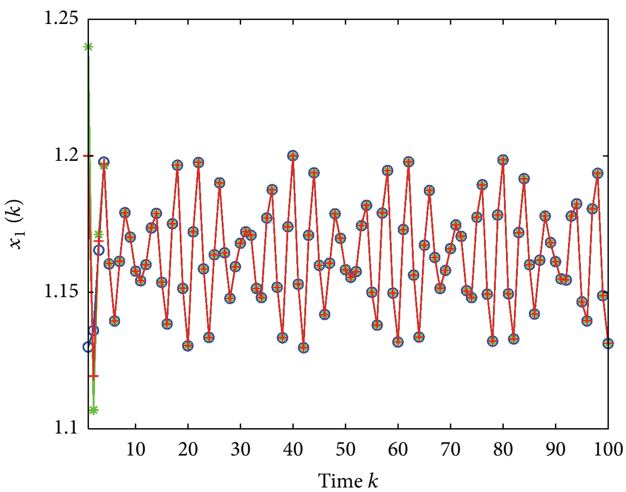

Figure 1: Dynamic behavior of the first component $x_1(k)$ of the solution $(x_1(k), x_2(k), x_3(k))$ to system (74) with the initial conditions (1.13, 1.17, 1.2), (1.24, 0.96, 0.95), and (1.2, 1.08, 1.14) for $k \in [1, 100]$, respectively.

$$+\left(0.03+0.0025\sin\left(\sqrt{2}k\right)\right) \times \frac{x_1(k)}{2.1+x_1(k)} + \left(0.028+0.0015\sin\left(\sqrt{3}k\right)\right) \times \frac{x_2(k)}{1.8+x_2(k)}\Bigg\}. \tag{74}$$

A computation shows that

$$\begin{aligned} m_1 &\approx 0.9538, & M_1 &\approx 1.257, \\ m_2 &\approx 0.9817, & M_2 &\approx 1.1185, \\ m_3 &\approx 0.818, & M_3 &\approx 1.1794, \end{aligned} \tag{75}$$

and, moreover, we have that

$$\rho_1 \approx 0.3857, \qquad \rho_2 \approx 0.2856, \qquad \rho_3 \approx 0.3947, \tag{76}$$

and that $\max\{\rho_1, \rho_2, \rho_3\} \approx 0.3947 < 1$. It is easy to see that condition (H2) is satisfied. Hence, there exists a unique globally attractive almost periodic solution of system (74). Our numerical simulations support our results (see Figures 1, 2, and 3).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper, and there is no financial conflict of interests between the authors and the commercial identity.

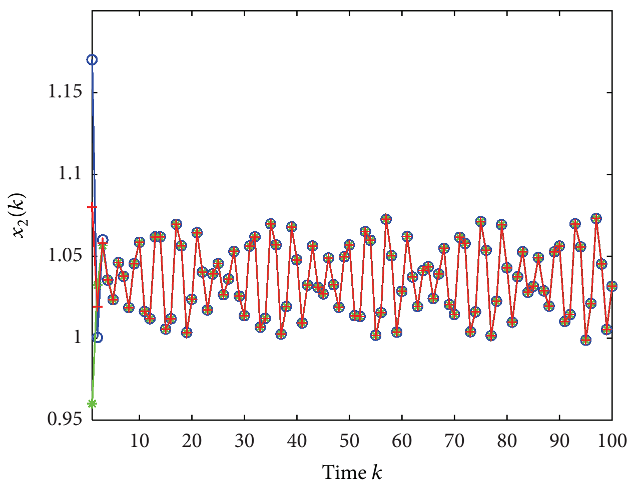

Figure 2: Dynamic behavior of the second component $x_2(k)$ of the solution $(x_1(k), x_2(k), x_3(k))$ to system (74) with the initial conditions (1.13, 1.17, 1.2), (1.24, 0.96, 0.95), and (1.2, 1.08, 1.14) for $k \in [1, 100]$, respectively.

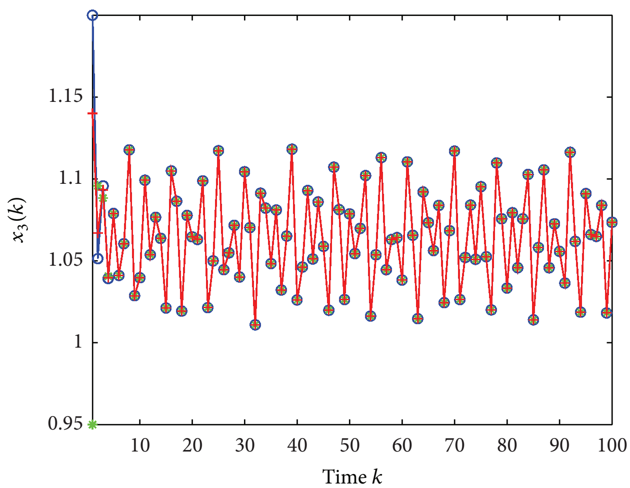

Figure 3: Dynamic behavior of the third component $x_3(k)$ of the solution $(x_1(k), x_2(k), x_3(k))$ to system (74) with the initial conditions (1.13, 1.17, 1.2), (1.24, 0.96, 0.95), and (1.2, 1.08, 1.14) for $k \in [1, 100]$, respectively.

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