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Research Article

Characterizing ξ-Lie Multiplicative Isomorphisms on Von Neumann Algebras

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Let $\mathscr{M}$ and $\mathscr{N}$ be von Neumann algebras without central summands of type I_1. Assume that $\xi \in \mathbb{C}$ with $\xi \neq 1$. In this paper, all maps $\Phi : \mathscr{M} \rightarrow \mathscr{N}$ satisfying $\Phi(AB - \xi BA) = \Phi(A)\Phi(B) - \xi\Phi(B)\Phi(A)$ are characterized.

1. Introduction

Let $\mathscr{A}$ and $\mathscr{A}'$ be two algebras over a field $\mathbb{F}$. Recall that a map $\Phi : \mathscr{A} \rightarrow \mathscr{A}'$ is called a multiplicative map if $\Phi(AB) = \Phi(A)\Phi(B)$ for all $A, B \in \mathscr{A}$; a Lie multiplicative map if $\Phi([A, B]) = [\Phi(A), \Phi(B)]$ for all $A, B \in \mathscr{A}$, where $[A, B] = AB - BA$; and a Jordan multiplicative map if $\Phi(AB + BA) = \Phi(A)\Phi(B) + \Phi(B)\Phi(A)$ for all $A, B \in \mathscr{A}$.

The question when a multiplicative map is additive is studied by many mathematicians. As the first result in this line, Matindale [1] proved that every multiplicative bijective map from a prime ring containing a nontrivial idempotent onto an arbitrary ring is additive and thus is a ring isomorphism. Recently, Matindale's result has been generalized in several directions, such as multiplicative maps and Jordan multiplicative maps between standard operator algebras or nest algebras (see [2, 3] and the references therein). For Lie multiplicative maps, Bai et al. [4] showed that if $\mathscr{R}$, $\mathscr{R}'$ are prime rings with $\mathscr{R}$ being unital and containing a nontrivial idempotent and if $\Phi : \mathscr{R} \rightarrow \mathscr{R}'$ is a Lie multiplicative bijective map, then $\Phi(T+S) = \Phi(T) + \Phi(S) + Z'_{T,S}$ for all $T, S \in \mathscr{R}$, where $Z'_{T,S}$ is an element in the center of $\mathscr{R}'$ depending on T and S. This result reveals that the Lie multiplicativity of a map does not imply its additivity anymore. Note that factor von Neumann algebras are prime. Later, the similar results were obtained on triangular algebras and certain Banach space nest algebras, respectively, in [5, 6].

Let $\mathscr{A}$ be an algebra over a field $\mathbb{F}$. For a scalar $\xi \in \mathbb{F}$ and for $A, B \in \mathscr{A}$, we say that A commutes with B up to a factor ξ if $AB = \xi BA$. The notion of commutativity up to a factor for pairs of operators is an important concept and has been studied in the context of operator algebras and quantum groups [7, 8]. Motivated by this, a binary operation $[A, B]_\xi = AB - \xi BA$, called ξ-Lie product of A and B, was introduced in [9]. Moreover, a concept of ξ-Lie multiplicative maps was introduced in [10], which unifies the above three kinds of maps. Recall that a map $\Phi : \mathscr{A} \rightarrow \mathscr{A}'$ is called a ξ-Lie multiplicative map if $\Phi([A, B]_\xi) = [\Phi(A), \Phi(B)]_\xi$ for all $A, B \in \mathscr{A}$. In addition, Φ is called a ξ-Lie multiplicative isomorphism if Φ is bijective and ξ-Lie multiplicative and is called a ξ-Lie ring isomorphism if Φ is bijective, additive, and ξ-Lie multiplicative. A linear (resp., conjugate linear) ξ-Lie ring isomorphism between two algebras is called a ξ-Lie isomorphism (resp., conjugate ξ-Lie isomorphism).

Recall that a standard operator algebra on a Banach space X is a subalgebra of the whole operator algebra $\mathscr{B}(X)$ containing the identity operator I and the ideal of all finite rank operators. Qi and Hou in [10] gave a characterization of all ξ-Lie multiplicative isomorphisms between standard operator algebras. Let $\mathscr{A}$ and $\mathscr{B}$ be standard operator algebras on infinite dimensional Banach spaces X and Y over the real or complex field $\mathbb{F}$, respectively. Assume that $\Phi : \mathscr{A} \rightarrow \mathscr{B}$ is a unital bijection and ξ is a scalar. The main result in [10] states that Φ is ξ-Lie multiplicative if and only if one of the following holds: (1) $\xi = 1$, there exists a functional $h : \mathscr{A} \to \mathbb{F}$ with $h([A, B]) = 0$ for all A, B, and either there exists an invertible bounded linear or conjugate linear operator $T : X \to Y$ such that $\Phi(A) = TAT^{-1} + h(A)I$ for all $A \in \mathscr{A}$ or there exists an invertible bounded linear or conjugate linear operator $T : X^* \to Y$ such that $\Phi(A) = -TA^*T^{-1} + h(A)I$ for all $A \in \mathscr{A}$; (2) $\xi = -1$, either there exists an invertible bounded linear or conjugate linear operator $T : X \to Y$ such that $\Phi(A) = TAT^{-1}$ for all $A \in \mathscr{A}$, or there exists an invertible bounded linear or conjugate linear operator $T : X^* \to Y$ such that $\Phi(A) = TA^*T^{-1}$ for all $A \in \mathscr{A}$; (3) $\xi \in \mathbb{R}\backslash\{\pm 1\}$, there exists an invertible bounded linear operator $T : X \to Y$ such that $\Phi(A) = TAT^{-1}$ for all $A \in \mathscr{A}$ if $\mathbb{F} = \mathbb{R}$; there exists an invertible bounded linear or conjugate linear operator $T : X \to Y$ such that $\Phi(A) = TAT^{-1}$ for all $A \in \mathscr{A}$ if $\mathbb{F} = \mathbb{C}$; (4) $\xi \in \mathbb{C} \setminus \mathbb{R}$, there exists an invertible bounded linear operator $T : X \to Y$ such that $\Phi(A) = TAT^{-1}$ for all $A \in \mathscr{A}$. A complete characterization of ξ-Lie multiplicative isomorphisms on matrix algebras and certain nest algebras was given, respectively, in [10] and [6]. These results reveal the structural properties of the involved operator algebras from some new aspects. However, we have not seen any description on the structure of the ξ-Lie multiplicative isomorphisms between nonfactor von Neumann algebras so far. The present paper considers this problem.

The purpose of this paper is to characterize the ξ-Lie multiplicative isomorphisms with $\xi \neq 1$ between certain quite general von Neumann algebras. Let $\mathscr{M}$ and $\mathscr{N}$ be two von Neumann algebras without central summands of type I_1. Denote by I_M and I_N the unit operators in $\mathscr{M}$ and $\mathscr{N}$, respectively. Assume that $\Phi : \mathscr{M} \to \mathscr{N}$ is a map and $\xi \in \mathbb{C}$ with $\xi \neq 1$. We show that Φ is a ξ-Lie multiplicative isomorphism if and only if one of the following statements is true: (1) $\xi = 0$, Φ is a ring isomorphism; (2) $\xi = -1$, there exist central projections $P \in \mathscr{M}$ and $Q \in \mathscr{N}$ such that $\Phi = \Phi_1 \oplus \Phi_2$, where $\Phi_1 = \Phi|_{P\mathscr{M}} : P\mathscr{M} \to Q\mathscr{N}$ is a ring isomorphism and $\Phi_2 = \Phi|_{(I_M - P)\mathscr{M}} : (I_M - P)\mathscr{M} \to (I_N - Q)\mathscr{N}$ is a ring anti-isomorphism; (3) $\xi \neq 0, -1$, there exist central projections $P \in \mathscr{M}$ and $Q \in \mathscr{N}$ such that $\Phi = \Phi_1 \oplus \Phi_2$, where $\Phi_1 : P\mathscr{M} \to Q\mathscr{N}$ is a ring isomorphism with $\Phi_1(\xi A_1) = \xi\Phi_1(A_1)$ for all $A_1 \in P\mathscr{M}$ and $-\xi\Phi_2 : (I_M - P)\mathscr{M} \to (I_N - Q)\mathscr{N}$ is a ring anti-isomorphism with $\Phi_2(\xi A_2) = (1/\xi)\Phi_2(A_2)$ for all $A_2 \in (I_M - P)\mathscr{M}$ (Theorem 1). It is clear from our result that, for $\xi \neq 1$, the ξ-Lie multiplicativity of a bijective map on von Neumann algebras without central summands of type I_1 implies its additivity. Particularly, multiplicative isomorphisms are ring isomorphisms and the Jordan multiplicative isomorphisms are Jordan ring isomorphisms. Moreover, for rational real number $\xi \notin \{1, -1\}$, ξ-Lie multiplicative isomorphisms are ring isomorphisms. However, ξ-Lie ring isomorphisms with nonrational ξ have some more complicated but still controllable algebraic structures. Assume further that $\mathscr{M}, \mathscr{N}$ act on some separable Hilbert spaces and $\mathscr{N}$ has no central summands of type I_n for any $n < \infty$; then Φ is a ξ-Lie multiplicative isomorphism if and only if one of the following statements is true: (1) $\xi \in \mathbb{R} \setminus \{-1\}$, $\Phi = \Phi_1 \oplus \Phi_2$, where Φ_1 is an algebraic isomorphism and Φ_2 is a conjugate algebraic isomorphism; (2) $\xi = -1$, $\Phi = \Phi_1 \oplus \cdots \oplus \Phi_4$, where Φ_1 is an algebraic isomorphism, Φ_2 is a conjugate algebraic isomorphism, Φ_3 is an algebraic anti-isomorphism, and Φ_4 is a conjugate algebraic anti-isomorphism; (3) $\xi \in \mathbb{C} \setminus \mathbb{R}$ with $|\xi| \neq 1$, Φ is an algebraic isomorphism; (4) $\xi \in \mathbb{C} \setminus \mathbb{R}$ with $|\xi| = 1$, $\Phi = \Phi_1 \oplus \Phi_2$, where Φ_1 is an algebraic isomorphism and $-\xi\Phi_2$ is a conjugate algebraic anti-isomorphism.

For the case of $\xi = 1$, the result and the approach are quite different and we will discuss it in another paper.

2. Main Result and Corollary

The following is our main result and its proof will be presented in the next section.

Theorem 1. *Let $\mathscr{M}$ and $\mathscr{N}$ be von Neumann algebras without central summands of type I_1. Assume that $\Phi : \mathscr{M} \to \mathscr{N}$ is a map and $\xi \in \mathbb{C}$ with $\xi \neq 1$. Then, Φ is a ξ-Lie multiplicative isomorphism, that is, Φ is bijective and satisfies $\Phi(AB - \xi BA) = \Phi(A)\Phi(B) - \xi\Phi(B)\Phi(A)$ for all $A, B \in \mathscr{M}$, if and only if one of the following statements holds:*

(1) $\xi = 0$, Φ is a ring isomorphism.

(2) $\xi = -1$, there exist central projections $P \in \mathscr{M}$ and $Q \in \mathscr{N}$ such that $\Phi = \Phi_1 \oplus \Phi_2$, where $\Phi_1 = \Phi|_{P\mathscr{M}} : P\mathscr{M} \to Q\mathscr{N}$ is a ring isomorphism and $\Phi_2 = \Phi|_{(I_M - P)\mathscr{M}} : (I_M - P)\mathscr{M} \to (I_N - Q)\mathscr{N}$ is a ring anti-isomorphism.

(3) $\xi \neq 0, -1$, there exist central projections $P \in \mathscr{M}$ and $Q \in \mathscr{N}$ such that $\Phi = \Phi_1 \oplus \Phi_2$, where $\Phi_1 : P\mathscr{M} \to Q\mathscr{N}$ is a ring isomorphism with $\Phi_1(\xi A_1) = \xi\Phi_1(A_1)$ for all $A_1 \in P\mathscr{M}$ and $-\xi\Phi_2 : (I_M - P)\mathscr{M} \to (I_N - Q)\mathscr{N}$ is a ring anti-isomorphism with $\Phi_2(\xi A_2) = (1/\xi)\Phi_2(A_2)$ for all $A_2 \in (I_M - P)\mathscr{M}$. Here, I_M denotes the unit in $\mathscr{M}$.

We remark that, from the above result, for bijective maps between von Neumann algebras without central summands of type I_1, the ξ-Lie multiplicativity ($\xi \neq 1$) will imply the additivity; moreover, if ξ is a rational real number and $\xi \notin \{0, \pm 1\}$, then by (3), $\xi\Phi_2(A_2) = \Phi_2(\xi A_2) = (1/\xi)\Phi_2(A_2)$ for all $A_2 \in (I_M - P)\mathscr{M}$, which forces that $\Phi_2 = 0$ and hence $\Phi = \Phi_1$ is a ring isomorphism.

In the sequel, we study in more detail on ξ-Lie multiplicative isomorphisms and get a corollary of Theorem 1. Firstly, let us give a lemma which is interesting in itself.

Lemma 2. *Let $\mathscr{N}$ be a factor von Neumann algebra. Assume that $\mathscr{M}$ is a von Neumann algebra and $\Psi : \mathscr{M} \to \mathscr{N}$ is a ring isomorphism (ring anti-isomorphism). If $\dim \mathscr{N} < \infty$, then there exists a field automorphism $\tau : \mathbb{C} \to \mathbb{C}$ such that Ψ is τ-linear; if $\dim \mathscr{N} = \infty$, then Ψ is linear or conjugate linear.*

Proof. We only deal in detail with the case that Ψ is a ring isomorphism. The other case can be proved similarly.

Assume that Ψ is a ring isomorphism. It is clear that $\Psi(\mathscr{Z}(\mathscr{M})) = \mathscr{Z}(\mathscr{N})$, which implies that $\Psi(\mathbb{C}I_M) \subseteq \mathbb{C}I_N$ as $\mathscr{N}$ is a factor. Then, since Ψ is a ring isomorphism, it is easy to check that Ψ is unital; that is, $\Psi(I_M) = I_N$.

If $\mathscr{M}$ is not a factor, there exists a nonzero central element $C \in \mathscr{Z}(\mathscr{M})$ such that C is not invertible. Thus, $\Psi(C) \in \mathbb{C}I_N$ is also not invertible, and so $\Psi(C) = 0$, a contradiction. Hence, $\mathscr{M}$ is a factor and $\Psi(\mathbb{C}I_M) = \mathbb{C}I_N$.

For any $\lambda \in \mathbb{C}$, let $\Psi(\lambda I_M) = \tau(\lambda) I_N$. Then,

$$\Psi(\lambda A) = \Psi(\lambda I_M A) = \tau(\lambda)\Psi(A) \quad \forall \lambda \in \mathbb{C} \text{ and } A \in \mathscr{M}. \tag{1}$$

We claim that $\tau : \mathbb{C} \to \mathbb{C}$ is a field automorphism. In fact, since Ψ is additive, $\tau : \mathbb{C} \to \mathbb{C}$ is additive. Note that

$$\tau(\lambda\gamma) I_N = \Psi(\lambda\gamma I_M) = \Psi(\lambda I_M)\Psi(\gamma I_M) = \tau(\lambda)\tau(\gamma) I_N. \tag{2}$$

This implies that $\tau(\lambda\gamma) = \tau(\lambda)\tau(\gamma)$; that is, τ is multiplicative.

In the following, we assume that $\mathscr{N}$ is infinite dimensional. We will show that τ is continuous. As $\dim \mathscr{N} = \infty$, there exists a sequence $\{P_n\}_{n=1}^{\infty}$ of projections which are orthogonal to each other. So, we have $0 = \Psi(P_n P_m) = \Psi(P_n)\Psi(P_m)$ for $n \neq m$ and $0 \neq \Psi(P_n) = \Psi(P_n P_n) = \Psi(P_n)^2$. If τ is not continuous, then τ is unbounded on any neighborhood of 0. So τ is unbounded on $\{z : |z| \leq (1/2)\}$ and hence there exists λ_1 with $|\lambda_1| < (1/2)$ such that

$$\|\Psi(\lambda_1 P_1)\| = |\tau(\lambda_1)| \|\Psi(P_1)\| > \|\Psi(P_1)\|. \tag{3}$$

Considering $\{z : |z| \leq (1/2^2)\}$ gives λ_2 with $|\lambda_2| < (1/2^2)$ such that

$$\|\Psi(\lambda_2 P_2)\| = |\tau(\lambda_2)| \|\Psi(P_2)\| > 2\|\Psi(P_2)\|. \tag{4}$$

Generally, for any n, there exists λ_n with $|\lambda_n| < (1/2^n)$ such that

$$\|\Psi(\lambda_n P_n)\| = |\tau(\lambda_n)| \|\Psi(P_n)\| > n\|\Psi(P_n)\|. \tag{5}$$

Let $A = \sum_{n=1}^{\infty} \lambda_n P_n$; then $\|A\| \leq 1$. This implies that $A \in \mathscr{M}$ and $\|\Psi(A)\| < \infty$. However,

$$\begin{aligned} \|\Psi(A)\| \|\Psi(P_n)\| &\geq \|\Psi(A)\Psi(P_n)\| \\ &= \|\Psi(AP_n)\| = \|\Psi(\lambda_n P_n)\| > n\|\Psi(P_n)\|, \end{aligned} \tag{6}$$

which implies that $\|\Psi(A)\| > n$ for any n, a contradiction. Hence, τ is continuous and by [11, pp. 52–57] is the identity or the conjugation. Therefore, Ψ is linear or conjugate linear. □

Lemma 3. *Let $\mathscr{M}$ be any von Neumann algebra and $\mathscr{N}$ a factor of infinite dimension. Assume that $-\xi\Psi : \mathscr{M} \to \mathscr{N}$ is a ring anti-isomorphism and $\xi \in \mathbb{C}$ with $\xi \neq 0, \pm 1$. Then, $\Psi(\xi I_M) = (1/\xi) I_N$ if and only if $-\xi\Psi$ is a conjugate algebra anti-isomorphism and $|\xi| = 1$.*

Proof. By Lemma 2, Ψ is linear or conjugate linear. So $\Psi(\xi I_M) = \xi I_N$ or $\Psi(\xi I_M) = \overline{\xi} I_N$. If $\Psi(\xi I_M) = (1/\xi) I_N$, it follows that $(1/\xi) I_N = \xi I_N$ or $(1/\xi) I_N = \overline{\xi} I_N$, which imply that $\xi^2 = 1$ or $\xi\overline{\xi} = 1$. Since $\xi \neq 0, \pm 1$, we see that Ψ must be conjugate linear and $|\xi| = 1$. The converse is obvious. □

Lemma 4. *Let $\mathscr{M}$ and $\mathscr{N}$ be von Neumann algebras acting on separable Hilbert spaces and assume that $\mathscr{N}$ has no central summands of type I_n for any $1 \leq n < \infty$.*

(1) $\Psi : \mathscr{M} \to \mathscr{N}$ is a ring isomorphism (resp., a ring anti-isomorphism) if and only if there exist central projections $P \in \mathscr{M}$ and $Q \in \mathscr{N}$, an algebraic isomorphism (resp., an algebraic anti-isomorphism) $\Psi_1 : P\mathscr{M} \to Q\mathscr{N}$ and a conjugate algebraic isomorphism (resp., a conjugate algebraic anti-isomorphism) $\Psi_2 : (I_M - P)\mathscr{M} \to (I_N - Q)\mathscr{N}$ such that $\Psi = \Psi_1 \oplus \Psi_2$.

(2) Assume that $-\xi\Psi : \mathscr{M} \to \mathscr{N}$ is a ring anti-isomorphism and $\xi \in \mathbb{C} \setminus \{0, \pm 1\}$. Then, $\Psi(\xi I_M) = (1/\xi) I_N$ if and only if $-\xi\Psi$ is a conjugate algebraic anti-isomorphism and $|\xi| = 1$.

Proof. (1) We consider the case of ring anti-isomorphism; the case of ring isomorphism is treated similarly.

Assume that Ψ is a ring anti-isomorphism. By [12, pp. 209, 236], there exists a positive measure space $(\mathbf{X}, \Omega, \mu)$ such that

$$\mathscr{N} = \int_{\mathbf{X}} \mathscr{N}_t d\mu_t, \quad \mathscr{M} = \int_{\mathbf{X}} \Psi_t^{-1}(\mathscr{N}_t) d\mu_t, \quad \Psi = \int_{\mathbf{X}} \Psi_t d\mu_t, \tag{7}$$

where every $\mathscr{N}_t$ is a factor and $\Psi_t : \mathscr{M}_t = \Psi^{-1}(\mathscr{N}_t) \to \mathscr{N}_t$ is a ring anti-isomorphism. Since $\mathscr{N}$ has no central summands of type I_n for any $1 \leq n < \infty$, $\mathscr{N}_t$ is a factor of infinite dimensional a.e. $[\mu]$. By Lemma 2, Ψ_t is linear or conjugate linear. If Ψ_t is not a conjugate algebraic anti-isomorphism a.e. $[\mu]$ for all t, then there exists a measurable subset with nonzero measure such that Ψ_t is an algebraic anti-isomorphism a.e. $[\mu]$ on it. It follows that there exists a proper central projection P_1 such that $\Psi|_{P_1\mathscr{M}}$ is an algebraic anti-isomorphism. Note that $\Psi(P_1)$ is a central projection. Now it is clear (e.g., using Zorn's Lemma) that there exist central projections $P \in \mathscr{M}$ and $Q \in \mathscr{N}$, an algebraic anti-isomorphism $\Psi_1 : P\mathscr{M} \to Q\mathscr{N}$ and a conjugate algebraic anti-isomorphism $\Psi_2 : (I_M - P)\mathscr{M} \to (I_N - Q)\mathscr{N}$ such that $\Psi = \Psi_1 \oplus \Psi_2$.

Conversely, if Ψ has the mentioned decomposition, then Ψ is clearly a ring anti-isomorphism.

(2) The "if" part is clear. To check the "only if" part, assume that $-\xi\Phi$ is a ring anti-isomorphism and $\Psi(\xi I_M) = (1/\xi)\Psi(I_M)$. For any $A \in \mathscr{M}$, writing $A = \int_{\mathbf{X}} A_t d\mu_t$, we have $\Psi(A) = \int_{\mathbf{X}} \Psi_t(A_t) d\mu_t$. Hence,

$$\frac{1}{\xi}\int_{\mathbf{X}} I_{tN} d\mu_t = \frac{1}{\xi} I_N = \Psi(\xi I_M) = \int_{\mathbf{X}} \Psi_t(\xi I_{tM}) d\mu_t. \tag{8}$$

It follows that $\Psi_t(\xi I_{tM}) = (1/\xi) I_{tN}$ a.e. $[\mu]$. By Lemma 3, we have $|\xi| = 1$ and $-\xi\Psi_t$ is conjugate linear a.e. $[\mu]$, and so $-\xi\Psi$ is a conjugate algebraic anti-isomorphism. □

Now, we are in a position to give the following corollary of Theorem 1.

Corollary 5. *Let $\mathscr{M}$ and $\mathscr{N}$ be any von Neumann algebras acting on separable Hilbert spaces without central summands of type I_1. Assume further that $\mathscr{N}$ has no any central summands of type I_n for $1 \leq n < \infty$. Let $\Phi : \mathscr{M} \to \mathscr{N}$ be a map and $\xi \in \mathbb{C}$ with $\xi \neq 1$. Then, Φ is a ξ-Lie multiplicative isomorphism if and only if one of the following statements is true.*

(1) $\xi \in \mathbb{R} \setminus \{-1\}$, there exist central projections $P \in \mathscr{M}$ and $Q \in \mathscr{N}$, and an algebraic isomorphism $\Phi_1 : P\mathscr{M} \to Q\mathscr{N}$ and a conjugate algebraic isomorphism $\Phi_2 : (I_M - P)\mathscr{M} \to (I_N - Q)\mathscr{N}$ such that $\Phi = \Phi_1 \oplus \Phi_2$.

(2) $\xi = -1$, there exist central projections $P_1, \ldots, P_4 \in \mathscr{M}$ and $Q_1, \ldots, Q_4 \in \mathscr{N}$ with $P_1 + \cdots + P_4 = I_M$ and $Q_1 + \cdots + Q_4 = I_N$ such that $\Phi = \Phi_1 \oplus \cdots \oplus \Phi_4$, where $\Phi_1 = \Phi|_{P_1\mathscr{M}} : P_1\mathscr{M} \to Q_1\mathscr{N}$ is an algebraic isomorphism, $\Phi_2 = \Phi|_{P_2\mathscr{M}} : P_2\mathscr{M} \to Q_2\mathscr{N}$ is a conjugate algebraic isomorphism, $\Phi_3 = \Phi|_{P_3\mathscr{M}} : P_3\mathscr{M} \to Q_3\mathscr{N}$ is an algebraic anti-isomorphism, and $\Phi_4 = \Phi|_{P_4\mathscr{M}} : P_4\mathscr{M} \to Q_4\mathscr{N}$ is a conjugate algebraic anti-isomorphism.

(3) $\xi \in \mathbb{C} \setminus \mathbb{R}$ with $|\xi| \neq 1$, Φ is an algebraic isomorphism.

(4) $\xi \in \mathbb{C} \setminus \mathbb{R}$ with $|\xi| = 1$, there exist central projections $P \in \mathscr{M}$ and $Q \in \mathscr{N}$ such that $\Phi = \Phi_1 \oplus \Phi_2$, where $\Phi_1 : P\mathscr{M} \to Q\mathscr{N}$ is an algebraic isomorphism and $-\xi\Phi_2 : (I_M - P)\mathscr{M} \to (I_N - Q)\mathscr{N}$ is a conjugate algebraic anti-isomorphism.

Proof. We only need to check the "only if" part. Assume that Φ is a ξ-Lie multiplicative isomorphism. By Theorem 1 and Lemma 4(1), if $\xi = -1$, (2) is true; if $\xi = 0$, (1) holds; if $\xi \neq 0, -1$, there exists central projections $P \in \mathscr{M}$ and $Q \in \mathscr{N}$ such that $\Phi = \Psi_1 \oplus \Psi_2$, where $\Psi_1 : P\mathscr{M} \to Q\mathscr{N}$ is a ring isomorphism with $\Psi_1(\xi A_1) = \xi\Psi_1(A_1)$ for all $A_1 \in P\mathscr{M}$ and $-\xi\Psi_2 : (I_M - P)\mathscr{M} \to (I_N - Q)\mathscr{N}$ is a ring anti-isomorphism with $\Psi_2(\xi A_2) = (1/\xi)\Psi_2(A_2)$ for all $A_2 \in (I_M - P)\mathscr{M}$.

For Ψ_1, by Lemma 4(1), there exist central projections $P_1 \in P\mathscr{M}$ and $Q_1 \in Q\mathscr{N}$ such that $\Psi_1 = \Phi_1 \oplus \Phi_2$, where $\Phi_1 = \Psi_1|_{P_1\mathscr{M}} : P_1\mathscr{M} \to Q_1\mathscr{N}$ is linear and $\Phi_2 = \Psi_1|_{(P-P_1)\mathscr{M}} : (P - P_1)\mathscr{M} \to (Q - Q_1)\mathscr{N}$ is conjugate linear. Note that $\Psi_1(\xi A_1) = \xi\Psi_1(A_1)$ for all $A_1 \in P\mathscr{M}$. This implies that $\Phi_2 = 0$ if $\xi \notin \mathbb{R}$.

For Ψ_2, by Lemma 4(2), $\Psi_2 = 0$ if $|\xi| \neq 1$ and $-\xi\Psi_2$ is a conjugate algebraic anti-isomorphism if $|\xi| = 1$. Thus, if $\xi \in \mathbb{R}\setminus\{0, -1\}$, Φ is a ring isomorphism and has the form (1); if $\xi \in \mathbb{C} \setminus \mathbb{R}$ with $|\xi| \neq 1$, then $\Psi_1 = \Phi_1$ is an algebraic isomorphism and $\Psi_2 = 0$, and consequently, Φ is an algebraic isomorphism, which implies the form (3); if $\xi \in \mathbb{C} \setminus \mathbb{R}$ with $|\xi| = 1$, then $\Psi_1 = \Phi_1$ is an algebraic isomorphism and $-\xi\Psi_2$ is a conjugate algebraic anti-isomorphism, which implies (4). $\square$

3. Proof of the Main Result

In this section, we present a proof of the main result Theorem 1. Before doing this, we need some notions. Let $\mathscr{M}$ be any von Neumann algebra and $A \in \mathscr{M}$. Recall that the central carrier of A, denoted by $\overline{A}$, is the intersection of all central projections P such that $PA = 0$. If A is self-adjoint, then the core of A, denoted by $\underline{A}$, is $\sup\{S \in \mathscr{Z}(\mathscr{M}) : S = S^*, S \leq A\}$. Particularly, if $A = P$ is a projection, it is clear that $\underline{P}$ is the largest central projection $\leq P$. A projection P is core-free if $\underline{P} = 0$. It is easy to see that $\underline{P} = 0$ if and only if $\overline{I_M - P} = I_M$ [13].

The following two lemmas are needed.

Lemma 6 (see [13]). *Let $\mathscr{M}$ be a von Neumann algebra without central summands of type I_1. Then, each nonzero central projection $C \in \mathscr{M}$ is the carrier of a core-free projection in $\mathscr{M}$. Particularly, there exists a nonzero core-free projection $P \in \mathscr{M}$ with $\overline{P} = I_M$.*

Lemma 7 (see [14]). *Let $\mathscr{M}$ be a von Neumann algebra without central summands of type I_1 or I_2. Then, the ideal $\mathscr{L}$ of $\mathscr{M}$ generated algebraically by $\{[A^2, C]B[A, C] - [A, C]B[A^2, C] : A, B, C \in \mathscr{M}\}$ is equal to $\mathscr{M}$.*

Proof of Theorem 1. For $\mathscr{M}$, by Lemma 6, we can find a nonzero central core-free projection $P_0 \in \mathscr{M}$ with central carrier I_M. By the definitions of core and central carrier, $I_M - P_0$ is core-free and $\overline{I_M - P_0} = I_M$. For the convenience, denote $\mathscr{M}_{ij} = P_i\mathscr{M}P_j$, $i, j \in \{1, 2\}$, where $P_1 = P_0$ and $P_2 = I_M - P_0$. Then, $\mathscr{M} = \mathscr{M}_{11} + \mathscr{M}_{12} + \mathscr{M}_{21} + \mathscr{M}_{22}$. In the sequel, when we write S_{ij}, we always indicate $S_{ij} \in \mathscr{M}_{ij}$.

If the statement (1) holds, it is clear that Φ is a multiplicative isomorphism. If the statement (2) holds, it is easy to check that Φ is a Jordan multiplicative isomorphism. If the statement (3) holds, then for any $A_1, B_1 \in P\mathscr{M}$, we have

$$\begin{aligned} \Phi_1(A_1B_1 - \xi B_1A_1) &= \Phi_1(A_1)\Phi_1(B_1) - \Phi_1(\xi B_1)\Phi_1(A_1) \\ &= \Phi_1(A_1)\Phi_1(B_1) - \xi\Phi_1(B_1)\Phi_1(A_1), \end{aligned} \tag{9}$$

and for any $A_2, B_2 \in (I_M - P)\mathscr{M}$, we have

$$\begin{aligned} -\xi\Phi_2(A_2B_2 - \xi B_2A_2) &= (-\xi\Phi_2)(B_2)(-\xi\Phi_2)(A_2) \\ &\quad - (-\xi\Phi_2)(A_2)(-\xi\Phi_2)(\xi B_2) \\ &= \xi^2\Phi_2(B_2)\Phi_2(A_2) \\ &\quad - \xi^2\Phi_2(A_2)\Phi_2(\xi B_2) \\ &= \xi^2\Phi_2(B_2)\Phi_2(A_2) \\ &\quad - \xi\Phi_2(A_2)\Phi_2(B_2) \\ &= -\xi(\Phi_2(A_2)\Phi_2(B_2) \\ &\quad - \xi\Phi_2(B_2)\Phi_2(A_2)), \end{aligned} \tag{10}$$

which implies that $\Phi_2(A_2B_2 - \xi B_2A_2) = \Phi_2(A_2)\Phi_2(B_2) - \xi\Phi_2(B_2)\Phi_2(A_2)$. Hence, for any $A, B \in \mathscr{M}$, one obtains that $\Phi(AB - \xi BA) = \Phi(A)\Phi(B) - \xi\Phi(B)\Phi(A)$. This completes the proof of "if" part.

We will prove the "only if" part by checking a series of claims.

Claim 1. Φ is additive. We will complete the proof of Claim 1 by nine steps.

Step 1. $\Phi(0) = 0$. Since Φ is surjective, there exists an element $A \in \mathscr{M}$ such that $\Phi(A) = 0$. So $\Phi(0) = \Phi(0A - \xi A0) = \Phi(0)\Phi(A) - \xi\Phi(A)\Phi(0) = 0$.

In the sequel, we will use a so-called *standard argument*: suppose that $S, A, B \in \mathscr{M}$ are such that $\Phi(S) = \Phi(A) + \Phi(B)$. Multiplying this equation by $\Phi(T)$ from the left and the right, respectively, we get $\Phi(T)\Phi(S) = \Phi(T)\Phi(A) + \Phi(T)\Phi(B)$ and $\Phi(S)\Phi(T) = \Phi(A)\Phi(T) + \Phi(B)\Phi(T)$. Then,

$$\begin{aligned}\Phi(T)\Phi(S) - \xi\Phi(S)\Phi(T) &= \Phi(T)\Phi(A) - \xi\Phi(A)\Phi(T) \\ &\quad + \Phi(T)\Phi(B) - \xi\Phi(B)\Phi(T), \\ \Phi(S)\Phi(T) - \xi\Phi(T)\Phi(S) &= \Phi(A)\Phi(T) - \xi\Phi(T)\Phi(A) \\ &\quad + \Phi(B)\Phi(T) - \xi\Phi(T)\Phi(B).\end{aligned} \tag{11}$$

It follows that

$$\begin{aligned}\Phi(TS - \xi ST) &= \Phi(TA - \xi AT) + \Phi(TB - \xi BT), \\ \Phi(ST - \xi TS) &= \Phi(AT - \xi TA) + \Phi(BT - \xi TB).\end{aligned} \tag{12}$$

Moreover, if we have $\Phi(TA - \xi AT) + \Phi(TB - \xi BT) = \Phi(TA - \xi AT + TB - \xi BT)$ and $\Phi(AT - \xi TA) + \Phi(BT - \xi TB) = \Phi(AT - \xi TA + BT - \xi TB)$, by the injectivity of Φ, one gets

$$\begin{aligned}TS - \xi ST &= TA - \xi AT + TB - \xi BT, \\ ST - \xi TS &= AT - \xi TA + BT - \xi TB.\end{aligned} \tag{13}$$

Step 2. For any $A_{ii} \in \mathscr{M}_{ii}$ and $A_{ij} \in \mathscr{M}_{ij}$, we have $\Phi(A_{ii} + A_{ij}) = \Phi(A_{ii}) + \Phi(A_{ij})$, $1 \leq i \neq j \leq 2$.

By the surjectivity of Φ, there is an element $S = S_{11} + S_{12} + S_{21} + S_{22} \in \mathscr{M}$ such that

$$\Phi(S) = \Phi(A_{ii}) + \Phi(A_{ij}). \tag{14}$$

For any $T_{jj} \in \mathscr{M}_{jj}$, applying the standard argument to (14), we obtain

$$\begin{aligned}\Phi(ST_{jj} - \xi T_{jj}S) &= \Phi(A_{ii}T_{jj} - \xi T_{jj}A_{ii}) \\ &\quad + \Phi(A_{ij}T_{jj} - \xi T_{jj}A_{ij}) = \Phi(A_{ij}T_{jj}),\end{aligned} \tag{15}$$

$$\Phi(T_{jj}S - \xi ST_{jj}) = \Phi(-\xi A_{ij}T_{jj}). \tag{16}$$

By (15) and the injectivity of Φ, one gets $ST_{jj} - \xi T_{jj}S = S_{ij}T_{jj} + S_{jj}T_{jj} - \xi T_{jj}S_{ji} - \xi T_{jj}S_{jj} = A_{ij}T_{jj}$ for all $T_{jj} \in \mathscr{M}_{jj}$, which implies that $\xi T_{jj}S_{ji} = 0$, $S_{ij}T_{jj} = A_{ij}T_{jj}$ and $S_{jj}T_{jj} = \xi T_{jj}S_{jj}$ for all $T_{jj} \in \mathscr{M}_{jj}$. Particularly, taking $T_{jj} = P_j$, one gets $S_{ij} = A_{ij}$ and, as $\xi \neq 1$, $S_{jj} = 0$. This, combining (16) and the injectivity of Φ, yields $T_{jj}S_{ji} = 0$, and so $S_{ji} = 0$.

For any $T_{ij} \in \mathscr{M}_{ij}$, applying the standard argument to (14) again, we get

$$\begin{aligned}\Phi(ST_{ij} - \xi T_{ij}S) &= \Phi(A_{ii}T_{ij} - \xi T_{ij}A_{ii}) \\ &\quad + \Phi(A_{ij}T_{ij} - \xi T_{ij}A_{ij}) = \Phi(A_{ii}T_{ij}).\end{aligned} \tag{17}$$

It follows from the injectivity of Φ that

$$ST_{ij} - \xi T_{ij}S = A_{ii}T_{ij} \tag{18}$$

for every $T_{ij} \in \mathscr{M}_{ij}$. Note that $S_{ji} = S_{jj} = 0$ and $S_{ij} = A_{ij}$. The above equation reduces to $S_{ii}T_{ij} = A_{ii}T_{ij}$; that is, $S_{ii}TP_j = A_{ii}TP_j$ holds for all $T \in \mathscr{M}$. Note that $\overline{P_j} = I_M$. It follows from the definition of the central carrier that span $\{TP_jx : T \in \mathscr{M}, x \in H\}$ is dense in H. Hence, $S_{ii} = A_{ii}$. Consequently, $S = A_{ii} + A_{ij}$.

Similarly, one can check that the following Step 3 holds.

Step 3. For any $A_{ii} \in \mathscr{M}_{ii}$ and $A_{ji} \in \mathscr{M}_{ji}$, we have $\Phi(A_{ii} + A_{ji}) = \Phi(A_{ii}) + \Phi(A_{ji})$, $1 \leq i \neq j \leq 2$.

Step 4. Φ is additive on $\mathscr{M}_{ij}$, $1 \leq i \neq j \leq 2$.

For any $A_{12}, B_{12} \in \mathscr{M}_{12}$, since

$$\begin{aligned}A_{12} + B_{12} &= (P_1 + B_{12})(A_{12} + P_2) \\ &= (P_1 + B_{12})(A_{12} + P_2) - \xi(A_{12} + P_2)(P_1 + B_{12}),\end{aligned} \tag{19}$$

by Steps 2-3, one can obtain

$$\begin{aligned}\Phi(A_{12} + B_{12}) &= \Phi(P_1 + B_{12})\Phi(A_{12} + P_2) \\ &\quad - \xi\Phi(A_{12} + P_2)\Phi(P_1 + B_{12}) \\ &= (\Phi(P_1) + \Phi(B_{12}))(\Phi(A_{12}) + \Phi(P_2)) \\ &\quad - \xi(\Phi(A_{12}) + \Phi(P_2)) \\ &\quad \times (\Phi(P_1) + \Phi(B_{12})) \\ &= \Phi(P_1A_{12} - \xi A_{12}P_1) + \Phi(P_1P_2 - \xi P_2P_1) \\ &\quad + \Phi(B_{12}A_{12} - \xi A_{12}B_{12}) \\ &\quad + \Phi(B_{12}P_2 - \xi P_2B_{12}) \\ &= \Phi(A_{12}) + \Phi(B_{12}).\end{aligned} \tag{20}$$

For any $A_{21}, B_{21} \in \mathscr{M}_{21}$, note that

$$\begin{aligned}A_{21} + B_{21} &= (A_{21} + P_2)(P_1 + B_{21}) \\ &= (A_{21} + P_2)(P_1 + B_{21}) - \xi(P_1 + B_{21})(A_{21} + P_2).\end{aligned} \tag{21}$$

By a similar computation as above, one can show $\Phi(A_{21} + B_{21}) = \Phi(A_{21}) + \Phi(B_{21})$.

Step 5. Φ is additive on $\mathscr{M}_{ii}$, $i = 1, 2$.

Take any $A_{ii}, B_{ii} \in \mathscr{M}_{ii}$ and choose $S = S_{11} + S_{12} + S_{21} + S_{22} \in \mathscr{M}$ such that

$$\Phi(S) = \Phi(A_{ii}) + \Phi(B_{ii}). \tag{22}$$

Let $j \neq i$. For $P_j \in \mathscr{M}_{jj}$, applying the standard argument to (22) and the injectivity of Φ, we get $0 = P_jS - \xi SP_j = S_{ji} + S_{jj} - \xi S_{ij} - \xi S_{jj}$ and $0 = SP_j - \xi P_jS = S_{ij} + S_{jj} - \xi S_{ji} - \xi S_{jj}$. Note that $\xi \neq 1$, these two equations imply that $S_{ij} = S_{ji} = S_{jj} = 0$.

For any $T_{ij} \in \mathscr{M}_{ij}$, applying the standard argument to (22), and by Step 4, one obtains

$$\begin{aligned}\Phi(ST_{ij} - \xi T_{ij}S) &= \Phi(A_{ii}T_{ij}) + \Phi(B_{ii}T_{ij}) \\ &= \Phi(A_{ii}T_{ij} + B_{ii}T_{ij}).\end{aligned} \tag{23}$$

Note that Φ is injective and $S_{ij} = S_{ji} = S_{jj} = 0$. The above equation implies $S_{ii}T_{ij} = (A_{ii} + B_{ii})T_{ij}$ for all $T_{ij} \in \mathscr{M}_{ij}$; that is, $S_{ii}TP_j = (A_{ii} + B_{ii})TP_j$ for all $T \in \mathscr{M}$. It follows from $\overline{P_j} = I_M$ that $S_{ii} = A_{ii} + B_{ii}$.

Step 6. Consider $\Phi(A_{12} + A_{21}) = \Phi(A_{12}) + \Phi(A_{21})$.

Choose $S = S_{11} + S_{12} + S_{21} + S_{22} \in \mathscr{M}$ such that

$$\Phi(S) = \Phi(A_{12}) + \Phi(A_{21}). \tag{24}$$

For any $T_{12} \in \mathscr{M}_{12}$, applying the standard argument to (24), and the injectivity of Φ, we have $ST_{12} - \xi T_{12}S = A_{21}T_{12} - \xi T_{12}A_{21}$ for all $T_{12} \in \mathscr{M}_{12}$. Multiplying this equation by P_2 from both sides, we get $S_{21}T_{12} = A_{21}T_{12}$ for all $T_{12} \in \mathscr{M}_{12}$, which implies that $S_{21} = A_{21}$.

Similarly, for any $T_{21} \in \mathscr{M}_{21}$, applying the standard argument to (24), one can prove that $S_{12} = A_{12}$.

For P_1 and P_2, applying the standard argument to (24), respectively, we have

$$\begin{aligned}\Phi(P_2S - \xi SP_2) &= \Phi(-\xi A_{12}) + \Phi(A_{21}),\\ \Phi(SP_1 - \xi P_1S) &= \Phi(-\xi A_{12}) + \Phi(A_{21}).\end{aligned} \tag{25}$$

Therefore, $\Phi(P_2S - \xi SP_2) = \Phi(SP_1 - \xi P_1S)$, which implies that $P_2S - \xi SP_2 = SP_1 - \xi P_1S$. A simple computation reveals that $S_{11} = S_{22} = 0$. Consequently, $S = A_{12} + A_{21}$.

Step 7. Consider $\Phi(A_{11} + A_{12} + A_{21}) = \Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21})$.

Let $S = S_{11} + S_{12} + S_{21} + S_{22} \in \mathscr{M}$ be such that $\Phi(S) = \Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21})$. Then, by Steps 2-3, we have

$$\Phi(S) = \Phi(A_{11} + A_{12}) + \Phi(A_{21}), \tag{26}$$

$$\Phi(S) = \Phi(A_{11} + A_{21}) + \Phi(A_{12}). \tag{27}$$

For any $T_{12} \in \mathscr{M}_{12}$, applying the standard argument to (27) and the injectivity of Φ, we get

$$ST_{12} - \xi T_{12}S = A_{11}T_{12} + A_{21}T_{12} - \xi T_{12}A_{21}. \tag{28}$$

Multiplying by P_2 from the left in (28), one obtains $S_{21}T_{12} = A_{21}T_{12}$ for each $T_{12} \in \mathscr{M}_{12}$, and so $S_{21} = A_{21}$. Multiplying by P_1 and P_2 from the left and the right, respectively, in (28), one gets

$$\xi T_{12}S_{22} = (S_{11} - A_{11})T_{12} \quad \forall T_{12} \in \mathscr{M}_{12}. \tag{29}$$

Similarly, for any $T_{21} \in \mathscr{M}_{21}$, applying the standard argument to (26), one can get $S_{12} = A_{12}$.

For P_2, applying the standard argument to (26), by Step 6, we have $\Phi(SP_2 - \xi P_2S) = \Phi(A_{12}) + \Phi(-\xi A_{21}) = \Phi(A_{12} - \xi A_{21})$, which implies that $SP_2 - \xi P_2S = A_{12} - \xi A_{21}$. As $S_{12} = A_{12}$ and $S_{21} = A_{21}$, a direct computation leads to $S_{22} = 0$. This fact and (29) yield $(S_{11} - A_{11})T_{12} = 0$ for all $T_{12} \in \mathscr{M}_{12}$, and so $S_{11} = A_{11}$. Consequently, $S = A_{11} + A_{12} + A_{21}$, as desired.

Step 8. Consider $\Phi(A_{11}+A_{12}+A_{21}+A_{22}) = \Phi(A_{11})+\Phi(A_{12})+\Phi(A_{21}) + \Phi(A_{22})$.

Let $S = S_{11} + S_{12} + S_{21} + S_{22} \in \mathscr{M}$ be such that

$$\Phi(S) = \Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21}) + \Phi(A_{22}). \tag{30}$$

For P_1, applying the standard argument to (30), by Step 7, we have

$$\begin{aligned}\Phi(P_1S - \xi SP_1) &= \Phi(P_1)\Phi(S) - \xi\Phi(S)\Phi(P_1)\\ &= \Phi((1-\xi)A_{11} + A_{12} - \xi A_{21})\end{aligned} \tag{31}$$

and $\Phi(SP_1 - \xi P_1S) = \Phi((1-\xi)A_{11} - \xi A_{12} + A_{21})$. It follows that $P_1S - \xi SP_1 = (1-\xi)A_{11} + A_{12} - \xi A_{21}$ and $SP_1 - \xi P_1S = (1-\xi)A_{11} - \xi A_{12} + A_{21}$. By a simple computation, one obtains $S_{11} = A_{11}$, $S_{12} = A_{12}$, and $S_{21} = A_{21}$.

For any $T_{12} \in \mathscr{M}_{12}$, applying the standard argument to (30), one gets

$$\begin{aligned}\Phi(T_{12}S - \xi ST_{12}) &= \Phi(-\xi A_{11}T_{12})\\ &\quad + \Phi(T_{12}A_{21} - \xi A_{21}T_{12}) + \Phi(T_{12}A_{22}).\end{aligned} \tag{32}$$

Furthermore, for P_1, applying the standard argument to the above equation, by Steps 2 and 4, one gets

$$\begin{aligned}&\Phi(P_1(T_{12}S - \xi ST_{12}) - \xi(T_{12}S - \xi ST_{12})P_1)\\ &\quad= \Phi(-\xi A_{11}T_{12}) + \Phi(T_{12}A_{21} - \xi T_{12}A_{21}) + \Phi(T_{12}A_{22})\\ &\quad= \Phi(-\xi A_{11}T_{12} + T_{12}A_{21} - \xi T_{12}A_{21} + T_{12}A_{22}).\end{aligned} \tag{33}$$

Thus, we have

$$\begin{aligned}&T_{12}S_{21} + T_{12}S_{22} - \xi S_{11}T_{12} - \xi T_{12}S_{21}\\ &\quad= -\xi A_{11}T_{12} + T_{12}A_{21} - \xi T_{12}A_{21} + T_{12}A_{22}.\end{aligned} \tag{34}$$

Note that $S_{11} = A_{11}$, $S_{12} = A_{12}$, and $S_{21} = A_{21}$. It follows that $T_{12}S_{22} = T_{12}A_{22}$ for all $T_{12} \in \mathscr{M}_{12}$, and hence $S_{22} = A_{22}$. Consequently, $S = A_{11} + A_{12} + A_{21} + A_{22}$.

Step 9. Φ is additive, and so Claim 1 is true.

For any $A, B \in \mathscr{M}$, write $A = A_{11} + A_{12} + A_{21} + A_{22}$ and $B = B_{11} + B_{12} + B_{21} + B_{22}$. By Steps 2–8, we have

$$\begin{aligned}\Phi(A + B) &= \Phi((A_{11} + B_{11}) + (A_{12} + B_{12})\\ &\qquad + (A_{21} + B_{21}) + (A_{22} + B_{22}))\\ &= \Phi(A_{11} + B_{11}) + \Phi(A_{12} + B_{12})\\ &\quad + \Phi(A_{21} + B_{21}) + \Phi(A_{22} + B_{22})\\ &= \Phi(A_{11}) + \Phi(B_{11}) + \Phi(A_{12}) + \Phi(B_{12})\\ &\quad + \Phi(A_{21}) + \Phi(B_{21}) + \Phi(A_{22}) + \Phi(B_{22})\\ &= \Phi(A_{11} + A_{12} + A_{21} + A_{22})\\ &\quad + \Phi(B_{11} + B_{12} + B_{21} + B_{22})\\ &= \Phi(A) + \Phi(B).\end{aligned} \tag{35}$$

Claim 2. The statements (1)-(2) hold in the theorem. By Claim 1, Φ is additive. So, in the case of $\xi = 0$, the statement (1) is true; in the case of $\xi = -1$, by [14] (also see [15]), it is also easy to see that the statement (2) is true.

In the sequel, we always assume that $\xi \neq 0, \pm 1$.

Claim 3. Consider $\Phi(I_M) \in \mathscr{Z}(\mathscr{N})$.

For any $A, B \in \mathscr{M}$, we have

$$\begin{aligned} &\Phi(AB - \xi BA) = \Phi(A)\Phi(B) - \xi\Phi(B)\Phi(A), \\ &\Phi(BA - \xi AB) = \Phi(B)\Phi(A) - \xi\Phi(A)\Phi(B). \end{aligned} \tag{36}$$

Note that, by Claim 1, Φ is additive. Thus, the above two equations imply that

$$\begin{aligned} &\Phi((1+\xi)(AB - BA)) \\ &\quad = (1+\xi)(\Phi(A)\Phi(B) - \Phi(B)\Phi(A)), \quad \forall A, B \in \mathscr{M}. \end{aligned} \tag{37}$$

As $\Phi(0) = 0$ and $\xi \neq 0, -1$, the above equation ensures that $AB = BA$ if and only if $\Phi(A)\Phi(B) = \Phi(B)\Phi(A)$. So $\Phi(A)\Phi(I_M) = \Phi(I_M)\Phi(A)$ holds for all $A \in \mathscr{M}$. It follows from the surjectivity of Φ that $\Phi(I_M) \in \mathscr{Z}(\mathscr{N})$.

Claim 4. $\Phi(I_M)$ is invertible.

For any $A \in \mathscr{M}$, by Claim 3, we have

$$\begin{aligned} \Phi((1-\xi)A) &= \Phi(A)\Phi(I_M) - \xi\Phi(I_M)\Phi(A) \\ &= (1-\xi)\Phi(I_M)\Phi(A) = (1-\xi)\Phi(A)\Phi(I_M). \end{aligned} \tag{38}$$

Taking $A = A_0 = (1/(1-\xi))\Phi^{-1}((1-\xi)I_N)$ in the above equation, one gets $(1-\xi)I_N = (1-\xi)\Phi(I_M)\Phi(A_0) = (1-\xi)\Phi(A_0)\Phi(I_M)$. It follows from the fact $\xi \neq 1$ that $\Phi(I_M)\Phi(A_0) = \Phi(A_0)\Phi(I_M) = I_N$. So $\Phi(I_M)$ is invertible and $\Phi(A_0)$ is its inverse. The claim holds.

Note that $\Phi(I_M)^{-1} \in \mathscr{Z}(\mathscr{N})$ as $\Phi(I_M) \in \mathscr{Z}(\mathscr{N})$. For any $A \in \mathscr{M}$, let $\Psi(A) = \Phi(I_M)^{-1}\Phi(A)$. Since

$$\begin{aligned} \Phi\left((1-\xi)A^2\right) &= \Phi\left(A^2\right)\Phi(I_M) - \xi\Phi(I_M)\Phi\left(A^2\right) \\ &= (1-\xi)\Phi(I_M)\Phi\left(A^2\right), \\ \Phi\left((1-\xi)A^2\right) &= \Phi(A)\Phi(A) - \xi\Phi(A)\Phi(A) \\ &= (1-\xi)\Phi(A)^2, \end{aligned} \tag{39}$$

we get $\Phi(A)^2 = \Phi(I_M)\Phi(A^2)$. So

$$\begin{aligned} \Psi(A)^2 &= \Phi(I_M)^{-1}\Phi(A)\Phi(I_M)^{-1}\Phi(A) \\ &= \Phi(I_M)^{-1}\Phi(I_M)^{-1}\Phi(A)^2 = \Phi(I_M)^{-1}\Phi\left(A^2\right) \\ &= \Psi\left(A^2\right). \end{aligned} \tag{40}$$

It follows that $\Psi : \mathscr{M} \to \mathscr{N}$ is a Jordan ring isomorphism and $\Phi(A) = \Phi(I_M)\Psi(A)$. Thus, by [14], there exist central projections $P \in \mathscr{M}$ and $Q \in \mathscr{N}$ such that $\Psi|_{P\mathscr{M}} : P\mathscr{M} \to Q\mathscr{N}$ is a ring isomorphism and $\Psi|_{(I_M-P)\mathscr{M}} : (I_M - P)\mathscr{M} \to (I_N - Q)\mathscr{N}$ is a ring anti-isomorphism.

For the convenience, write $\Phi|_{P\mathscr{M}} = \Phi_1$, $\Phi|_{(I_M-P)\mathscr{M}} = \Phi_2$, $\Psi|_{P\mathscr{M}} = \Psi_1$, and $\Psi|_{(I_M-P)\mathscr{M}} = \Psi_2$. Then, we may write $\Phi = \Phi_1 \oplus \Phi_2$.

Note that $\Phi(I_M) = \Phi(P) + \Phi(I_M - P) \in \mathscr{Z}(\mathscr{N})$, $\Phi(P) = \Phi(I_M)\Psi(P) \in Q\mathscr{N}$, and $\Phi(I_M - P) = \Phi(I_M)\Psi(I_M - P) \in (I_N - Q)\mathscr{N}$. It is easy to check that $\Phi(P) \in \mathscr{Z}(Q\mathscr{N})$, $\Phi(I_M - P) \in \mathscr{Z}((I_N - Q)\mathscr{N})$, $\Phi_1(P)^{-1} \in Q\mathscr{N}$, and $\Phi_2(I_M - P)^{-1} \in (I_N - Q)\mathscr{N}$. Hence,

$$\begin{aligned} \Psi_1(A_1) &= \Phi(I_M)^{-1}\Phi_1(A_1) = \Phi_1(P)^{-1}\Phi_1(A_1) \\ &\qquad \forall A_1 \in P\mathscr{M}, \\ \Psi_2(A_2) &= \Phi(I_M)^{-1}\Phi_2(A_2) = \Phi_2(I_M - P)^{-1}\Phi_2(A_2) \\ &\qquad \forall A_2 \in (I_M - P)\mathscr{M}. \end{aligned} \tag{41}$$

Claim 5. $\Phi_1 = \Phi|_{P\mathscr{M}} : P\mathscr{M} \to Q\mathscr{N}$ is a ring isomorphism satisfying $\Phi_1(\xi A_1) = \xi\Phi_1(A_1)$ for all $A_1 \in P\mathscr{M}$.

For any $A_1, B_1 \in P\mathscr{M}$, since Ψ_1 is a ring isomorphism, we have

$$\begin{aligned} \Psi_1(A_1B_1 - \xi B_1A_1) &= \Psi_1(A_1)\Psi_1(B_1) - \Psi_1(B_1)\Psi_1(\xi A_1) \\ &= \Psi_1(A_1)\Psi_1(B_1) \\ &\quad - \Psi_1(B_1)(\Psi_1(A_1) - \Psi_1((1-\xi)A_1)) \\ &= \Psi_1(A_1)\Psi_1(B_1) \\ &\quad - \Psi_1(B_1)(\Psi_1(A_1) \\ &\quad - \Phi_1(P)^{-1}\Phi_1(PA_1 - \xi A_1P)) \\ &= \Psi_1(A_1)\Psi_1(B_1) \\ &\quad - \Psi_1(B_1)(\Psi_1(A_1) - (1-\xi)\Phi_1(A_1)) \\ &= \Psi_1(A_1)\Psi_1(B_1) - \Psi_1(B_1) \\ &\quad \times (\Psi_1(A_1) - (1-\xi)\Phi_1(P)\Psi_1(A_1)) \\ &= \Psi_1(A_1)\Psi_1(B_1) - \Psi_1(B_1)\Psi_1(A_1) \\ &\quad + (1-\xi)\Phi_1(P)\Psi_1(B_1)\Psi_1(A_1), \\ \Psi_1(A_1B_1 - \xi B_1A_1) &= \Phi_1(P)^{-1}\Phi_1(A_1B_1 - \xi B_1A_1) \\ &= \Phi_1(P)^{-1}(\Phi_1(A_1)\Phi_1(B_1) \\ &\qquad - \xi\Phi_1(B_1)\Phi_1(A_1)) \\ &= \Phi_1(P)\Psi_1(A_1)\Psi_1(B_1) \\ &\quad - \xi\Phi_1(P)\Psi_1(B_1)\Psi_1(A_1). \end{aligned} \tag{42}$$

These imply that $[\Psi_1(A_1), \Psi_1(B_1)] = \Phi_1(P)[\Psi_1(A_1), \Psi_1(B_1)]$ holds for all $A_1, B_1 \in P\mathscr{M}$. It follows from the surjectivity of Ψ_1 that $(Q - \Phi_1(P))[T, S] = 0$ holds for all $T, S \in Q\mathscr{N}$. Furthermore, it is easily checked that $(Q-\Phi_1(P))([T^2, S]X[T, S]-$

$[T,S]X[T^2,S]) = 0$ holds for every $T,S,X \in Q\mathcal{N}$. Note that $Q\mathcal{N}$ is a von Neumann algebra without central summands of type I_1. By Lemma 7, one obtains $\Phi_1(P) = Q$. Hence, $\Phi_1(A_1) = \Phi_1(P)\Psi_1(A_1) = Q\Psi_1(A_1) = \Psi_1(A_1)$ for all $A_1 \in P\mathcal{M}$; that is, $\Phi_1 = \Phi|_{P\mathcal{M}} : P\mathcal{M} \to Q\mathcal{N}$ is a ring isomorphism.

Also note that, for any $A_1 \in P\mathcal{M}$, we have

$$\begin{aligned}\Phi_1(A_1) - \Phi_1(\xi A_1) &= \Phi_1(A_1P - \xi PA_1)\\ &= \Phi_1(A_1)\Phi_1(P) - \xi\Phi_1(P)\Phi_1(A_1)\\ &= \Phi_1(A_1) - \xi\Phi_1(A_1).\end{aligned} \tag{43}$$

This leads to $\Phi_1(\xi A_1) = \xi\Phi_1(A_1)$ which completes the proof of Claim 5.

Claim 6. $-\xi\Phi_2 : (I_M - P)\mathcal{M} \to (I_N - Q)\mathcal{N}$ is a ring anti-isomorphism and $\Phi_2(\xi A_2) = (1/\xi)\Phi_2(A_2)$ for all $A_2 \in (I_M - P)\mathcal{M}$.

For every $A_2, B_2 \in (I_M - P)\mathcal{M}$, we have

$$\begin{aligned}\Psi_2(A_2B_2 - \xi B_2A_2) &= \Psi_2(B_2)\Psi_2(A_2)\\ &\quad - \Psi_2(A_2)\Psi_2(\xi B_2)\\ &= \Psi_2(B_2)\Psi_2(A_2) - \Psi_2(A_2)\\ &\quad \times (\Psi_2(B_2) - (1-\xi)\\ &\qquad \times \Phi_2(I_M - P)\Psi_2(B_2))\\ &= \Psi_2(B_2)\Psi_2(A_2) - \Psi_2(A_2)\\ &\quad \times \Psi_2(B_2)\\ &\quad + (1-\xi)\Phi_2(I_M - P)\Psi_2(A_2)\\ &\quad \times \Psi_2(B_2),\\ \Psi_2(A_2B_2 - \xi B_2A_2) &= \Phi_2(I_M - P)^{-1}\Phi_2(A_2B_2 - \xi B_2A_2)\\ &= \Phi_2(I_M - P)^{-1}(\Phi_2(A_2)\Phi_2(B_2)\\ &\qquad - \xi\Phi_2(B_2)\Phi_2(A_2))\\ &= \Phi_2(I_M - P)\Psi_2(A_2)\Psi_2(B_2)\\ &\quad - \xi\Phi_2(I_M - P)\Psi_2(B_2)\Psi_2(A_2).\end{aligned} \tag{44}$$

Then, $[\Psi_2(B_2), \Psi_2(A_2)] = -\xi\Phi_2(I_M - P)[\Psi_2(B_2), \Psi_2(A_2)]$, and so $((I_N - Q) + \xi\Phi_2(I_M - P))[T,S] = 0$ holds for all $T,S \in (I_N - Q)\mathcal{N}$. It is easily checked that $((I_N - Q) + \xi\Phi_2(I_M - P))([T^2,S]X[T,S] - [T,S]X[T^2,S]) = 0$ holds for each $T,S,X \in (I_N - Q)\mathcal{N}$. It follows from Lemma 7 that $\Phi_2(I_M - P) = -\xi^{-1}(I_N - Q)$. Hence, $-\xi\Phi_2(A_2) = -\xi\Phi_2(I - P)\Psi_2(A_2) = (I_N - Q)\Psi_2(A_2) = \Psi_2(A_2)$, and $-\xi\Phi_2$ is a ring anti-isomorphism.

Since

$$\begin{aligned}\Phi_2(A_2) - \Phi_2(\xi A_2) &= \Phi_2((1-\xi)A_2)\\ &= (1-\xi)\Phi_2(I_M - P)\Phi_2(A_2)\\ &= \Phi_2(A_2) - \frac{1}{\xi}\Phi_2(A_2),\end{aligned} \tag{45}$$

we get $\Phi_2(\xi A_2) = (1/\xi)\Phi_2(A_2)$ for all $A_2 \in (I_M - P)\mathcal{M}$. Hence, Claim 6 is true.

Combining Claims 3–6, one sees that the statement (3) in Theorem 1 holds.

The proof of the theorem is therefore completed. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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