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Research Article

Fixed Points for Weak α-ψ-Contractions in Partial Metric Spaces

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Recently, Samet et al. (2012) introduced the notion of α-ψ-contractive mappings and established some fixed point results in the setting of complete metric spaces. In this paper, we introduce the notion of weak α-ψ-contractive mappings and give fixed point results for this class of mappings in the setting of partial metric spaces. Also, we deduce fixed point results in ordered partial metric spaces. Our results extend and generalize the results of Samet et al.

1. Introduction

The notion of partial metric is one of the most useful and interesting generalizations of the classical concept of metric. The partial metric spaces were introduced in 1994 by Matthews [1] as a part of the study of denotational semantics of data for networks, showing that the contraction mapping principle can be generalized to the partial metric context for applications in program verification. Later on, many authors studied the existence of several connections between partial metrics and topological aspects of domain theory (see [2–8] and the references therein). On the other hand, some researchers [9, 10] investigated the characterization of partial metric 0-completeness in terms of fixed point theory, extending the characterization of metric completeness [11–14].

Recently, Samet et al. [15] introduced the notion of α-ψ-contractive mappings and established some fixed point results in the setting of complete metric spaces. In this paper, we introduce the notion of weak α-ψ-contractive mappings and give fixed point results for this class of mappings in the setting of partial metric spaces. Also, we deduce fixed point results in ordered partial metric spaces. Our results extend and generalize Theorems 2.1–2.3 of [15] and many others. An application to ordinary differential equations concludes the paper.

2. Preliminaries

In this section, we recall some definitions and some properties of partial metric spaces that can be found in [1, 5, 10, 16, 17]. A partial metric on a nonempty set X is a function $p : X \times X \to [0, +\infty)$ such that, for all $x, y, z \in X$, we have

(p_1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$,

(p_2) $p(x, x) \le p(x, y)$,

(p_3) $p(x, y) = p(y, x)$,

(p_4) $p(x, y) \le p(x, z) + p(z, y) - p(z, z)$.

A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X. It is clear that if $p(x, y) = 0$, then from (p_1) and (p_2) it follows that $x = y$. But if $x = y$, $p(x, y)$ may not be 0. A basic example of a partial metric space is the pair $([0, +\infty), p)$, where $p(x, y) = \max\{x, y\}$ for all $x, y \in [0, +\infty)$. Other examples of partial metric spaces which are interesting from a computational point of view can be found in [1].

Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p-balls $\{B_p(x, \varepsilon) : x \in X,\ \varepsilon > 0\}$, where

$$B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\} \quad (1)$$

for all $x \in X$ and $\varepsilon > 0$.

Let (X, p) be a partial metric space. A sequence $\{x_n\}$ in (X, p) converges to a point $x \in X$ if and only if $p(x, x) = \lim_{n\to+\infty} p(x, x_n)$.

A sequence $\{x_n\}$ in (X, p) is called a Cauchy sequence if there exists (and is finite) $\lim_{n,m\to+\infty} p(x_n, x_m)$.

A partial metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = \lim_{n,m\to+\infty} p(x_n, x_m)$.

A sequence $\{x_n\}$ in (X, p) is called 0-Cauchy if $\lim_{n,m\to+\infty} p(x_n, x_m) = 0$. We say that (X, p) is 0-complete if every 0-Cauchy sequence in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = 0$.

On the other hand, the partial metric space $(\mathbb{Q} \cap [0, +\infty), p)$, where $\mathbb{Q}$ denotes the set of rational numbers and the partial metric p is given by $p(x, y) = \max\{x, y\}$, provides an example of a 0-complete partial metric space which is not complete.

It is easy to see that every closed subset of a complete partial metric space is complete.

Notice that if p is a partial metric on X, then the function $p^s : X \times X \to [0, +\infty)$ given by

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y) \tag{2}$$

is a metric on X. Furthermore, $\lim_{n\to+\infty} p^s(x_n, x) = 0$ if and only if

$$p(x, x) = \lim_{n\to+\infty} p(x_n, x) = \lim_{n,m\to+\infty} p(x_n, x_m). \tag{3}$$

Lemma 1 (see [1, 16]). *Let (X, p) be a partial metric space. Then*

(a) *$\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, p^s),*

(b) *a partial metric space (X, p) is complete if and only if the metric space (X, p^s) is complete.*

Let X be a non-empty set. If (X, p) is a partial metric space and $(X, \preceq)$ is a partially ordered set, then $(X, p, \preceq)$ is called an ordered partial metric space. Then $x, y \in X$ are called comparable if $x \preceq y$ or $y \preceq x$ holds. Let $(X, \preceq)$ be a partially ordered set, and let $T : X \to X$ be a mapping. T is a non-decreasing mapping if $Tx \preceq Ty$ whenever $x \preceq y$ for all $x, y \in X$.

Definition 2 (see [15]). Let $T : X \to X$ and $\alpha : X \times X \to [0, +\infty)$. One says that T is α-admissible if

$$x, y \in X, \quad \alpha(x, y) \geq 1 \Longrightarrow \alpha(Tx, Ty) \geq 1. \tag{4}$$

Example 3. Let $X = [0, +\infty)$, and define the function $\alpha : X \times X \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} e^{x-y} & \text{if } x \geq y, \\ 0 & \text{if } x < y. \end{cases} \tag{5}$$

Then, every non-decreasing mapping $T : X \to X$ is α-admissible. For example the mappings defined by $Tx = \ln(1 + x)$ and $Tx = x/(1 + x)$ for all $x \in X$ are α-admissible.

3. Main Results

Throughout this paper, the standard notations and terminologies in nonlinear analysis are used. We start the main section by presenting the new notion of weak α-ψ-contractive mappings.

Denote by Ψ the family of non-decreasing functions $\psi : [0, +\infty) \to [0, +\infty)$ such that $\psi(t) > 0$ and $\lim_{n\to+\infty} \psi^n(t) = 0$ for each $t > 0$, where ψ^n is the nth iterate of ψ.

Remark 4. Notice that the family Ψ used in this paper is larger (less restrictive) than the corresponding family of functions defined in [15], see also next Examples 12–13.

Lemma 5. *For every function $\psi \in \Psi$, one has $\psi(t) < t$ for each $t > 0$.*

Definition 6. Let (X, p) be a partial metric space, and let $T : X \to X$ be a given mapping. We say that T is a weak α-ψ-contractive mapping if there exist two functions $\alpha : X \times X \to [0, +\infty)$ and $\psi \in \Psi$ such that

$$\begin{aligned} &\alpha(x, y) p(Tx, Ty) \\ &\quad \leq \psi(\max\{p(x, y), p(x, Tx), p(y, Ty)\}), \end{aligned} \tag{6}$$

for all $x, y \in X$. If

$$\alpha(x, y) p(Tx, Ty) \leq \psi(p(x, y)), \tag{7}$$

for all $x, y \in X$, then T is an α-ψ-contractive mapping.

Remark 7. If $T : X \to X$ satisfies the contraction mapping principle, then T is a weak α-ψ-contractive mapping, where $\alpha(x, y) = 1$ for all $x, y \in X$ and $\psi(t) = kt$ for all $t \geq 0$ and some $k \in [0, 1)$.

In the sequel, we consider the following property of regularity. Let (X, p) be a partial metric space, and let $\alpha : X \times X \to [0, +\infty)$ be a function. Then

(r) X is α-regular if for each sequence $\{x_n\} \subset X$, such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$, we have that $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$,

(c) X has the property (C) with respect to α if for each sequence $\{x_n\} \subset X$, such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, there exists $n_0 \in \mathbb{N}$ such that $\alpha(x_m, x_n) \geq 1$ for all $n > m \geq n_0$.

Remark 8. Let X be a non-empty set, and let $\alpha : X \times X \to [0, +\infty)$ be a function. Denote

$$\mathscr{R} := \{(x, y) : \alpha(x, y) \geq 1\}. \tag{8}$$

If $\mathscr{R}$ is a transitive relation on X, then X has the property (C) with respect to α.

In fact, if $\{x_n\} \subset X$ is a sequence such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, then $(x_n, x_{n+1}) \in \mathscr{R}$ for all $n \in \mathbb{N}$. Now, fix $m \geq 1$ and show that

$$\alpha(x_m, x_n) \geq 1 \quad \forall n > m. \tag{9}$$

Obviously, (9) holds if $n = m + 1$. Assume that (9) holds for some $n > m$. From $(x_m, x_n), (x_n, x_{n+1}) \in \mathscr{R}$, since $\mathscr{R}$ is transitive, we get $(x_m, x_{n+1}) \in \mathscr{R}$. This implies that $\alpha(x_m, x_{n+1}) \geq 1$, and so $\alpha(x_m, x_n) \geq 1$ for all $n > m$; that is, X has the property (C) with respect to α.

Remark 9. Let $(X, p, \preceq)$ be an ordered partial metric space, and let $\alpha : X \times X \to [0, +\infty)$ be a function defined by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x \preceq y, \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

Then X has the property (C) with respect to α. Moreover, if for each sequence $\{x_n\}$, such that $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$ convergent to some $x \in X$, we have $x_n \preceq x$ for all $n \in \mathbb{N}$, and then X is α-regular.

By Remark 8, X has the property (C) with respect to α. Now, let $\{x_n\}$ be a sequence such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ convergent to some $x \in X$, and then $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$, and hence $x_n \preceq x$ for all $n \in \mathbb{N}$. This implies that $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$, and so X is α-regular.

Our first result is the following theorem that generalizes Theorem 2.1 of [15].

Theorem 10. *Let (X, p) be a complete partial metric space, and let $T : X \to X$ be a weak α-ψ-contractive mapping satisfying the following conditions:*

(i) *T is α-admissible,*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$,*

(iii) *X has the property (C) with respect to α,*

(iv) *T is continuous on (X, p^s).*

Then, T has a fixed point, that is; there exists $x^* \in X$ such that $Tx^* = x^*$.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Define the sequence $\{x_n\}$ in X by

$$x_{n+1} = Tx_n, \quad \forall n \in \mathbb{N}. \tag{11}$$

If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then $x^* = x_n$ is a fixed point for T. Assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Since T is α-admissible, we have

$$\begin{aligned} &\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1 \\ &\Longrightarrow \alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1. \end{aligned} \tag{12}$$

By induction, we get

$$\alpha(x_n, x_{n+1}) \geq 1, \quad \forall n \in \mathbb{N}. \tag{13}$$

Applying inequality (6) with $x = x_{n-1}$ and $y = x_n$ and using (13), we obtain

$$\begin{aligned} p(x_n, x_{n+1}) &= p(Tx_{n-1}, Tx_n) \\ &\leq \alpha(x_{n-1}, x_n) p(Tx_{n-1}, Tx_n) \\ &\leq \psi(\max\{p(x_{n-1}, x_n), \\ &\qquad p(x_{n-1}, Tx_{n-1}), p(x_n, Tx_n)\}) \\ &= \psi(\max\{p(x_{n-1}, x_n), p(x_n, x_{n+1})\}). \end{aligned} \tag{14}$$

If $\max\{p(x_{n-1}, x_n), p(x_n, x_{n+1})\} = p(x_n, x_{n+1})$, from

$$\begin{aligned} p(x_n, x_{n+1}) &\leq \psi(p(x_n, x_{n+1})) \\ &< p(x_n, x_{n+1}), \end{aligned} \tag{15}$$

we obtain a contradiction; therefore, $\max\{p(x_{n-1}, x_n), p(x_n, x_{n+1})\} = p(x_{n-1}, x_n)$, and hence

$$p(x_n, x_{n+1}) \leq \psi(p(x_{n-1}, x_n)), \quad \forall n \in \mathbb{N}. \tag{16}$$

By induction, we get

$$p(x_n, x_{n+1}) \leq \psi^n(p(x_0, x_1)), \quad \forall n \in \mathbb{N}. \tag{17}$$

This implies that

$$\lim_{n \to +\infty} p(x_n, x_{n+1}) = 0. \tag{18}$$

Fix $\varepsilon > 0$, and let $n(\varepsilon) \in \mathbb{N}$ such that

$$p(x_m, x_{m+1}) < \varepsilon - \psi(\varepsilon), \quad \forall m \geq n(\varepsilon). \tag{19}$$

Since X has the property (C) with respect to α, there exists $n_0 \in \mathbb{N}$ such that $\alpha(x_m, x_n) \geq 1$ for all $n > m \geq n_0$. Let $m \in \mathbb{N}$ with $m \geq \max\{n_0, n(\varepsilon)\}$, and we show that

$$p(x_m, x_{n+1}) < \varepsilon, \quad \forall n \geq m. \tag{20}$$

Note that (20) holds for $n = m$. Assume that (20) holds for some $n > m$, then

$$\begin{aligned} p(x_m, x_{n+2}) &\leq p(x_m, x_{m+1}) \\ &\quad + p(x_{m+1}, x_{n+2}) - p(x_{m+1}, x_{m+1}) \\ &\leq p(x_m, x_{m+1}) + p(Tx_m, Tx_{n+1}) \\ &\leq p(x_m, x_{m+1}) + \alpha(x_m, x_{n+1}) \\ &\quad \times p(Tx_m, Tx_{n+1}) \\ &\leq p(x_m, x_{m+1}) \\ &\quad + \psi(\max\{p(x_m, x_{n+1}), \ p(x_m, x_{m+1}), \\ &\qquad p(x_{n+1}, x_{n+2})\}) \\ &< \varepsilon - \psi(\varepsilon) + \psi(\varepsilon) = \varepsilon. \end{aligned} \tag{21}$$

This implies that (20) holds for $n \geq m$, and hence

$$\lim_{m,n \to +\infty} p(x_m, x_n) = 0. \tag{22}$$

Thus, we proved that $\{x_n\}$ is a Cauchy sequence in the partial metric space (X, p) and hence, by Lemma 1, in the metric space (X, p^s). Since (X, p) is complete, by Lemma 1, also (X, p^s) is complete. This implies that there exists $x^* \in X$ such that $p^s(x_n, x^*) \to 0$ as $n \to +\infty$; that is,

$$p(x^*, x^*) = \lim_{n\to+\infty} p(x^*, x_n) = \lim_{m,n\to+\infty} p(x_m, x_n) = 0. \tag{23}$$

From the continuity of T on (X, p^s), it follows that $x_{n+1} = Tx_n \to Tx^*$ as $n \to +\infty$. By the uniqueness of the limit, we get $x^* = Tx^*$; that is, x^* is a fixed point of T. □

In the next theorem, which is a proper generalization of Theorem 2.2 in [15], we omit the continuity hypothesis of T. Moreover, we assume 0-completeness of the space.

Theorem 11. *Let (X, p) be a 0-complete partial metric space, and let $T : X \to X$ be a weak α-ψ-contractive mapping satisfying the following conditions:*

(i) *T is α-admissible,*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$,*

(iii) *X has the property (C) with respect to α,*

(iv) *X is α-regular.*

Then, T has a fixed point.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$. Define the sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$, for all $n \in \mathbb{N}$. Following the proof of Theorem 10, we know that $\alpha(x_n, x_{n+1}) \ge 1$ for all $n \in \mathbb{N}$ and that $\{x_n\}$ is a 0-Cauchy sequence in the 0-complete partial metric space (X, p). Consequently, there exists $x^* \in X$ such that

$$p(x^*, x^*) = \lim_{n\to+\infty} p(x^*, x_n) = \lim_{m,n\to+\infty} p(x_m, x_n) = 0. \tag{24}$$

On the other hand, from $\alpha(x_n, x_{n+1}) \ge 1$ for all $n \in \mathbb{N}$ and the hypothesis (iv), we have

$$\alpha(x_n, x^*) \ge 1, \quad \forall n \in \mathbb{N}. \tag{25}$$

Now, using the triangular inequality, (6) and (25), we get

$$\begin{aligned} &p(Tx^*, x^*) \\ &\le p(Tx^*, Tx_n) \\ &\quad + p(x_{n+1}, x^*) - p(x_{n+1}, x_{n+1}) \\ &\le \alpha(x_n, x^*)\, p(Tx_n, Tx^*) \\ &\quad + p(x_{n+1}, x^*) \\ &\le \psi(\max\{p(x_n, x^*), p(x_n, x_{n+1}), \\ &\qquad p(x^*, Tx^*)\}) \\ &\quad + p(x_{n+1}, x^*). \end{aligned} \tag{26}$$

Since $p(x_n, x^*)$, $p(x_n, x_{n+1}) \to 0$ as $n \to +\infty$, for n great enough, we have

$$\max\{p(x_n, x^*), p(x_n, x_{n+1}), p(x^*, Tx^*)\} = p(x^*, Tx^*), \tag{27}$$

and hence

$$p(Tx^*, x^*) \le \psi(p(x^*, Tx^*)) < p(x^*, Tx^*). \tag{28}$$

This is a contradiction, and so we obtain $p(Tx^*, x^*) = 0$; that is, $Tx^* = x^*$. □

The following example illustrates the usefulness of Theorem 10.

Example 12. Let $X = [0, +\infty)$ and $p : X \times X \to [0, +\infty)$ be defined by $p(x, y) = \max\{x, y\}$ for all $x, y \in X$. Clearly, (X, p) is a complete partial metric space. Define the mapping $T : X \to X$ by

$$Tx = \begin{cases} 2x - \dfrac{3}{2} & \text{if } x > 1, \\ \dfrac{x}{1+x} & \text{if } 0 \le x \le 1. \end{cases} \tag{29}$$

At first, we observe that we cannot find $k \in [0, 1)$ such that

$$p(Tx, Ty) \le k \max\{p(x, y), p(x, Tx), p(y, Ty)\} \tag{30}$$

for all $x, y \in X$, since we have

$$\begin{aligned} &p(T1, T2) \\ &= \max\left\{\frac{1}{2}, \frac{5}{2}\right\} = \frac{5}{2} \\ &> k\frac{5}{2} = k \max\left\{\max\{1, 2\}, \max\left\{1, \frac{1}{2}\right\},\right. \\ &\qquad\qquad \left.\max\left\{2, \frac{5}{2}\right\}\right\} \end{aligned} \tag{31}$$

for all $k \in [0, 1)$. Now, we define the function $\alpha : X \times X \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \in [0, 1], \\ 0 & \text{otherwise.} \end{cases} \tag{32}$$

Clearly T is a weak α-ψ-contractive mapping with $\psi(t) = t/(1 + t)$ for all $t \ge 0$. In fact, for all $x, y \in X$, we have

$$\begin{aligned} &\alpha(x, y)\, p(Tx, Ty) \\ &= \max\left\{\frac{x}{1+x}, \frac{y}{1+y}\right\} \\ &= \psi(p(x, y)) \\ &\le \psi(\max\{p(x, y), p(x, Tx), \\ &\qquad p(y, Ty)\}). \end{aligned} \tag{33}$$

Moreover, there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. In fact, for $x_0 = 1$, we have

$$\alpha(1, T1) = \alpha\left(1, \frac{1}{2}\right) = 1. \tag{34}$$

Obviously, T is continuous on (X, p^s) since $p^s(x, y) = |x - y|$, and so we have to show that T is α-admissible. In doing so, let $x, y \in X$ such that $\alpha(x, y) \geq 1$. This implies that $x, y \in [0, 1]$, and by the definitions of T and α, we have

$$Tx = \frac{x}{1+x} \in [0,1], \quad Ty = \frac{y}{1+y} \in [0,1], \quad \alpha(Tx, Ty) = 1. \tag{35}$$

Then, T is α-admissible. Moreover, if $\{x_n\}$ is a sequence such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, then $x_n \in [0, 1]$ for all $n \in \mathbb{N}$, and hence $\alpha(x_m, x_n) \geq 1$ for all $n > m \geq 1$. Thus, X has the property (C) with respect to α.

Now, all the hypotheses of Theorem 10 are satisfied, and so T has a fixed point. Notice that Theorem 10 (also Theorem 11) guarantees only the existence of a fixed point but not the uniqueness. In this example, 0 and 3/2 are two fixed points of T.

Moreover, $\sum_{n=1}^{+\infty} \psi^n(t) = \sum_{n=1}^{+\infty}(t/(1 + nt)) \nless +\infty$, and so T is not an α-ψ-contractive mapping in the sense of [15] with respect to the complete metric space (X, p^s); that is, Theorem 2.1 of [15] cannot be applied in this case.

Now, we give an example involving a mapping T that is not continuous. Also, this example shows that our Theorem 11 is a proper generalization of Theorem 2.2 in [15].

Example 13. Let $X = \mathbb{Q} \cap [0, +\infty)$ and p as in Example 12. Clearly, (X, p) is a 0-complete partial metric space which is not complete. Then, Theorem 10 is not applicable in this case. Define the mapping $T : X \to X$ by

$$Tx = \begin{cases} 2x - \dfrac{5}{2} & \text{if } x > 2, \\ \dfrac{x}{1+x} & \text{if } 0 \leq x \leq 2. \end{cases} \tag{36}$$

It is clear that T is not continuous at $x = 2$ with respect to the metric p^s. Define the function $\alpha : X \times X \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \in [0, 2], \\ 0 & \text{otherwise.} \end{cases} \tag{37}$$

Clearly T is a weak α-ψ-contractive mapping with $\psi(t) = t/(1 + t)$ for all $t \geq 0$. In fact, for all $x, y \in X$, we have

$$\begin{aligned} &\alpha(x, y)\, p(Tx, Ty) \\ &\leq \psi(p(x, y)) \\ &\leq \psi(\max\{p(x, y), \\ &\qquad p(x, Tx), p(y, Ty)\}). \end{aligned} \tag{38}$$

Proceeding as in Example 12, the reader can show that all the required hypotheses of Theorem 11 are satisfied, and so T has a fixed point. Here, 0 and 5/2 are two fixed points of T.

Moreover, since (X, p^s) is not complete, where $p^s(x, y) = |x - y|$ for all $x, y \in X$, we conclude that neither Theorem 2.1 nor Theorem 2.2 of [15] can be applied to cover this case, also because $\sum_{n=1}^{+\infty} \psi^n(t) \nless +\infty$.

To ensure the uniqueness of the fixed point, we will consider the following hypothesis:

(H) for all $x, y \in X$ with $\alpha(x, y) < 1$, there exists $z \in X$ such that $\alpha(x, z) \geq 1$, $\alpha(y, z) \geq 1$, and $\lim_{n\to+\infty} p(T^{n-1}z, T^n z) = 0$.

Theorem 14. *Adding condition (H) to the hypotheses of Theorem 10 (resp., Theorem 11), one obtain the uniqueness of the fixed point of T.*

Proof. Suppose that x^* and y^* are two fixed points of T with $x^* \neq y^*$. If $\alpha(x^*, y^*) \geq 1$, using (6), we get

$$\begin{aligned} p(x^*, y^*) &\leq \alpha(x^*, y^*)\, p(Tx^*, Ty^*) \\ &= \psi(p(x^*, y^*)) < p(x^*, y^*), \end{aligned} \tag{39}$$

which is a contradiction, and so $x^* = y^*$. If $\alpha(x^*, y^*) < 1$ by (H), there exists $z \in X$ such that

$$\alpha(x^*, z) \geq 1, \quad \alpha(y^*, z) \geq 1. \tag{40}$$

Since T is α-admissible, from (40), we get

$$\alpha(x^*, T^n z) \geq 1, \quad \alpha(y^*, T^n z) \geq 1, \quad \forall n \in \mathbb{N}. \tag{41}$$

Let $z_n = T^n z$ for all $n \in \mathbb{N}$. Using (41) and (6), we have

$$\begin{aligned} p(x^*, z_n) &= p(Tx^*, Tz_{n-1}) \\ &\leq \alpha(x^*, z_{n-1})\, p(Tx^*, Tz_{n-1}) \\ &\leq \psi(\max\{p(x^*, z_{n-1}), p(x^*, Tx^*), \\ &\qquad p(z_{n-1}, Tz_{n-1})\}) \\ &= \psi(\max\{p(x^*, z_{n-1}), p(z_{n-1}, z_n)\}). \end{aligned} \tag{42}$$

Now, let $J = \{n \in \mathbb{N} : \max\{p(x^*, z_{n-1}), p(z_{n-1}, z_n)\} = p(z_{n-1}, z_n)\}$. If J is an infinite subset of $\mathbb{N}$, then

$$p(x^*, z_n) \leq \psi(p(z_{n-1}, z_n)) < p(z_{n-1}, z_n) \quad \forall n \in J. \tag{43}$$

Then, letting $n \to +\infty$ with $n \in J$ in the previous inequality, we get

$$\lim_{n\to+\infty} p(x^*, z_n) = 0. \tag{44}$$

If J is a finite subset of $\mathbb{N}$, then there exists $n_0 \in \mathbb{N}$ such that

$$\max\{p(x^*, z_{n-1}), p(z_{n-1}, z_n)\} = p(x^*, z_{n-1}) \quad \forall n > n_0. \tag{45}$$

This implies that

$$p(x^*, z_n) \le \psi^{n-n_0}\left(p\left(x^*, z_{n_0}\right)\right), \quad \forall n > n_0. \tag{46}$$

Then, letting $n \to +\infty$, we get

$$\lim_{n\to+\infty} p(x^*, z_n) = 0. \tag{47}$$

Similarly, using (41) and (6), we get

$$\lim_{n\to+\infty} p(y^*, z_n) = 0. \tag{48}$$

Since $p^s(x, y) \le 2p(x, y)$, using (47) and (48), we deduce that

$$\lim_{n\to+\infty} p^s(x^*, z_n) = \lim_{n\to+\infty} p^s(y^*, z_n) = 0. \tag{49}$$

Now, the uniqueness of the limit gives us $x^* = y^*$. This finishes the proof. □

From Theorems 10 and 11, we obtain the following corollaries.

Corollary 15. *Let (X, p) be a complete partial metric space, and let $T : X \to X$ be an α-ψ-contractive mapping satisfying the following conditions:*

(i) *T is α-admissible,*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$,*

(iii) *X has the property (C) with respect to α,*

(iv) *T is continuous on (X, p^s).*

Then, T has a fixed point.

Corollary 16. *Let (X, p) be a 0-complete partial metric space, and let $T : X \to X$ be an α-ψ-contractive mapping satisfying the following conditions:*

(i) *T is α-admissible,*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$,*

(iii) *X has the property (C) with respect to α,*

(iv) *X is α-regular.*

Then, T has a fixed point.

From the proof of Theorem 14, we deduce the following corollaries.

Corollary 17. *One adds to the hypotheses of Corollary 15 (resp., Corollary 16) the following condition:*

(HC) *for all $x, y \in X$ with $\alpha(x, y) < 1$, there exists $z \in X$ such that $\alpha(x, z) \ge 1$ and $\alpha(y, z) \ge 1$,*

and one obtains the uniqueness of the fixed point of T.

4. Consequences

Now, we show that many existing results in the literature can be deduced easily from our theorems.

4.1. Contraction Mapping Principle

Theorem 18 (Matthews [1]). *Let (X, p) be a 0-complete partial metric space, and let $T : X \to X$ be a given mapping satisfying*

$$p(Tx, Ty) \le kp(x, y) \tag{50}$$

for all $x, y \in X$, where $k \in [0, 1)$. Then T has a unique fixed point.

Proof. Let $\alpha : X \times X \to [0, +\infty)$ be defined by $\alpha(x, y) = 1$, for all $x, y \in X$, and let $\psi : [0, +\infty) \to [0, +\infty)$ be defined by $\psi(t) = kt$. Then T is an α-ψ-contractive mapping. It is easy to show that all the hypotheses of Corollaries 16 and 17 are satisfied. Consequently, T has a unique fixed point. □

Remark 19. In Example 12, Theorem 18 cannot be applied since $p(T1, T2) > p(2, 1)$. However, using our Corollary 15, we obtain the existence of a fixed point of T.

4.2. Fixed Point Results in Ordered Metric Spaces. The existence of fixed points in partially ordered sets has been considered in [18]. Later on, some generalizations of [18] are given in [19–24]. Several applications of these results to matrix equations are presented in [18]; some applications to periodic boundary value problems and particular problems are given in [22, 23], respectively.

In this section, we will show that many fixed point results in ordered metric spaces can be deduced easily from our presented theorems.

4.2.1. Ran and Reurings Type Fixed Point Theorem. In 2004, Ran and Reurings proved the following theorem.

Theorem 20 (Ran and Reurings [18]). *Let $(X, \preceq)$ be a partially ordered set, and suppose that there exists a metric d in X such that the metric space (X, d) is complete. Let $T : X \to X$ be a continuous and non-decreasing mapping with respect to $\preceq$. Suppose that the following two assertions hold:*

(i) *there exists $k \in [0, 1)$ such that $d(Tx, Ty) \le k\, d(x, y)$ for all $x, y \in X$ with $x \preceq y$,*

(ii) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$,*

(iii) *T is continuous.*

Then, T has a fixed point.

From Theorem 10, we deduce the following generalization and extension of the Ran and Reurings theorem in the framework of ordered complete partial metric spaces.

Theorem 21. *Let $(X, p, \preceq)$ be an ordered complete partial metric space, and let $T : X \to X$ be a non-decreasing mapping with respect to $\preceq$. Suppose that the following assertions hold:*

(i) *there exists $\psi \in \Psi$ such that $p(Tx, Ty) \le \psi(\max\{p(x, y), p(x, Tx), p(y, Ty)\})$ for all $x, y \in X$ with $x \preceq y$,*

(ii) *there exists* $x_0 \in X$ *such that* $x_0 \preceq Tx_0$,

(iii) T *is continuous on* (X, p^s).

Then, T has a fixed point.

Proof. Define the function $\alpha : X \times X \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x \preceq y, \\ 0 & \text{otherwise.} \end{cases} \quad (51)$$

From (i), we have

$$\alpha(x, y) p(Tx, Ty) \leq \psi(\max\{p(x, y), p(x, Tx), p(y, Ty)\}), \quad \forall x, y \in X. \quad (52)$$

Then, T is a weak α-ψ-contractive mapping. Now, let $x, y \in X$ such that $\alpha(x, y) \geq 1$. By the definition of α, this implies that $x \preceq y$. Since T is a non-decreasing mapping with respect to $\preceq$, we have $Tx \preceq Ty$, which gives us that $\alpha(Tx, Ty) = 1$. Then T is α-admissible. From (ii), there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, and so $\alpha(x_0, Tx_0) = 1$. Moreover, by Remark 9, X has the property (C) with respect to α.

Therefore, all the hypotheses of Theorem 10 are satisfied, and so T has a fixed point. □

Example 22. Let $X = [0, +\infty)$ and $p : X \times X \to [0, +\infty)$ be defined by $p(x, y) = \max\{x, y\}$ for all $x, y \in X$. Clearly, (X, p) is a complete partial metric space. Define the mapping $T : X \to X$ by

$$Tx = 2x, \quad \forall x \in X. \quad (53)$$

Clearly T is a continuous mapping with respect to the metric p^s. We endow X with the usual order of real numbers. Now, condition (i) of Theorem 21 is not satisfied for $x = 1 \leq 3 = y$. In fact, if we assume the contrary, then

$$p(T1, T3) = 6 \leq \psi(p(1, 3)) = \psi(3) < 3, \quad (54)$$

which is a contradiction. Then, we cannot apply Theorem 21 to prove the existence of a fixed point of T.

Define the function $\alpha : X \times X \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} \dfrac{1}{4} & \text{if } (x, y) \neq (0, 0), \\ 1 & \text{if } (x, y) = (0, 0). \end{cases} \quad (55)$$

It is clear that

$$\alpha(x, y) p(Tx, Ty) \leq \frac{1}{2} p(x, y), \quad \forall x, y \in X. \quad (56)$$

Then, T is a weak α-ψ-contractive mapping with $\psi(t) = t/2$ for all $t \geq 0$. Now, let $x, y \in X$ such that $\alpha(x, y) \geq 1$. By the definition of α, this implies that $x = y = 0$. Then we have $\alpha(Tx, Ty) = \alpha(0, 0) = 1$, and so T is α-admissible. Also, for $x_0 = 0$, we have $\alpha(x_0, Tx_0) = 1$. Consequently, all the hypotheses of Theorem 10 are satisfied, then we deduce the existence of a fixed point of T. Here 0 is a fixed point of T.

4.2.2. Nieto and Rodríguez-López Type Fixed Point Theorem. In 2005, Nieto and Rodríguez-López proved the following theorem.

Theorem 23 (Nieto and Rodríguez-López [22]). *Let $(X, \preceq)$ be a partially ordered set, and suppose that there exists a metric d in X such that the metric space (X, d) is complete. Let $T : X \to X$ be a non-decreasing mapping with respect to $\preceq$. Suppose that the following assertions hold:*

(i) *there exists* $k \in [0, 1)$ *such that* $d(Tx, Ty) \leq k\, d(x, y)$ *for all* $x, y \in X$ *with* $x \preceq y$,

(ii) *there exists* $x_0 \in X$ *such that* $x_0 \preceq Tx_0$,

(iii) *if* $\{x_n\}$ *is a non-decreasing sequence in* X *such that* $x_n \to x \in X$ *as* $n \to +\infty$, *then* $x_n \preceq x$ *for all* n.

Then, T has a fixed point.

From Theorem 11, we deduce the following generalization and extension of the Nieto and Rodríguez-López theorem in the framework of ordered 0-complete partial metric spaces.

Theorem 24. *Let $(X, p, \preceq)$ be an ordered 0-complete partial metric space, and let $T : X \to X$ be a non-decreasing mapping with respect to $\preceq$. Suppose that the following assertions hold:*

(i) *there exists* $\psi \in \Psi$ *such that* $p(Tx, Ty) \leq \psi(\max\{p(x, y), p(x, Tx), p(y, Ty)\})$ *for all* $x, y \in X$ *with* $x \preceq y$,

(ii) *there exists* $x_0 \in X$ *such that* $x_0 \preceq Tx_0$,

(iii) *if* $\{x_n\}$ *is a non-decreasing sequence in* X *such that* $x_n \to x \in X$ *as* $n \to +\infty$, *then* $x_n \preceq x$ *for all* n.

Then, T has a fixed point.

Proof. Define the function $\alpha : X \times X \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x \preceq y, \\ 0 & \text{otherwise.} \end{cases} \quad (57)$$

The reader can show easily that T is a weak α-ψ-contractive and α-admissible mapping. Now, by Remark 9, X has the property (C) with respect to α and is α-regular. Thus all the hypotheses of Theorem 11 are satisfied, and T has a fixed point. □

Remark 25. In, Example 22, also Theorem 24 cannot be applied since condition (i) is not satisfied.

Remark 26. To establish the uniqueness of the fixed point, Ran and Reurings, Nieto and Rodríguez-López [18, 22] considered the following hypothesis:

(u) for all $x, y \in X$, there exists $z \in X$ such that $x \preceq z$ and $y \preceq z$.

Notice that in establishing the uniqueness it is enough to assume that (u) holds for all $x, y \in X$ that are not comparable. This result is also a particular case of Corollary 17. Precisely, if $x, y \in X$ are not comparable, then there exists $z \in X$ such that $x \preceq z$ and $y \preceq z$. This implies that $\alpha(x,z) \geq 1$ and $\alpha(y,z) \geq 1$, and here, we consider the same function α used in the previous proof. Then, hypothesis (HC) of Corollary 17 is satisfied, and so we deduce the uniqueness of the fixed point. For establishing the uniqueness of the fixed point in Theorems 21 and 24, we consider the following hypothesis:

(U) for all $x, y \in X$ that are not comparable, there exists $z \in X$ such that $x \preceq z$, $y \preceq z$, and $\lim_{n\to+\infty} p(T^{n-1}z, T^n z) = 0$.

5. Application to Ordinary Differential Equations

In this section, we present a typical application of fixed point results to ordinary differential equations. In fact, in the literature there are many papers focusing on the solution of differential problems approached via fixed point theory (see, e.g., [15, 25, 26] and the references therein). For such a case, even without any additional problem structure, the optimal strategy can be obtained by finding the fixed point of an operator T which satisfies a contractive condition in certain spaces.

Here, we consider the following two-point boundary value problem for second order differential equation:

$$-\frac{d^2x}{dt^2} = f(t, x(t)), \quad t \in [0,1]$$
$$x(0) = x(1) = 0, \tag{58}$$

where $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Recall that the Green's function associated to (58) is given by

$$G(t,s) = \begin{cases} t(1-s) & 0 \leq t \leq s \leq 1, \\ s(1-t) & 0 \leq s \leq t \leq 1. \end{cases} \tag{59}$$

Let $C(I)$ ($I = [0,1]$) be the space of all continuous functions defined on I. It is well known that such a space with the metric given by

$$d(x,y) = \|x-y\|_\infty = \max_{t\in I} |x(t) - y(t)| \tag{60}$$

is a complete metric space.

Now, we consider the following conditions:

(i) for all $t \in I$, for all $a, b \in \mathbb{R}$ with $|a|, |b| \leq 1$, we have

$$|f(t,a) - f(t,b)| \leq 8\psi(|a-b|), \tag{61}$$

where $\psi \in \Psi$,

(ii) there exists $x_0 \in C(I)$ such that $\|x_0\|_\infty \leq 1$,

(iii) for all $x \in C(I)$,

$$\|x\|_\infty \leq 1 \Longrightarrow \left\| \int_0^1 G(t,s) f(s, x(s))\, ds \right\|_\infty \leq 1. \tag{62}$$

Theorem 27. *Suppose that conditions (i)–(iii) hold. Then (58) has at least one solution $x^* \in C^2(I)$.*

Proof. Consider $C(I)$ endowed with the partial metric given by

$$p(x,y) = \begin{cases} \|x-y\|_\infty & \text{if } \|x\|_\infty, \|y\|_\infty \leq 1, \\ \|x-y\|_\infty + \rho & \text{otherwise}, \end{cases} \tag{63}$$

where $\rho > 0$. It is easy to show that $(C(I), p)$ is 0-complete but is not complete. In fact,

$$p^s(x,y) = \begin{cases} 2\|x-y\|_\infty & \text{if } (\|x\|_\infty, \|y\|_\infty \leq 1) \\ & \text{or } (\|x\|_\infty, \|y\|_\infty > 1), \\ 2\|x-y\|_\infty + \rho & \text{otherwise}, \end{cases} \tag{64}$$

and consequently $(C(I), p^s)$ is not complete.

On the other hand, it is well known that $x \in C(I)$, and is a solution of (58), is equivalent to $x \in C(I)$ is a solution of the integral equation

$$x(t) = \int_0^1 G(t,s) f(s, x(s))\, ds, \qquad \forall t \in I. \tag{65}$$

Define the operator $T : C(I) \to C(I)$ by

$$Tx(t) = \int_0^1 G(t,s) f(s, x(s))\, ds, \qquad \forall t \in I. \tag{66}$$

Then solving problem (58) is equivalent to finding $x^* \in C(I)$ that is a fixed point of T. Now, let $x, y \in C(I)$ such that $\|x\|_\infty, \|y\|_\infty \leq 1$. From (i), we have

$$\begin{aligned} &|Tx(t) - Ty(t)| \\ &= \left| \int_0^1 G(t,s) \times [f(s,x(s)) - f(s,y(s))]\, ds \right| \\ &\leq \int_0^1 G(t,s) |f(s,x(s)) - f(s,y(s))|\, ds \\ &\leq 8 \int_0^1 G(t,s) \psi(|x(s) - y(s)|)\, ds \\ &\leq 8 \left(\sup_{t\in I} \int_0^1 G(t,s)\, ds \right) \\ &\quad \times \psi(\|x-y\|_\infty) \\ &\leq \psi(\|x-y\|_\infty). \end{aligned} \tag{67}$$

Note that for all $t \in I$, $\int_0^1 G(t,s)ds = (-t^2/2) + (t/2)$, which implies that

$$\sup_{t\in I} \int_0^1 G(t,s)\, ds = \frac{1}{8}. \tag{68}$$

Then, for all $x, y \in C(I)$ such that $\|x\|_\infty, \|y\|_\infty \leq 1$, we have

$$\|Tx - Ty\|_\infty \leq \psi(\|x-y\|_\infty). \tag{69}$$

Define the function $\alpha : C(I) \times C(I) \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } \|x\|_\infty, \|y\|_\infty \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{70}$$

For all $x, y \in C(I)$, we have

$$\alpha(x, y) \|Tx - Ty\|_\infty \le \psi(\|x - y\|_\infty). \tag{71}$$

Then, T is an α-ψ-contractive mapping. From condition (iii), for all $x, y \in C(I)$, we get

$$\begin{aligned} \alpha(x, y) \ge 1 &\Longrightarrow \|x\|_\infty, \|y\|_\infty \le 1 \\ &\Longrightarrow \|Tx\|_\infty, \|Ty\|_\infty \le 1 \\ &\Longrightarrow \alpha(Tx, Ty) \ge 1. \end{aligned} \tag{72}$$

Then, T is α-admissible. From conditions (ii) and (iii), there exists $x_0 \in C(I)$ such that $\alpha(x_0, Tx_0) \ge 1$. Thus, all the conditions of Corollary 16 are satisfied, and hence we deduce the existence of $x^* \in C(I)$ such that $x^* = Tx^*$; that is, x^* is a solution to (58). □

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