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Research Article

Refinements on the Hermite-Hadamard Inequalities for r-Convex Functions

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We give some new generalizations of the well-known Hermite-Hadamard inequality for r-convex functions.

1. Introduction

Let $f : I \subset R \to R$ be a convex function on the interval I; then for any $a, b \in I$ with $a \neq b$, we have the following double inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)\,dt \leq \frac{f(a)+f(b)}{2}. \quad (1)$$

This remarkable result is well known in the literature as the Hermite-Hadamard inequality. Note that some of the classical inequalities for means can be derived from (1) for appropriate particular selections of the mapping f. Both inequalities hold in the reversed direction if f is concave. Some refinements of the Hermite-Hadamard inequality on convex functions have been extensively investigated by a number of authors (e.g., [1–9]). The Hermite-Hadamard inequality was generalized in [10] to an r-convex positive function which is defined on an interval $[a, b]$. A positive function f is called r-convex on $[a, b]$, if for each $x, y \in [a, b]$ and $t \in [0, 1]$

$$f(tx + (1-t)y) \leq \begin{cases} \left[tf^r(x) + (1-t)f^r(y)\right]^{1/r} \\ [f(x)]^t[f(y)]^{1-t}. \end{cases} \quad (2)$$

It is obvious that 0-convex functions are simply log-convex functions and 1-convex functions are ordinary convex functions. If f is a positive r-concave function, then inequality (2) is reversed. We note that if f and g are convex and g is increasing, then $g \circ f$ is convex; moreover, since $f = \exp(\log f)$, it follows that a log-convex function is convex. This follows directly from (2) because, by the arithmetic-geometric mean inequality, we have

$$[f(x)]^t[f(y)]^{1-t} \leq tf(x) + (1-t)f(y). \quad (3)$$

In [6], Gill et al. proved the following two theorems.

Theorem 1. *Suppose that f is a positive r-convex function on $[a, b]$. Then*

$$\frac{1}{b-a}\int_a^b f(t)\,dt \leq \frac{r}{r+1}\left(\frac{f^{r+1}(b)-f^{r+1}(a)}{f^r(b)-f^r(a)}\right). \quad (4)$$

If f is a positive r-concave function, then the inequality is reversed.

Theorem 2. *Suppose that f is a positive log-convex function on $[a, b]$. Then*

$$\frac{1}{b-a}\int_a^b f(t)\,dt \leq \frac{f(b)-f(a)}{\ln f(b)-\ln f(a)}. \quad (5)$$

If f is a positive log-concave function, then the inequality is reversed.

In [11], Sulaiman obtained the following result for r-convex functions.

Theorem 3. *Let f be a positive r-convex function on $[a,b]$, $0 < r \leq 1$. Set*

$$w(t) = \left[t f^r \left(\frac{(2-t)a + tb}{2} \right) + (1-t) f^r \left(\frac{(1-t)a + (1+t)b}{2} \right) \right]^{1/r},$$
$$W(t) = \frac{r}{r+1} \left[t \frac{f^{r+1}((1-t)a + tb) - f^{r+1}(a)}{f^r((1-t)a + tb) - f^r(a)} + (1-t) \frac{f^{r+1}(b) - f^{r+1}((1-t)a + tb)}{f^r(b) - f^r((1-t)a + tb)} \right]. \tag{6}$$

Then, the following inequality holds:

$$f\left(\frac{a+b}{2}\right) \leq w(t) \leq \frac{1}{b-a} \int_a^b f(x)\,dx \leq W(t) \leq \frac{r}{r+1} \left(\frac{f^{r+1}(b) - f^{r+1}(a)}{f^r(b) - f^r(a)} \right) \leq \frac{f(a) + f(b)}{2}. \tag{7}$$

In [12], Sulaiman obtained the following result for log-convex functions.

Theorem 4. *Assume that $f : I \to R$ is an increasing log-convex function. Then for all $t \in [0,1]$, one has*

$$f\left(\frac{a+b}{2}\right) \leq w(a,b) \leq \frac{1}{b-a} \int_a^b f(x)\,dx \leq W(t) \leq \frac{f(b) - f(a)}{\ln f(b) - \ln f(a)} \leq \frac{f(a) + f(b)}{2}, \tag{8}$$

where

$$w(a,b) = \sqrt{f\left(\frac{3a+b}{4}\right) f\left(\frac{a+3b}{4}\right)} \tag{9}$$

$$W(t) = (1-t) \frac{f(ta + (1-t)b) - f(a)}{\ln f(ta + (1-t)b) - \ln f(a)} + t \frac{f(b) - f(ta + (1-t)b)}{\ln f(b) - \ln f(ta + (1-t)b)}. \tag{10}$$

For recent results and generalizations concerning the Hermite-Hadamard inequality, see [7, 8] and the references given therein.

In this note, we establish some generalizations of the above results for the class of r-convex functions.

2. Lemmas

Lemma 5. *If $a, b > 0$, then*

$$\lim_{r \to 0} \left(t a^r + (1-t) b^r \right)^{1/r} = a^t b^{1-t}. \tag{11}$$

Remark 6. Applying Lemma 5 and (2), for an r-convex f, let $r \to 0$; then f is log-convex. Now we let $r \to 0$ in Theorem 1; then we get Theorem 2 from

$$\lim_{r \to 0} \frac{r}{r+1} \left(\frac{f^{r+1}(b) - f^{r+1}(a)}{f^r(b) - f^r(a)} \right) = \frac{f(b) - f(a)}{\ln f(b) - \ln f(a)}. \tag{12}$$

Lemma 7. *If $0 < a < b < c$, then*

$$\frac{b-a}{\ln b - \ln a} \leq \frac{c-a}{\ln c - \ln a}, \qquad \frac{c-b}{\ln c - \ln b} \leq \frac{c-a}{\ln c - \ln a}. \tag{13}$$

Lemma 8. *If $0 < a < b < c$ and $r > 0$, then*

$$\frac{c^{r+1} - b^{r+1}}{c^r - b^r} \leq \frac{c^{r+1} - a^{r+1}}{c^r - a^r}, \qquad \frac{b^{r+1} - a^{r+1}}{b^r - a^r} \leq \frac{c^{r+1} - a^{r+1}}{c^r - a^r}. \tag{14}$$

Lemma 9. *If $a, b > 0$, then the following inequality holds:*

$$\frac{b-a}{\ln b - \ln a} \leq \frac{a+b}{2}. \tag{15}$$

Lemma 10 (see [11, Theorem 3.1]). *Let $f : [a,b] \to (0,\infty)$ be r-convex on $[a,b]$ and $0 < r \leq 1$; then*

$$f\left(\frac{a+b}{2}\right) \leq \left[\frac{1}{(b-a)} \int_a^b f^r(x)\,dx \right]^{1/r} \leq \frac{1}{b-a} \int_a^b f(x)\,dx \leq \frac{r}{r+1} \left(\frac{f^{r+1}(b) - f^{r+1}(a)}{f^r(b) - f^r(a)} \right) \leq \frac{f(a) + f(b)}{2}. \tag{16}$$

3. Main Results

Theorem 11. *Let $f : [a,b] \to (0,\infty)$ be r-convex and non-decreasing on $[a,b]$ and $0 < r \leq 1$; for $n \in N$, $\lambda_0 = 0$, $\lambda_{n+1} = 1$, and arbitrary $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \lambda_n \leq 1$, the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq w(\lambda_1, \ldots, \lambda_n) \leq \frac{1}{b-a} \int_a^b f(x)\,dx \leq W(\lambda_1, \ldots, \lambda_n) \leq \frac{r}{r+1} \left(\frac{f^{r+1}(b) - f^{r+1}(a)}{f^r(b) - f^r(a)} \right) \leq \frac{f(a) + f(b)}{2}, \tag{17}$$

where

$$w(\lambda_1,\dots,\lambda_n)=\left[\sum_{k=0}^{n}(\lambda_{k+1}-\lambda_k)f^r\times(2-\lambda_{k+1}-\lambda_k)a+(\lambda_{k+1}+\lambda_k)b\times(2)^{-1}\right]^{1/r},$$

$$W(\lambda_1,\dots,\lambda_n)=\frac{r}{r+1}\times\left[\sum_{k=0}^{n}(\lambda_{k+1}-\lambda_k)\times\left(f^{r+1}((1-\lambda_{k+1})a+\lambda_{k+1}b)-f^{r+1}((1-\lambda_k)a+\lambda_kb)\right)\times(f^r((1-\lambda_{k+1})a+\lambda_{k+1}b)-f^r((1-\lambda_k)a+\lambda_kb))^{-1}\right].\tag{18}$$

Proof. Observing that $\sum_{k=0}^{n}(\lambda_{k+1}-\lambda_k)=1$, $\sum_{k=0}^{n}(\lambda_{k+1}^2-\lambda_k^2)=1$ and Jensen's inequality for $f^r(x)$, we have

$$\begin{aligned}f^r\left(\frac{a+b}{2}\right)&=f^r\left(\sum_{k=0}^{n}(\lambda_{k+1}-\lambda_k)\times\frac{(2-\lambda_{k+1}-\lambda_k)a+(\lambda_{k+1}+\lambda_kb)}{2}\right)\\&\le\sum_{k=0}^{n}(\lambda_{k+1}-\lambda_k)\times f^r\left[\frac{(2-\lambda_{k+1}-\lambda_k)a+(\lambda_{k+1}+\lambda_kb)}{2}\right]\\&=w^r(\lambda_1,\dots,\lambda_n)\\&\le\sum_{k=0}^{n}(\lambda_{k+1}-\lambda_k)\times\left[\frac{1}{(\lambda_{k+1}-\lambda_k)(b-a)}\times\int_{(1-\lambda_k)a+\lambda_kb}^{(1-\lambda_{k+1})a+\lambda_{k+1}b}f^r(x)\,dx\right]\\&=\sum_{k=0}^{n}\left[\frac{1}{(b-a)}\int_{(1-\lambda_k)a+\lambda_kb}^{(1-\lambda_{k+1})a+\lambda_{k+1}b}f^r(x)\,dx\right]\\&=\frac{1}{(b-a)}\int_a^b f^r(x)\,dx,\end{aligned}\tag{19}$$

where the second inequality follows from replacing a and b by $(1-\lambda_k)a+\lambda_kb$ and $(1-\lambda_{k+1})a+\lambda_{k+1}b$, respectively, and (1) for $f^r(x)$. Therefore

$$\begin{aligned}f\left(\frac{a+b}{2}\right)&\le\left[\frac{1}{(b-a)}\int_a^b f^r(x)\,dx\right]^{1/r}\\&\le\frac{1}{b-a}\int_a^b f(x)\,dx\\&=\frac{1}{b-a}\sum_{k=0}^{n}\int_{(1-\lambda_k)a+\lambda_kb}^{(1-\lambda_{k+1})a+\lambda_{k+1}b}f(x)\,dx\\&=\sum_{k=0}^{n}(\lambda_{k+1}-\lambda_k)\frac{1}{(\lambda_{k+1}-\lambda_k)(b-a)}\times\int_{(1-\lambda_k)a+\lambda_kb}^{(1-\lambda_{k+1})a+\lambda_{k+1}b}f(x)\,dx\\&\le\frac{r}{r+1}\times\left[\sum_{k=0}^{n}(\lambda_{k+1}-\lambda_k)\times\left(f^{r+1}((1-\lambda_{k+1})a+\lambda_{k+1}b)-f^{r+1}((1-\lambda_k)a+\lambda_kb)\right)\times(f^r((1-\lambda_{k+1})a+\lambda_{k+1}b)-f^r((1-\lambda_k)a+\lambda_kb))^{-1}\right]\\&=W(\lambda_1,\dots,\lambda_n)\\&\le\frac{r}{r+1}\left[\sum_{k=0}^{n}(\lambda_{k+1}-\lambda_k)\times\frac{f^{r+1}(b)-f^{r+1}(a)}{f^r(b)-f^r(a)}\right]\\&=\frac{r}{r+1}\frac{f^{r+1}(b)-f^{r+1}(a)}{f^r(b)-f^r(a)}\\&\le\frac{f(a)+f(b)}{2},\end{aligned}\tag{20}$$

where the third inequality follows from replacing a and b by $(1-\lambda_k)a+\lambda_kb$ and $(1-\lambda_{k+1})a+\lambda_{k+1}b$, respectively, and (4). The proof is completed. □

Remark 12. Applying Theorem 11 for $n=1$, we get Theorem 3.

Corollary 13. *With the above notations, if $f : [a,b] \to (0,\infty)$ is r-convex and nondecreasing on $[a,b]$ and $0 < r \le 1$, one has the following inequality:*

$$\sup_{0\le\lambda_1\le\cdots\le\lambda_n\le 1} w(\lambda_1,\ldots,\lambda_n) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \inf_{0\le\lambda_1\le\cdots\le\lambda_n\le 1} W(\lambda_1,\ldots,\lambda_n), \quad (21)$$

where $W(\lambda_1,\ldots,\lambda_n)$ and $w(\lambda_1,\ldots,\lambda_n)$ are defined in Theorem 11.

Theorem 14. *Let $f : [a,b] \to (0,\infty)$ be log-convex and nondecreasing on $[a,b]$; for $n \in N$, $\lambda_0 = 0$, $\lambda_{n+1} = 1$, and arbitrary $0 \le \lambda_1 \le \lambda_2 \le \cdots \lambda_n \le 1$, the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \le m(t) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le M(\lambda_1,\ldots,\lambda_n) \le \frac{f(b)-f(a)}{\ln f(b) - \ln f(a)} \le \frac{f(a)+f(b)}{2}, \quad (22)$$

where

$$m(t) = \left[f\left(\frac{(2-t)a+tb}{2}\right)\right]^t \left[f\left(\frac{(1-t)a+(1+t)b}{2}\right)\right]^{1-t}, \quad (23)$$

$$\begin{aligned} M(\lambda_1,\ldots,\lambda_n) = \sum_{k=0}^{n} (\lambda_{k+1}-\lambda_k) &\times (f((1-\lambda_{k+1})a+\lambda_{k+1}b) \\ &- f((1-\lambda_k)a+\lambda_k b)) \\ &\times (\ln f((1-\lambda_{k+1})a+\lambda_{k+1}b) \\ &- \ln f((1-\lambda_k)a+\lambda_k b))^{-1}. \end{aligned} \quad (24)$$

Proof. Let f be a positive r-convex function on $[a,b]$, $0 < r \le 1$, by Theorem 3: then

$$\begin{aligned} w(t) &= \left[t f^r\left(\frac{(2-t)a+tb}{2}\right) + (1-t) f^r\left(\frac{(1-t)a+(1+t)b}{2}\right)\right]^{1/r} \\ &\le \frac{1}{b-a}\int_a^b f(x)\,dx. \end{aligned} \quad (25)$$

Applying Lemma 5, let $r \to 0$; then f is log-convex and

$$\begin{aligned} \lim_{r\to 0} w(t) &= m(t) \le \frac{1}{b-a}\int_a^b f(x)\,dx. \\ f\left(\frac{a+b}{2}\right) &= f\left(t\frac{(2-t)a+tb}{2} + (1-t) \times \frac{(1-t)a+(1+t)b}{2}\right) \\ &\le \left[f\left(\frac{(2-t)a+tb}{2}\right)\right]^t \times \left[f\left(\frac{(1-t)a+(1+t)b}{2}\right)\right]^{1-t} \\ &= m(t). \end{aligned} \quad (26)$$

Observing that $\sum_{k=0}^{n}(\lambda_{k+1}-\lambda_k) = 1$,

$$\begin{aligned} \frac{1}{(b-a)}\int_a^b f(x)\,dx &= \frac{1}{b-a}\left[\sum_{k=0}^{n}\int_{(1-\lambda_k)a+\lambda_k b}^{(1-\lambda_{k+1})a+\lambda_{k+1}b} f(x)\,dx\right] \\ &= \sum_{k=0}^{n}(\lambda_{k+1}-\lambda_k)\left[\frac{1}{(\lambda_{k+1}-\lambda_k)(b-a)} \times \int_{(1-\lambda_k)a+\lambda_k b}^{(1-\lambda_{k+1})a+\lambda_{k+1}b} f(x)\,dx\right] \\ &\le \sum_{k=0}^{n}(\lambda_{k+1}-\lambda_k) \times (f((1-\lambda_{k+1})a+\lambda_{k+1}b) - f((1-\lambda_k)a+\lambda_k b)) \\ &\quad \times (\ln f((1-\lambda_{k+1})a+\lambda_{k+1}b) - \ln f((1-\lambda_k)a+\lambda_k b))^{-1} \\ &= M(\lambda_1,\ldots,\lambda_n) \\ &\le \sum_{k=0}^{n}(\lambda_{k+1}-\lambda_k)\frac{f(b)-f(a)}{\ln f(b)-\ln f(a)} \\ &= \frac{f(b)-f(a)}{\ln f(b)-\ln f(a)} \\ &\le \frac{f(a)+f(b)}{2}, \end{aligned} \quad (27)$$

where the first inequality follows from replacing a and b by $(1-\lambda_k)a+\lambda_k b$ and $(1-\lambda_{k+1})a+\lambda_{k+1}b$, respectively, and (5). The proof is completed. □

Remark 15. Applying Theorem 14 for $t = 1/2$ and $n = 1$, $m(1/2) = w(a,b) = \sqrt{f((3a+b)/4)f((a+3b)/4)}$, we get Theorem 4.

Corollary 16. *With the above notations, suppose that* $f : [a,b] \to (0,\infty)$ *is log-convex and nondecreasing on* $[a,b]$*; one has the following inequality:*

$$\sup_{0\le t\le 1} m(t) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \inf_{0\le\lambda_1\le\cdots\le\lambda_n\le 1} M(\lambda_1,\ldots,\lambda_n), \quad (28)$$

where $m(t)$ *and* $M(\lambda_1,\ldots,\lambda_n)$ *are defined in Theorem 14.*

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