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Research Article

Some Generalized Difference Sequence Spaces Defined by Ideal Convergence and Musielak-Orlicz Function

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In the present paper we introduced the ideal convergence of generalized difference sequence spaces combining de La Vallée-Poussin mean and Musielak-Orlicz function over n-normed spaces. We also study some topological properties and inclusion relation between these spaces.

1. Introduction

Throughout the paper ω, ℓ_∞, c, c_0, and ℓ_p denote the classes of all, bounded, convergent, null, and p-absolutely summable sequences of complex numbers. The sets of natural numbers and real numbers will be denoted by $\mathbb{N}$, $\mathbb{R}$, respectively. Many authors studied various sequence spaces using normed or seminormed linear spaces. In this paper, using de La Vallée-Poussin mean and the notion of ideal, we aimed to introduce some new sequence spaces with respect to generalized difference operator Δ^s_m and Musielak-Orlicz function in n-normed linear spaces. By an ideal we mean a family $I \subset 2^Y$ of subsets of a nonempty set Y satisfying (i) $\phi \in I$; (ii) $A, B \in I$ imply $A \cup B \in I$; (iii) $A \in I$, $B \subset A$ imply $B \in I$, while an admissible ideal I of Y further satisfies $\{x\} \in I$ for each $x \in Y$. The notion of ideal convergence was introduced first by Kostyrko et al. [1] as a generalization of statistical convergence. The concept of 2-normed spaces was initially introduced by Gähler [2] in the 1960s, while that of n-normed spaces can be found in [3], and this concept has been studied by many authors; see for instance [4–7]. The notion of ideal convergence in 2-normed space was initially introduced by Gürdal [8]. Later on, it was extended to n-normed spaces by Gürdal and Şahiner [9]. Given $I \subset 2^{\mathbb{N}}$ is a nontrivial ideal in $\mathbb{N}$, the sequence $(x_n)_{n\in\mathbb{N}}$ in a normed space $(X; \|\cdot\|)$ is said to be I-convergent to $x \in X$, if for each $\varepsilon > 0$,

$$A(\varepsilon) = \{n \in \mathbb{N} : \|x_n - x\| \geq \varepsilon\} \in I. \tag{1}$$

A sequence (x_k) in a normed space $(X, \|\cdot\|)$ is said to be I-bounded if there exists $L > 0$ such that

$$\{k \in \mathbb{N} : \|x_k\| > L\} \in I. \tag{2}$$

A sequence (x_k) in a normed space $(X, \|\cdot\|)$ is said to be I-Cauchy if for each $\varepsilon > 0$, there exists a positive integer $m = m(\varepsilon)$ such that

$$\{k \in \mathbb{N} : \|x_k - x_m\| \geq \varepsilon\} \in I. \tag{3}$$

An Orlicz function is a function $M : [0,\infty) \to [0,\infty)$ which is continuous, nondecreasing, and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$, as $x \to \infty$. If convexity of M is replaced by $M(x + y) \leq M(x) + M(y)$, then it is called a modulus function, introduced by Nakano [10]. Ruckle [11] and Maddox [12] used the idea of a modulus function to construct some spaces of complex sequences. An Orlicz function M is said to satisfy Δ_2-condition for all values of $x \geq 0$, if there exists a constant $k > 0$, such that

$M(2x) \leq kM(x)$. The Δ_2-condition is equivalent to $M(lx) \leq klM(x)$ for all values of x and for $l > 1$. Lindenstrauss and Tzafriri [13] used the idea of an Orlicz function to define the following sequence spaces:

$$\ell_M = \left\{ x \in \omega : \sum_{k=1}^{\infty} M\left(\frac{|x(k)|}{\rho}\right) < \infty \right\} \quad (4)$$

which is a Banach space with the Luxemburg norm defined by

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x(k)|}{\rho}\right) \leq 1 \right\}. \quad (5)$$

The space ℓ_M is closely related to the space ℓ_p, which is an Orlicz sequence space with $M(x) = x^p$ for $1 \leq p < \infty$.

Recently different classes of sequences have been introduced using Orlicz functions. See [7, 9, 14–16].

A sequence $M = (M_k)$ of Orlicz functions M_k for all $k \in \mathbb{N}$ is called a Musielak-Orlicz function, for a given Musielak-Orlicz function M. Kızmaz [17] defined the difference sequence spaces $\ell_\infty(\Delta)$, $c(\Delta)$, and $c_0(\Delta)$ as follows: $Z(\Delta) = \{x = (x_k) : (\Delta x_k) \in Z\}$, for $Z = \ell_\infty$, c, and c_0, where $\Delta x = (x_k - x_{k+1})$, for all $k \in \mathbb{N}$. The above spaces are Banach spaces, normed by $\|x\| = |x_1| + \sup_k |\Delta x_k|$. The notion of difference sequence spaces was generalized by Et and Colak [18] as follows: $Z(\Delta^s) = \{x = (x_k) : (\Delta^s x_k) \in Z\}$, for $Z = \ell_\infty$, c, and c_0, where $s \in \mathbb{N}$, $(\Delta^s x_k) = (\Delta^{s-1} x_k - \Delta^{s-1} x_{k+1})$ and so that $\Delta^s x_k = \sum_{n=0}^{s} (-1)^n C_n^s x_{k+n}$. Tripathy and Esi [19] introduced the following new type of difference sequence spaces.

$Z(\Delta_m) = \{x = (x_k) : (\Delta_m x_k) \in Z\}$, $Z = \ell_\infty$, c, and c_0, where $\Delta_m x_k = (x_k - x_{k+m})$, for all $k \in \mathbb{N}$. Tripathy et al. [20] generalized the above notions and unified them as follows. Let m, s be nonnegative integers, then for Z a given sequence space we have

$$Z(\Delta_m^s) = \{x = (x_k) : (\Delta_m^s x_k) \in Z\}, \quad (6)$$

where

$$\Delta_m^s x_k = \sum_{n=0}^{s} (-1)^n C_n^s x_{k+mn}. \quad (7)$$

Also let m, s be nonnegative integers, then for Z a given sequence space we have

$$Z\left(\Delta_m^{(s)}\right) = \left\{x = (x_k) : \left(\Delta_m^{(s)} x_k\right) \in Z\right\}, \quad (8)$$

where

$$\Delta_m^{(s)} x_k = \sum_{n=0}^{s} (-1)^n C_n^s x_{k-mn}, \quad (9)$$

where $x_k = 0$, for $k < 0$

2. Definitions and Preliminaries

Let $n \in \mathbb{N}$ and X be a linear space over the field K of dimension d, where $d \geq n \geq 2$ and K is the field of real or complex numbers. A real valued function $\|\cdot, \ldots, \cdot\|$ on X^n satisfies the following four conditions:

(1) $\|x_1, x_2, \ldots, x_n\| = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent in X;

(2) $\|x_1, x_2, \ldots, x_n\|$ is invariant under permutation;

(3) $\|\alpha x_1, x_2, \ldots, x_n\| = |\alpha| \|x_1, x_2, \ldots, x_n\|$ for any $\alpha \in K$;

(4) $\|x + x', x_2, \ldots, x_n\| \leq \|x, x_2, \ldots, x_n\| + \|x', x_2, \ldots, x_n\|$; is called an n-norm on X and the pair $(X; \|\cdot, \ldots, \cdot\|)$ is called an n-normed space over the field K. For example, we may take $X = \mathbb{R}^n$ being equipped with the n-norm $\|x_1, x_2, \ldots, x_n\|_E =$ the volume of the n-dimensional parallelepiped spanned by the vectors $x_1, x_2, \ldots, x_n$ which may be given explicitly by the formula

$$\|x_1, x_2, \ldots, x_n\|_E = \left|\det\left(x_{ij}\right)\right| = \text{abs}\left(\begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{vmatrix}\right), \quad (10)$$

where $x_i = (x_{i1}, x_{i2}, \ldots, x_{in})$ for each $i \in \mathbb{N}$.

Let $(X, \|\cdot, \ldots, \cdot\|)$ be an n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, a_3, \ldots, a_n\}$ a linearly independent set in X. Then, the function $\|\cdot, \ldots, \cdot\|_\infty$ on X^{n-1} defined by

$$\|x_1, x_2, \ldots, x_n\|_\infty = \max_{1 \leq i \leq n} \|x_1, x_2, \ldots, x_{n-1}, a_i\|, \quad (11)$$

defines an $(n-1)$-norm on X with respect to $a_1, a_2, a_3, \ldots, a_n$ and this is known as the derived $(n-1)$-norm. The standard (n)-norm on X, a real inner product space of dimension $d \geq n$, is as follows:

$$\|x_1, x_2, \ldots, x_n\|_S = \text{abs}\left(\begin{vmatrix} \langle x_1, x_1\rangle & \langle x_1, x_2\rangle & \cdots & \langle x_1, x_n\rangle \\ \langle x_2, x_1\rangle & \langle x_2, x_2\rangle & \cdots & \langle x_2, x_n\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle x_n, x_1\rangle & \langle x_n, x_2\rangle & \cdots & \langle x_n, x_n\rangle \end{vmatrix}\right)^{1/2}, \quad (12)$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product on X. If we take $X = \mathbb{R}^n$, then

$$\|x_1, x_2, \ldots, x_n\|_E = \|x_1, x_2, \ldots, x_n\|_S. \quad (13)$$

For $n = 1$, this n-norm is the usual norm $\|x_1\| = \sqrt{\langle x_1, x_1\rangle}$.

Definition 1. A sequence (x_k) in an n-normed space is said to be convergent to $x \in X$ if,

$$\lim_{k \to \infty} \|(z_1, z_2, \ldots, z_{n-1}, x_k - x)\|_n = 0, \quad \forall z_1, z_2, \ldots, z_{n-1} \in X. \quad (14)$$

Definition 2. A sequence (x_k) in an n-normed space is called Cauchy (with respect to n-norm) if,

$$\lim_{k,j \to \infty} \left\|\left(z_1, z_2, \ldots, z_{n-1}, x_k - x_j\right)\right\|_n = 0, \quad \forall z_1, z_2, \ldots, z_{n-1} \in X. \quad (15)$$

If every Cauchy sequence in X converges to an $x \in X$, then X is said to be complete (with respect to the n-norm). A complete n-normed space is called n-Banach space.

Definition 3. A sequence (x_k) in an n-normed space $(X, \|\cdot, \ldots, \cdot\|)$ is said to be I-convergent to $x_0 \in X$ with respect to n-norm, if for each $\varepsilon > 0$, the set

$$\{k \in \mathbb{N} : \|x_k - x_0, z_1, z_2, \ldots, z_{n-1}\| \geq \varepsilon, \quad \text{for every } z_1, z_2, \ldots, z_{n-1}\} \in I. \tag{16}$$

Definition 4. A sequence (x_k) in an n-normed space $(X, \|\cdot, \ldots, \cdot\|)$ is said to be I-Cauchy if for each $\varepsilon > 0$, there exists a positive integer $m = m(\varepsilon)$ such that the set

$$\{k \in \mathbb{N} : \|x_k - x_m, z_1, z_2, \ldots, z_{n-1}\| \geq \varepsilon, \quad \text{for every } z_1, z_2, \ldots, z_{n-1}\} \in I. \tag{17}$$

Let $x = (x_k)$ be a sequence; then $S(x)$ denotes the set of all permutations of the elements of (x_k); that is, $S(x) = (x_{\pi(n)}) : \pi$ is a permutation of $\mathbb{N}$.

Definition 5. A sequence space E is said to be symmetric if $S(x) \subset E$ for all $x \in E$.

Definition 6. A sequence space E is said to be normal (or solid) if $(\alpha_k x_k) \in E$, whenever $(x_k) \in E$ and for all sequence (α_k) of scalars with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$.

Definition 7. A sequence space E is said to be a sequence algebra if $x, y \in E$ then $x \cdot y = (x_k y_k) \in E$.

Lemma 8. *Every n-normed space is an $(n-r)$-normed space for all $r = 1, 2, 3, \ldots, n-1$. In particular, every n-normed space is a normed space.*

Lemma 9. *On a standard n-normed space X, the derived $(n-1)$-norm $\|\cdot, \ldots, \cdot\|_\infty$ defined with respect to the orthogonal set $\{e_1, e_2, \ldots, e_n\}$ is equivalent to the standard $(n-1)$-norm $\|\cdot, \ldots, \cdot\|_S$. To be precise, one has*

$$\|x_1, x_2, \ldots, x_{n-1}\|_\infty \leq \|\cdot, \ldots, \cdot\|_S \leq \sqrt{n}\|x_1, x_2, \ldots, x_{n-1}\|_\infty, \tag{18}$$

for all $x_1, x_2, \ldots, x_{n-1} \in X$, where $\|x_1, x_2, \ldots, x_{n-1}\|_\infty =$ $\max_{1 \leq i \leq n}\{\|x_1, x_2, \ldots, x_{n-1}, e_i\|_S\}$.

Let $\Lambda = (\lambda_k)$ be a nondecreasing sequence of positive real numbers tending to infinity and let $\lambda_1 = 1$ and $\lambda_{k+1} \leq \lambda_k + 1$. In summability theory, de La Vallée-Poussin mean was first used to define the (V, λ)-summability by Leindler [21]. Also the (V, λ)-summable sequence spaces have been studied by many authors including [22, 23]. The generalized de La Vallée-Poussin's mean of a sequence $x = (x_k)$ is defined as follows: $t_k(x) = (1/\lambda_k)\sum_{j \in I_k}|x_j|$, where $I_k = [k - \lambda_k + 1, k]$ for $k \in \mathbb{N}$. We write
$[V, \lambda]_0 = \{x \in \omega : \lim_{k \to \infty}(1/\lambda_k)\sum_{j \in I_k}|x_j| = 0\}$,
$[V, \lambda] = \{x \in \omega : \lim_{k \to \infty}(1/\lambda_k)\sum_{j \in I_k}|x_j - l| = 0$ *for some* $l \in \mathbb{C}\}$,
$[V, \lambda]_\infty = \{x \in \omega : \sup_k(1/\lambda_k)\sum_{j \in I_k}|x_j| < \infty\}$.

For the sequence spaces that are strongly summable to zero, strongly summable and strongly bounded by the de La Vallée-Poussin's method, respectively. In the special case where $\lambda_k = k$ for $k \in \mathbb{N}$ the spaces $[V, \lambda]_0$, $[V, \lambda]$, and $[V, \lambda]_\infty$ reduce to the spaces v_0, v, and v_∞ introduced by Maddox [24]. The following new paranormed sequence space is defined in [22]:

$V(\lambda, p) = \{x \in \omega : \sum_{k=1}^{\infty}((1/\lambda_k)\sum_{j \in I_k}|x_j|)^{p_k} < \infty\}$. *If one takes $p_k = p$ for all $k \in \mathbb{N}$; the space $V(\lambda, p)$ reduced to normed space $V_p(\lambda)$ defined by $V_p(\lambda) = \{x \in \omega : \sum_{k=1}^{\infty}((1/\lambda_k)\sum_{j \in I_k}|x_j|)^p < \infty\}$. The details of the sequence spaces mentioned above can be found in [23].*

For any bounded sequence (p_n) of positive numbers, one has the following well-known inequality.

If $0 \leq p_k \leq \sup_k p_k = G$ and $D = \max(1, 2^{G-1})$, then $|a_n + b_n|^{p_n} \leq D(|a_n|^{p_n} + |b_n|^{p_n})$, for all k and $a_k, b_k \in \mathbb{C}$.

3. Main Results

In this section, we define some new ideal convergent sequence spaces and investigate their linear topological structures. We find out some relations related to these sequence spaces. Let I be an admissible ideal of $\mathbb{N}$, $\mathcal{M} = (M_j)$ be a Musielak-Orlicz function, and $(X, \|\cdot, \ldots, \cdot\|)$ an n-normed space. Further, let $p = (p_k)$ be any bounded sequence of positive real numbers,

$$V[\lambda, \mathcal{M}, \|\cdot, \ldots, \cdot\|, p, \Delta_m^s]^I = \Bigg\{x \in \omega(n-X) : \Bigg\{k \in \mathbb{N} : \lambda_k^{-1} \times \sum_{j \in I_k}\left[M_j\left(\left\|\frac{\Delta_m^s x_j - l}{\rho}, z_1, z_2, \ldots, z_{n-1}\right\|\right)\right]^{p_j} \geq \varepsilon\Bigg\} \in I, \text{ for some } \rho > 0, \; l \in X \text{ and each } z_1, z_2, \ldots, z_{n-1} \in X\Bigg\},$$

$$V[\lambda, \mathcal{M}, \|\cdot, \ldots, \cdot\|, p, \Delta_m^s]_0^I = \Bigg\{x \in \omega(n-X) : \Bigg\{k \in \mathbb{N} : \lambda_k^{-1}$$

$$\times \sum_{j\in I_k}\left[M_j\left(\left\|\frac{\Delta^s_m x_j}{\rho},z_1,z_2,\dots,z_{n-1}\right\|\right)\right]^{p_j}$$

$$\left.\geq \varepsilon\right\}\in I,$$

$$\left.\text{for some } \rho>0, \text{ and each } z_1,z_2,\dots,z_{n-1}\in X\right\},$$

$$V[\lambda,\mathcal{M},\|\cdot,\dots,\cdot\|,p,\Delta^s_m]_\infty$$

$$=\Big\{x\in\omega(n-X):$$

$$\times \sup_k \lambda_k^{-1}\sum_{j\in I_k}\left[M_j\left(\left\|\frac{\Delta^s_m x_j}{\rho},z_1,z_2,\dots,z_{n-1}\right\|\right)\right]^{p_j}$$

$$<\infty,$$

$$\left.\text{for some } \rho>0, \text{ and each } z_1,z_2,\dots,z_{n-1}\in X\right\},$$

$$V[\lambda,\mathcal{M},\|\cdot,\dots,\cdot\|,p,\Delta^s_m]^I_\infty$$

$$=\Big\{x\in\omega(n-X):\exists K>0, \text{ s.t.}$$

$$\Big\{k\in\mathbb{N}:\lambda_k^{-1}$$

$$\times \sum_{j\in I_k}\left[M_j\left(\left\|\frac{\Delta^s_m x_j}{\rho},z_1,z_2,\dots,z_{n-1}\right\|\right)\right]^{p_j}$$

$$\left.\geq K\right\}\in I,$$

$$\left.\text{for some } \rho>0, \text{ and each } z_1,z_2,\dots,z_{n-1}\in X\right\}. \tag{19}$$

The above sequence spaces contain some unbounded sequences for $s\geq 1$. If $M_k(x)=x$, $m=1$, $\lambda_k=k$ for all $k\in\mathbb{N}$ and $p_k=1$ for all $k\in\mathbb{N}$, then $(k^s)\in V[\lambda,M,\|\cdot,\dots,\cdot\|,p,\Delta^s_m]_\infty$ but $(k^s)\notin \ell_\infty$.

Let us consider a few special cases of the above sets.

(1) If $n=2$, $m=1$, and $M_k(x)=M(x)$, then the above classes of sequences are denoted by $V[\lambda,M,\|\cdot,\dots,\cdot\|,p,\Delta^s]^I$, $V[\lambda,M,\|,\cdot,\|,p,\Delta^s]^I_0$, $V[\lambda,M,\|,\cdot,\|,p,\Delta^s]_\infty$, and $V[\lambda,M,\|,\cdot,\|,p,\Delta^s]^I_\infty$, respectively, which were defined and studied by Savaş [25].

(2) If $M_k(x)=M(x)$, then the above classes of sequences are denoted by $V[\lambda,M,\|\cdot,\dots,\cdot\|,p,\Delta^s_m]^I$, $V[\lambda,M,\|\cdot,\dots,\cdot\|,p,\Delta^s_m]^I_0$, $V[\lambda,M,\|\cdot,\dots,\cdot\|,p,\Delta^s_m]_\infty$, and $V[\lambda,M,\|\cdot,\dots,\cdot\|,p,\Delta^s_m]^I_\infty$, respectively.

(3) If $M_k(x)=x$, for all $k\in\mathbb{N}$, then the above classes of sequences are denoted by $V[\lambda,\|\cdot,\dots,\cdot\|,p,\Delta^s_m]^I$, $V[\lambda,\|\cdot,\dots,\cdot\|,p,\Delta^s_m]^I_0$, $V[\lambda,\|\cdot,\dots,\cdot\|,p,\Delta^s_m]_\infty$, and $V[\lambda,\|\cdot,\dots,\cdot\|,p,\Delta^s_m]^I_\infty$, respectively.

(4) If $p_k=1$, for all $k\in\mathbb{N}$, then we denote the above classes of sequences by $V[\lambda,\|\cdot,\dots,\cdot\|,\Delta^s_m]^I$, $V[\lambda,\|\cdot,\dots,\cdot\|,\Delta^s_m]^I_0$, $V[\lambda,\|\cdot,\dots,\cdot\|,\Delta^s_m]_\infty$, and $V[\lambda,\|\cdot,\dots,\cdot\|,\Delta^s_m]^I_\infty$, respectively.

(5) If $M_k(x)=M(x)$, $m=1$, and $\lambda_k=k$ for all $k\in\mathbb{N}$, then the above classes of sequences are denoted by $V[M,\|\cdot,\dots,\cdot\|,p,\Delta^s]^I$, $V[M,\|\cdot,\dots,\cdot\|,p,\Delta^s]^I_0$, $V[M,\|\cdot,\dots,\cdot\|,p,\Delta^s]_\infty$, and $V[M,\|\cdot,\dots,\cdot\|,p,\Delta^s]^I_\infty$, respectively, which were defined and studied by Savaş [7].

Theorem 10. *The spaces* $V[\lambda,\mathcal{M},\|\cdot,\dots,\cdot\|,p,\Delta^s_m]^I$, $V[\lambda,\mathcal{M},\|\cdot,\dots,\cdot\|,p,\Delta^s_m]^I_0$, *and* $V[\lambda,\mathcal{M},\|\cdot,\dots,\cdot\|,p,\Delta^s_m]^I_\infty$ *are linear spaces.*

Theorem 11. *The spaces* $V[\lambda,\mathcal{M},\|\cdot,\dots,\cdot\|,p,\Delta^s_m]^I$, $V[\lambda,\mathcal{M},\|\cdot,\dots,\cdot\|,p,\Delta^s_m]^I_0$, *and* $V[\lambda,\mathcal{M},\|\cdot,\dots,\cdot\|,p,\Delta^s_m]^I_\infty$ *are paranormed spaces (not totally paranormed) with respect to the paranorm* g_Δ *defined by*

$$g_\Delta(x)=\sum_{j=1}^{ms}\|x_j,z_1,z_2,\dots,z_{n-1}\|$$

$$+\inf\left\{\rho^{p_k/H}:\sup_j M_j\left(\left\|\frac{\Delta^s_m x_j}{\rho},z_1,z_2,\dots,z_{n-1}\right\|\right)\right.$$

$$\leq 1, \quad\text{for some } \rho>0,$$

$$\left.\text{and each } z_1,z_2,\dots,z_{n-1}\in X\right\}, \tag{20}$$

where $H=\max\{1,\sup_k p_k\}$.

Proof. Clearly $g_\Delta(-x)=g_\Delta(x)$ and $g_\Delta(\theta)=0$. Let $x=(x_k)$ and $y=(y_k)\in V[\lambda,\mathcal{M},\|\cdot,\dots,\cdot\|,p,\Delta^s_m]^I_0$. Then, for $\rho>0$ we set

$$A_1=\left\{\rho:\sup_j M_j\left(\left\|\frac{\Delta^s_m x_j}{\rho},z_1,z_2,\dots,z_{n-1}\right\|\right)\leq 1,\right.$$

$$\left.\text{for each } z_1,z_2,\dots,z_{n-1}\in X\right\},$$

$$A_2 = \left\{\rho : \sup_j M_j\left(\left\|\frac{\Delta_m^s y_j}{\rho}, z_1, z_2, \ldots, z_{n-1}\right\|\right) \leq 1, \right.$$
$$\left. \text{for each } z_1, z_2, \ldots, z_{n-1} \in X\right\}. \tag{21}$$

Let $\rho_1 \in A_1$, $\rho_2 \in A_2$ and $\rho = \rho_1 + \rho_2$, then we have

$$\begin{aligned}
&M_j\left(\left\|\frac{\Delta_m^s(x_j + y_j)}{\rho}, z_1, z_2, \ldots, z_{n-1}\right\|\right) \\
&\quad \leq \frac{\rho_1}{\rho_1+\rho_2} M_j\left(\left\|\frac{\Delta_m^s(x_j)}{\rho_1}, z_1, z_2, \ldots, z_{n-1}\right\|\right) \\
&\qquad + \frac{\rho_2}{\rho_1+\rho_2} M_j\left(\left\|\frac{\Delta_m^s(y_j)}{\rho_2}, z_1, z_2, \ldots, z_{n-1}\right\|\right) \leq 1, \\
&g_\Delta(x+y) \\
&\quad = \sum_{j=1}^{ms}\left\|x_j + y_j, z_1, z_2, \ldots, z_{n-1}\right\| \\
&\qquad + \inf\left\{(\rho_1+\rho_2)^{p_j/H} : \rho_1 \in A_1, \rho_2 \in A_2\right\} \\
&\quad \leq \sum_{j=1}^{ms}\left\|x_j, z_1, z_2, \ldots, z_{n-1}\right\| \\
&\qquad + \inf\left\{(\rho_1)^{p_j/H} : \rho_1 \in A_1\right\} \\
&\qquad + \sum_{j=1}^{ms}\left\|y_j, z_1, z_2, \ldots, z_{n-1}\right\| \\
&\qquad + \inf\left\{(\rho_2)^{p_j/H} : \rho_2 \in A_2\right\} \\
&\quad = g_\Delta(x) + g_\Delta(y).
\end{aligned} \tag{22}$$

Let $\lambda^t \to \lambda$ where $\lambda^t, \lambda \in \mathbb{C}$, and let $g_\Delta(x^t - x) \to 0$ as $t \to \infty$. We have to show that $g_\Delta(\lambda^t x^t - \lambda x) \to 0$ as $t \to \infty$. We set

$$\begin{aligned}
A_3 &= \left\{\rho_t : \sup_j M_j\left(\left\|\frac{\Delta_m^s x_j}{\rho_t}, z_1, z_2, \ldots, z_{n-1}\right\|\right) \leq 1, \right. \\
&\qquad \left. \text{for each } z_1, z_2, \ldots, z_{n-1} \in X\right\}, \\
A_4 &= \left\{\rho_t^1 : \sup_j M_j\left(\left\|\frac{\Delta_m^s y_j}{\rho_t^1}, z_1, z_2, \ldots, z_{n-1}\right\|\right) \leq 1, \right. \\
&\qquad \left. \text{for each } z_1, z_2, \ldots, z_{n-1} \in X\right\}.
\end{aligned} \tag{23}$$

If $\rho_t \in A_3$ and $\rho_t^1 \in A_4$, by using nondecreasing and convexity of the Orlicz function M_j for all $j \in \mathbb{N}$ that

$$\begin{aligned}
&M_j\left(\left\|\frac{\Delta_m^s(\lambda^t x_j^t - \lambda x_j)}{|\lambda^t - \lambda|\rho_t + |\lambda|\rho_t^1}, z_1, z_2, \ldots, z_{n-1}\right\|\right) \\
&\quad \leq M_j\left[\left\|\frac{(\Delta_m^s \lambda^t x_j^t - \lambda x_j^t)}{|\lambda^t - \lambda|\rho_t + |\lambda|\rho_t^1}, z_1, z_2, \ldots, z_{n-1}\right\|\right. \\
&\qquad \left. + \left\|\frac{\Delta_m^s(\lambda x_j^t - \lambda x_j)}{|\lambda^t - \lambda|\rho_t + |\lambda|\rho_t^1}, z_1, z_2, \ldots, z_{n-1}\right\|\right] \\
&\quad \leq \frac{|\lambda^t - \lambda|\rho_t}{|\lambda^t - \lambda|\rho_t + |\lambda|\rho_t^1} M_j\left(\left\|\frac{\Delta_m^s x_j^t}{\rho_t}, z_1, z_2, \ldots, z_{n-1}\right\|\right) \\
&\qquad + \frac{|\lambda|\rho_t^1}{|\lambda^t - \lambda|\rho_t + |\lambda|\rho_t^1} M_j \\
&\qquad \times \left(\left\|\frac{\Delta_m^s(x_j^t - x_j)}{\rho_t^1}, z_1, z_2, \ldots, z_{n-1}\right\|\right).
\end{aligned} \tag{24}$$

From the above inequality, it follows that

$$\sup_j M_j\left(\left\|\frac{\Delta_m^s(\lambda^t x_j^t - \lambda x_j)}{|\lambda^t - \lambda|\rho_t + |\lambda|\rho_t^1}, z_1, z_2, \ldots, z_{n-1}\right\|\right) \leq 1, \tag{25}$$

and consequently

$$\begin{aligned}
&g_\Delta(\lambda^t x^t - \lambda x) \\
&\quad = \sum_{j=1}^{ms}\left\|\lambda^t x_j^t - \lambda x_j, z_1, z_2, \ldots, z_{n-1}\right\| \\
&\qquad + \inf\left\{\left(\left|\lambda^t - \lambda\right|\rho_t + |\lambda|\rho_t^1\right)^{p_j/H} : \rho_t \in A_3, \rho_t^1 \in A_4\right\} \\
&\quad \leq \left|\lambda^t - \lambda\right| \sum_{j=1}^{ms}\left\|x_j^t, z_1, z_2, \ldots, z_{n-1}\right\| \\
&\qquad + \left|\lambda^t - \lambda\right|^{p_j/H} \inf\left\{(\rho_t)^{p_j/H} : \rho_t \in A_3\right\} \\
&\qquad + |\lambda| \sum_{j=1}^{ms}\left\|\lambda^t x_j^t - \lambda x_j, z_1, z_2, \ldots, z_{n-1}\right\| \\
&\qquad + |\lambda|^{p_j/H} \inf\left\{\left(\rho_t^1\right)^{p_j/H} : \rho_t^1 \in A_4\right\} \\
&\quad \leq \max\left\{\left|\lambda^t - \lambda\right|, \left|\lambda^t - \lambda\right|^{p_j/H}\right\} g_\Delta(x^t) \\
&\qquad + \max\left\{|\lambda|, |\lambda|^{p_j/H}\right\} g_\Delta(x^t - x).
\end{aligned} \tag{26}$$

Note that $g_\Delta(x^t) \leq g_\Delta(x) + g_\Delta(x^t - x)$, for all $t \in \mathbb{N}$. Hence, by our assumption, the right hand of (26) tends to

0 as $t \to \infty$, and the result follows. This completes the proof of the theorem. □

Theorem 12. *Let $\mathcal{M} = (M_j)$, $\mathcal{M}' = (M'_j)$, and $\mathcal{M}'' = (M''_j)$ be Musielak-Orlicz functions. Then, the following hold:*
(a) $V[\lambda, \mathcal{M}', \|\cdot, \dots, \cdot\|, p, \Delta^s_m]^I_0 \subseteq V[\lambda, \mathcal{M} \cdot \mathcal{M}', \|\cdot, \dots, \cdot\|, p, \Delta^s_m]^I_0$, *provided $p = (p_k)$ be such that $G_0 = \inf p_k > 0$,*
(b) $V[\lambda, \mathcal{M}', \|\cdot, \dots, \cdot\|, p, \Delta^s_m]^I_0 \cap V[\lambda, \mathcal{M}'', \|\cdot, \dots, \cdot\|, p, \Delta^s_m]^I_0 \subseteq V[\lambda, \mathcal{M}' + \mathcal{M}'', \|\cdot, \dots, \cdot\|, p, \Delta^s_m]^I_0$.

Proof. (a) Let $\varepsilon > 0$ be given. Choose $\varepsilon_1 > 0$ such that $\max\{\varepsilon_1^G, \varepsilon_1^{G^0}\} < \varepsilon$. Using the continuity of the Orlicz function M, choose $0 < \delta < 1$ such that $0 < t < \delta$ implies that $M(t) < \varepsilon_1$. Let $x = (x_k)$ be any element in $V[\lambda, \mathcal{M}', \|\cdot, \dots, \cdot\|, p, \Delta^n_m]^I_0$, put

$$A_\delta = \left\{ k \in \mathbb{N} : \lambda_k^{-1} \sum_{j \in I_k} \left[M'_j \left(\left\| \frac{\Delta^s_m x_j}{\rho_1}, z_1, z_2, \dots, z_{n-1} \right\| \right) \right]^{p_j} \geq \delta^G \right\}. \tag{27}$$

Then, by definition of ideal convergent, we have the set $A_\delta \in I$. If $n \notin A_\delta$ then we have

$$\begin{aligned} &\lambda_k^{-1} \sum_{j \in I_k} \left[M'_j \left(\left\| \frac{\Delta^s_m x_j}{\rho_1}, z_1, z_2, \dots, z_{n-1} \right\| \right) \right]^{p_j} < \delta^G \\ &\Longrightarrow \left[M'_j \left(\left\| \frac{\Delta^s_m x_j}{\rho_1}, z_1, z_2, \dots, z_{n-1} \right\| \right) \right]^{p_j} < \lambda_k \delta^G, \quad \forall j \in I_k \\ &\Longrightarrow \left[M'_j \left(\left\| \frac{\Delta^s_m x_j}{\rho_1}, z_1, z_2, \dots, z_{n-1} \right\| \right) \right]^{p_j} < \delta^G \\ &\Longrightarrow \left[M'_j \left(\left\| \frac{\Delta^s_m x_j}{\rho_1}, z_1, z_2, \dots, z_{n-1} \right\| \right) \right] < \delta. \end{aligned} \tag{28}$$

Using the continuity of the Orlicz function M_j for all j and the relation (28), we have

$$M_j \left[M'_j \left(\left\| \frac{\Delta^s_m x_j}{\rho_1}, z_1, z_2, \dots, z_{n-1} \right\| \right) \right] < \varepsilon_1, \quad \forall j \in I_k. \tag{29}$$

Consequently, we get

$$\begin{aligned} &\sum_{j \in I_k} M_j \left[M'_j \left(\left\| \frac{\Delta^s_m x_j}{\rho_1}, z_1, z_2, \dots, z_{n-1} \right\| \right) \right]^{p_j} \\ &\quad < \lambda_j \max\left\{\varepsilon_1^G, \varepsilon_1^{G_0}\right\} < \lambda_j \varepsilon \\ &\Longrightarrow \lambda_j^{-1} \sum_{j \in I_k} M_j \left[M'_j \left(\left\| \frac{\Delta^s_m x_j}{\rho_1}, z_1, z_2, \dots, z_{n-1} \right\| \right) \right]^{p_j} \\ &\quad < \varepsilon. \end{aligned} \tag{30}$$

This shows that

$$\left\{ k \in \mathbb{N} : \sum_{j \in I_k} M_j \left[M'_j \left(\left\| \frac{\Delta^s_m x_j}{\rho_1}, z_1, z_2, \dots, z_{n-1} \right\| \right) \right]^{p_j} \geq \varepsilon \right\} \subseteq A_\delta \in I. \tag{31}$$

This proves the assertion.
(b) Let $x = (x_k)$ be any element in $V[\lambda, \mathcal{M}', \|\cdot, \dots, \cdot\|, p, \Delta^s_m]^I_0 \cap V[\lambda, \mathcal{M}'', \|\cdot, \dots, \cdot\|, p, \Delta^s_m]^I_0$. Then, by the following inequality, the results follow:

$$\begin{aligned} &\lambda_k^{-1} \sum_{j \in I_k} \left[\left(M'_j + M''_j \right) \left(\left\| \frac{\Delta^s_m x_j}{\rho_1}, z_1, z_2, \dots, z_{n-1} \right\| \right) \right]^{p_j} \\ &\quad \leq D \lambda_k^{-1} \sum_{j \in I_k} \left[M'_j \left(\left\| \frac{\Delta^s_m x_j}{\rho_1}, z_1, z_2, \dots, z_{n-1} \right\| \right) \right]^{p_j} \\ &\qquad + D \lambda_k^{-1} \sum_{j \in I_k} \left[M''_j \left(\left\| \frac{\Delta^s_m x_j}{\rho_1}, z_1, z_2, \dots, z_{n-1} \right\| \right) \right]^{p_j}. \end{aligned} \tag{32}$$

□

Theorem 13. *The inclusions $Z[\lambda, \mathcal{M}, \|\cdot, \dots, \cdot\|, \Delta^{s-1}_m] \subseteq Z[\lambda, \mathcal{M}, \|\cdot, \dots, \cdot\|, \Delta^s_m]$ are strict for $s, m \geq 1$ in general where $Z = V^I$, V^I_0, and V^I_∞.*

Proof. We will give the proof for $V[\lambda, \mathcal{M}, \|\cdot, \dots, \cdot\|, \Delta^{s-1}_m]^I_0 \subseteq V[\lambda, \mathcal{M}, \|\cdot, \dots, \cdot\|, \Delta^s_m]^I_0$ only. The others can be proved by similar arguments. Let $x = (x_k) \in V[\lambda, \mathcal{M}, \|\cdot, \dots, \cdot\|, \Delta^{s-1}_m]^I_0$. Then let $\varepsilon > 0$ be given; there exists $\rho > 0$ such that

$$\left\{ k \in \mathbb{N} : \lambda_k^{-1} \sum_{j \in I_k} M_j \left(\left\| \frac{\Delta^{s-1}_m x_j}{\rho}, z_1, z_2, \dots, z_{n-1} \right\| \right) \geq \frac{\varepsilon}{2} \right\} \in I. \tag{33}$$

Since M_j for all $j \in \mathbb{N}$ is nondecreasing and convex, it follows that

$$\begin{aligned} &\lambda_k^{-1} \sum_{j \in I_k} M_j \left(\left\| \frac{\Delta^s_m x_j}{2\rho}, z_1, z_2, \dots, z_{n-1} \right\| \right) \\ &\quad = \lambda_k^{-1} \sum_{j \in I_k} M_j \left(\left\| \frac{\Delta^{s-1}_m x_{j+1} - \Delta^{s-1}_m x_j}{2\rho}, z_1, z_2, \dots, z_{n-1} \right\| \right) \end{aligned}$$

$$\leq \lambda_k^{-1} \sum_{j \in I_k} \frac{1}{2} M_j \left(\left\| \frac{\Delta_m^{s-1} x_{j+1}}{\rho}, z_1, z_2, \ldots, z_{n-1} \right\| \right)$$
$$+ \lambda_k^{-1} \sum_{j \in I_k} \frac{1}{2} M_j \left(\left\| \frac{\Delta_m^{s-1} x_j}{\rho}, z_1, z_2, \ldots, z_{n-1} \right\| \right)$$
$$\leq \frac{1}{2} \lambda_k^{-1} \sum_{j \in I_k} M_j \left(\left\| \frac{\Delta_m^{s-1} x_{j+1}}{\rho}, z_1, z_2, \ldots, z_{n-1} \right\| \right)$$
$$+ \frac{1}{2} \lambda_k^{-1} \sum_{j \in I_k} M_j \left(\left\| \frac{\Delta_m^{s-1} x_j}{\rho}, z_1, z_2, \ldots, z_{n-1} \right\| \right), \quad (34)$$

then we have

$$\left\{ k \in \mathbb{N} : \lambda_k^{-1} \sum_{j \in I_k} M_j \left(\left\| \frac{\Delta_m^s x_j}{2\rho}, z_1, z_2, \ldots, z_{n-1} \right\| \right) \geq \varepsilon \right\}$$
$$\subseteq \left\{ k \in \mathbb{N} : \frac{1}{2} \lambda_k^{-1} \sum_{j \in I_k} M_j \left(\left\| \frac{\Delta_m^{s-1} x_{j+1}}{\rho}, z_1, z_2, \ldots, z_{n-1} \right\| \right) \geq \frac{\varepsilon}{2} \right\}$$
$$\cup \left\{ k \in \mathbb{N} : \frac{1}{2} \lambda_k^{-1} \sum_{j \in I_k} M_j \left(\left\| \frac{\Delta_m^{s-1} x_j}{\rho}, z_1, z_2, \ldots, z_{n-1} \right\| \right) \geq \frac{\varepsilon}{2} \right\}. \quad (35)$$

Let $M_k(x) = M(x) = x$ for all $x \in [0, \infty[$, $k \in \mathbb{N}$ and $\lambda_k = k$ for all $k \in \mathbb{N}$. Consider a sequence $x = (x_k) = (k^s)$. Then, $x \in V[\lambda, \mathcal{M}, \|\cdot, \ldots, \cdot\|, \Delta_m^s]_0^I$ but does not belong to $V[\lambda, \mathcal{M}, \|\cdot, \ldots, \cdot\|, \Delta_m^{s-1}]_0^I$, for $s = m = 1$. This shows that the inclusion is strict. □

Theorem 14. *Let $0 < p_k \leq q_k$ for all $k \in \mathbb{N}$, then $V[\lambda, \mathcal{M}, \|\cdot, \ldots, \cdot\|, p, \Delta_m^s]_\infty \subseteq V[\lambda, \mathcal{M}, \|\cdot, \ldots, \cdot\|, q, \Delta_m^s]_\infty$.*

Proof. Let $x = (x_j) \in V[\lambda, \mathcal{M}, \|\cdot, \ldots, \cdot\|, p, \Delta_m^s]_\infty$, then there exists some $\rho > 0$ such that

$$\sup_k \lambda_k^{-1} \sum_{j \in I_k} \left[M_j \left(\left\| \frac{\Delta_m^s x_j}{\rho}, z_1, z_2, \ldots, z_{n-1} \right\| \right) \right]^{p_j} < \infty. \quad (36)$$

This implies that

$$M_j \left(\left\| \frac{\Delta_m^s x_j}{\rho}, z_1, z_2, \ldots, z_{n-1} \right\| \right) < 1, \quad (37)$$

for sufficiently large value of j. Since M_j for all $j \in \mathbb{N}$ is nondecreasing, we get

$$\sup_k \lambda_k^{-1} \sum_{j \in I_k} \left[M_j \left(\left\| \frac{\Delta_m^s x_j}{\rho}, z_1, z_2, \ldots, z_{n-1} \right\| \right) \right]^{q_j}$$
$$\leq \sup_k \lambda_k^{-1} \sum_{j \in I_k} \left[M_j \left(\left\| \frac{\Delta_m^s x_j}{\rho}, z_1, z_2, \ldots, z_{n-1} \right\| \right) \right]^{p_j} \quad (38)$$
$$< \infty.$$

Thus, $x \in [\lambda, \mathcal{M}, \|\cdot, \ldots, \cdot\|, q, \Delta_m^s]_\infty$. This completes the proof of the theorem. □

Theorem 15. (i) *If $0 < \inf p_k \leq p_k < 1$, then $V[\lambda, \mathcal{M}, \|\cdot, \ldots, \cdot\|, p, \Delta_m^s]_\infty \subseteq V[\lambda, \mathcal{M}, \|\cdot, \ldots, \cdot\|, \Delta_m^s]_\infty$.*

(ii) *If $0 < p_k \leq \sup_k p_k < \infty$, then $V[\lambda, \mathcal{M}, \|\cdot, \ldots, \cdot\|, \Delta_m^s]_\infty \subseteq V[\lambda, \mathcal{M}, p, \|\cdot, \ldots, \cdot\|, \Delta_m^s]_\infty$.*

Theorem 16. *For any sequence of Orlicz functions $\mathcal{M} = (M_j)$ which satisfies Δ_2-condition, one has $V[\lambda, \|\cdot, \ldots, \cdot\|, p, \Delta_m^s]^I \subset V[\lambda, \mathcal{M}, \|\cdot, \ldots, \cdot\|, p, \Delta_m^s]^I$.*

Theorem 17. *Let $0 < p_n \leq q_n < 1$ and (q_n/p_n) be bounded; then*

$$V[\lambda, \mathcal{M}, \|\cdot, \ldots, \cdot\|, q, \Delta_m^s]^I \subseteq V[\lambda, \mathcal{M}, \|\cdot, \ldots, \cdot\|, p, \Delta_m^s]^I. \quad (39)$$

Theorem 18. *For any two sequences $p = (p_k)$ and $q = (q_k)$ of positive real numbers and for any two n-norms $\|\cdot, \ldots, \cdot\|_1$ and $\|\cdot, \ldots, \cdot\|_2$ on X, the following holds:*

$$Z[\lambda, \mathcal{M}, \|\cdot, \ldots, \cdot\|_1, p, \Delta_m^s] \cap Z[\lambda, \mathcal{M}, \|\cdot, \ldots, \cdot\|_2, q, \Delta_m^s] \neq \phi, \quad (40)$$

where $Z = V^I$, V_0^I, V_∞^I, and V_∞.

Proof. Proof of the theorem is obvious, because the zero element belongs to each of the sequence spaces involved in the intersection. □

Theorem 19. *The sequence spaces $V[\lambda, \mathcal{M}, \|\cdot, \ldots, \cdot\|, p, \Delta_m^s]^I$, $V[\lambda, \mathcal{M}, \|\cdot, \ldots, \cdot\|, p, \Delta_m^s]_0^I$, $V[\lambda, \mathcal{M}, \|\cdot, \ldots, \cdot\|, p, \Delta_m^s]_\infty$, and $V[\lambda, \mathcal{M}, \|\cdot, \ldots, \cdot\|, p, \Delta_m^s]_\infty^I$ are neither solid nor symmetric, nor sequence algebras for $s, m \geq 1$ in general.*

Proof. The proof is obtained by using the same techniques of Et [26, Theorems 3.6, 3.8, and 3.9]. □

Remark 20. If we replace the difference operator Δ_m^s by $\Delta_m^{(s)}$, then for each $\varepsilon > 0$ we get the following sequence spaces:

$$V\left[\lambda, \mathscr{M}, \|\cdot, \dots, \cdot\|, p, \Delta_m^{(s)}\right]^I = \Bigg\{ x \in \omega(n-X) : \Bigg\{ k \in \mathbb{N} : \lambda_k^{-1} \times \sum_{j \in I_k} \left[M_j \left(\left\| \frac{\Delta_m^{(s)} x_j - l}{\rho}, z_1, z_2, \dots, z_{n-1} \right\| \right) \right]^{p_j} \geq \varepsilon \Bigg\} \in I, \text{ for some } \rho > 0, \; l \in X \text{ and each } z_1, z_2, \dots, z_{n-1} \in X \Bigg\},$$

$$V\left[\lambda, \mathscr{M}, \|\cdot, \dots, \cdot\|, p, \Delta_m^{(s)}\right]_0^I = \Bigg\{ x \in \omega(n-X) : \Bigg\{ k \in \mathbb{N} : \lambda_k^{-1} \sum_{j \in I_k} \left[M_j \left(\left\| \frac{\Delta_m^{(s)} x_j}{\rho}, z_1, z_2, \dots, z_{n-1} \right\| \right) \right]^{p_j} \geq \varepsilon \Bigg\} \in I, \text{ for some } \rho > 0 \text{ and each } z_1, z_2, \dots, z_{n-1} \in X \Bigg\},$$

$$V\left[\lambda, \mathscr{M}, \|\cdot, \dots, \cdot\|, p, \Delta_m^{(s)}\right]_\infty = \Bigg\{ x \in \omega(n-X) : \times \sup_k \lambda_k^{-1} \sum_{j \in I_k} \left[M_j \left(\left\| \frac{\Delta_m^{(s)} x_j}{\rho}, z_1, z_2, \dots, z_{n-1} \right\| \right) \right]^{p_j} < \infty, \text{ for some } \rho > 0 \text{ and each } z_1, z_2, \dots, z_{n-1} \in X \Bigg\},$$

$$V\left[\lambda, \mathscr{M}, \|\cdot, \dots, \cdot\|, p, \Delta_m^{(s)}\right]_\infty^I = \Bigg\{ x \in \omega(n-X) : \exists K > 0, \text{ s.t. } \Bigg\{ k \in \mathbb{N} : \lambda_k^{-1} \times \sum_{j \in I_k} \left[M_j \left(\left\| \frac{\Delta_m^{(s)} x_j}{\rho}, z_1, z_2, \dots, z_{n-1} \right\| \right) \right]^{p_j} \geq K \Bigg\} \in I, \text{ for some } \rho > 0 \text{ and each } z_1, z_2, \dots, z_{n-1} \in X \Bigg\}. \quad (41)$$

Note. It is clear from definitions that $V[\lambda, \mathscr{M}, \|\cdot, \dots, \cdot\|, p, \Delta_m^{(s)}]_0^I \subseteq V[\lambda, \mathscr{M}, \|\cdot, \dots, \cdot\|, p, \Delta_m^{(s)}]^I \subseteq V[\lambda, \mathscr{M}, \|\cdot, \dots, \cdot\|, p, \Delta_m^{(s)}]_\infty^I$.

Corollary 21. *The sequence spaces $Z[\lambda, \mathscr{M}, \|\cdot, \dots, \cdot\|, p, \Delta_m^{(s)}]$, where $Z = V^I, V_0^I, V_\infty^I$, and V_∞ are paranormed spaces (not totally paranormed) with respect to the paranorm h_Δ defined by*

$$h_\Delta(x) = \inf \Bigg\{ \rho^{p_k/H} : \sup_j M_j \left(\left\| \frac{\Delta_m^{(s)} x_j}{\rho}, z_1, z_2, \dots, z_{n-1} \right\| \right) \leq 1, \text{ for some } \rho > 0, \text{ and each } z_1, z_2, \dots, z_{n-1} \in X \Bigg\}, \quad (42)$$

where $H = \max\{1, \sup_k p_k\}$ and $Z = V^I, V_0^I, V_\infty^I$, and V_∞. Also it is clear that the paranorm g_Δ and h_Δ are equivalent. We state the following theorem in view of Lemma 9. Let X be a standard n-normed space and $\{e_1, e_2, \dots, e_n\}$ an orthogonal set in X. Then, the following hold:

(a) $V[\lambda, \mathscr{M}, \|\cdot, \dots, \cdot\|_\infty, p, \Delta_m^{(s)}]^I = V[\lambda, \mathscr{M}, \|\cdot, \dots, \cdot\|_{n-1}, p, \Delta_m^{(s)}]^I$;

(b) $V[\lambda, \mathscr{M}, \|\cdot, \dots, \cdot\|_\infty, p, \Delta_m^{(s)}]_0^I = V[\lambda, \mathscr{M}, \|\cdot, \dots, \cdot\|_{n-1}, p, \Delta_m^{(s)}]_0^I$;

(c) $V[\lambda, \mathscr{M}, \|\cdot, \dots, \cdot\|_\infty, p, \Delta_m^{(s)}]_\infty = V[\lambda, \mathscr{M}, \|\cdot, \dots, \cdot\|_{n-1}, p, \Delta_m^{(s)}]_\infty$;

(d) $V[\lambda, \mathscr{M}, \|\cdot, \dots, \cdot\|_\infty, p, \Delta_m^{(s)}]_\infty^I = V[\lambda, \mathscr{M}, \|\cdot, \dots, \cdot\|_{n-1}, p, \Delta_m^{(s)}]_\infty^I$,

where $\|\cdot, \dots, \cdot\|_\infty$ is the derived $(n-1)$-norm defined with respect to the set $\{e_1, e_2, \dots, e_n\}$ and $\|\cdot, \dots, \cdot\|_{n-1}$ is the standard $(n-1)$-norm on X.

Theorem 22. *The spaces* $Z[\lambda, \mathscr{M}, \|\cdot, \ldots, \cdot\|_\infty, p, \Delta_m^{(s)}]$ *and* $Z[\lambda, \mathscr{M}, \|\cdot, \ldots, \cdot\|_\infty, p]$ *are equivalent as topological spaces, where* $Z = V^I, V_0^I, V_\infty^I$, *and* V_∞.

Proof. Consider the mapping $T : Z[\lambda, \mathscr{M}, \|\cdot, \ldots, \cdot\|_\infty, p, \Delta_m^{(s)}] \to Z[\lambda, \mathscr{M}, \|\cdot, \ldots, \cdot\|_\infty, p]$ defined by $T(x) = (\Delta_m^{(s)} x_k)$ for each $x = (x_k) \in Z[\lambda, \mathscr{M}, \|\cdot, \ldots, \cdot\|_\infty, p, \Delta_m^{(s)}]$. Then, clearly T is a linear homeomorphism and the proof follows. □

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References

[1] P. Kostyrko, T. Šalát, and W. Wilczyński, "On I-convergence," *Real Analysis Exchange*, vol. 26, no. 2, pp. 669–685, 2000-2001.

[2] S. Gähler, "Lineare n-normierte Räume," *Mathematische Nachrichten*, vol. 28, pp. 1–43, 1964.

[3] A. Misiak, "p-inner product spaces," *Mathematische Nachrichten*, vol. 140, pp. 299–319, 1989.

[4] H. Gunawan, "The space of p-summable sequences and its natural n-norm," *Bulletin of the Australian Mathematical Society*, vol. 64, no. 1, pp. 137–147, 2001.

[5] H. Gunawan and M. Mashadi, "On n-normed spaces," *International Journal of Mathematics and Mathematical Sciences*, vol. 27, no. 10, pp. 631–639, 2001.

[6] H. Gunawan and M. Mashadi, "On finite-dimensional 2-normed spaces," *Soochow Journal of Mathematics*, vol. 27, no. 3, pp. 321–329, 2001.

[7] E. Savaş, "Δ^m-strongly summable sequences spaces in 2-normed spaces defined by ideal convergence and an Orlicz function," *Applied Mathematics and Computation*, vol. 217, no. 1, pp. 271–276, 2010.

[8] M. Gürdal, "On ideal convergent sequences in 2-normed spaces," *Thai Journal of Mathematics*, vol. 4, no. 1, pp. 85–91, 2006.

[9] M. Gürdal and A. Şahiner, "New sequence spaces in n-normed spaces with respect to an Orlicz function," *The Aligarh Bulletin of Mathematics*, vol. 27, no. 1, pp. 53–58, 2008.

[10] H. Nakano, "Concave modulars," *Journal of the Mathematical Society of Japan*, vol. 5, pp. 29–49, 1953.

[11] W. H. Ruckle, "FK spaces in which the sequence of coordinate vectors is bounded," *Canadian Journal of Mathematics*, vol. 25, pp. 973–978, 1973.

[12] I. J. Maddox, "Sequence spaces defined by a modulus," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 100, no. 1, pp. 161–166, 1986.

[13] J. Lindenstrauss and L. Tzafriri, "On Orlicz sequence spaces," *Israel Journal of Mathematics*, vol. 10, pp. 379–390, 1971.

[14] M. Güngör and M. Et, "Δ^r-strongly almost summable sequences defined by Orlicz functions," *Indian Journal of Pure and Applied Mathematics*, vol. 34, no. 8, pp. 1141–1151, 2003.

[15] M. Et, Y. Altin, B. Choudhary, and B. C. Tripathy, "On some classes of sequences defined by sequences of Orlicz functions," *Mathematical Inequalities & Applications*, vol. 9, no. 2, pp. 335–342, 2006.

[16] B. C. Tripathy, Y. Altin, and M. Et, "Generalized difference sequence spaces on seminormed space defined by Orlicz functions," *Mathematica Slovaca*, vol. 58, no. 3, pp. 315–324, 2008.

[17] H. Kızmaz, "On certain sequence spaces," *Canadian Mathematical Bulletin*, vol. 24, no. 2, pp. 169–176, 1981.

[18] M. Et and R. Colak, "On some generalized difference sequence spaces," *Soochow Journal of Mathematics*, vol. 21, no. 4, pp. 377–386, 1995.

[19] B. C. Tripathy and A. Esi, "A new type of sequence spaces," *International Journal of Food Science and Technology*, vol. 1, no. 1, pp. 11–14, 2006.

[20] B. C. Tripathy, A. Esi, and B. Tripathy, "On a new type of generalized difference Cesàro sequence spaces," *Soochow Journal of Mathematics*, vol. 31, no. 3, pp. 333–340, 2005.

[21] L. Leindler, "Über die verallgemeinerte de la Vallée-Poussinsche Summierbarkeit allgemeiner Orthogonalreihen," *Acta Mathematica Academiae Scientiarum Hungaricae*, vol. 16, no. 3-4, pp. 375–387, 1965.

[22] N. Şimşek, E. Savaş, and V. Karakaya, "Some geometric and topological properties of a new sequence space defined by de la Vallée-Poussin mean," *Journal of Computational Analysis and Applications*, vol. 12, no. 4, pp. 768–779, 2010.

[23] N. Şimşek, "On some geometric properties of sequence space defined by de la Vallée-Poussin mean," *Journal of Computational Analysis and Applications*, vol. 13, no. 3, pp. 565–573, 2011.

[24] I. J. Maddox, "On Kuttner's theorem," *Journal of the London Mathematical Society*, vol. 43, pp. 285–290, 1968.

[25] E. Savaş, "A-sequence spaces in 2-normed space defined by ideal convergence and an Orlicz function," *Abstract and Applied Analysis*, vol. 2011, Article ID 741382, 9 pages, 2011.

[26] M. Et, "Spaces of Cesàro difference sequences of order r defined by a modulus function in a locally convex space," *Taiwanese Journal of Mathematics*, vol. 10, no. 4, pp. 865–879, 2006.