

## Research Article

# Strong Convergence Theorem for Bregman Strongly Nonexpansive Mappings and Equilibrium Problems in Reflexive Banach Spaces

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By using a new hybrid method, a strong convergence theorem for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of Bregman strongly nonexpansive mappings in a reflexive Banach space is proved.

## 1. Introduction

Throughout this paper, we denote by  $\mathbb{R}$  and  $\mathbb{R}^+$  the set of all real numbers and all nonnegative real numbers, respectively. We also assume that  $E$  is a real reflexive Banach space,  $E^*$  is the dual space of  $E$ ,  $C$  is a nonempty closed convex subset of  $E$ , and  $\langle \cdot, \cdot \rangle$  is the pairing between  $E$  and  $E^*$ . Let  $\Theta$  be a bifunction from  $C \times C \rightarrow \mathbb{R}$ . The equilibrium problem is to find

$$x^* \in C \text{ such that } \Theta(x^*, y) \geq 0, \quad \forall y \in C. \quad (1)$$

The set of such solutions  $x^*$  is denoted by  $EP(\Theta)$ .

Recall that a mapping  $T : C \rightarrow C$  is said to be nonexpansive, if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \quad (2)$$

We denote by  $F(T)$  the set of fixed points of  $T$ .

Numerous problems in physics, optimization, and economics reduce to find a solution of the equilibrium problem. Some methods have been proposed to solve the equilibrium problem in a Hilbert spaces; see, for instance, Blum and Oettli [1], Combettes and Hirstoaga [2], and Moudafi [3]. Recently, Tada and Takahashi [4, 5] and S. Takahashi and W. Takahashi [6] obtained weak and strong convergence theorems for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of

a nonexpansive mapping in a Hilbert space. In particular, Tada and Takahashi [4] established a strong convergence theorem for finding a common element of two sets by using the hybrid method introduced by Nakajo and Takahashi [7]. The authors also proved such a strong convergence theorem in a uniformly convex and uniformly smooth Banach space.

In this paper, motivated by Takahashi et al. [8], we prove a strong convergence theorem for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a Bregman strongly nonexpansive mapping in a real reflexive Banach space by using the shrinking projection method. Using this theorem, we obtain two new strong convergence results for finding a solution of an equilibrium problem and a fixed point of Bregman strongly nonexpansive mappings in a real reflexive Banach space.

## 2. Preliminaries and Lemmas

In the sequel, we begin by recalling some preliminaries and lemmas which will be used in the proof.

Let  $E$  be a real reflexive Banach space with the norm  $\|\cdot\|$  and  $E^*$  the dual space of  $E$ . Throughout this paper,  $f : E \rightarrow (-\infty, +\infty]$  is a proper, lower semicontinuous, and convex function. We denote by  $\text{dom } f$  the domain of  $f$ , that is, the set  $\{x \in E : f(x) < +\infty\}$ .

Let  $x \in \text{int dom } f$ . The subdifferential of  $f$  at  $x$  is the convex set defined by

$$\begin{aligned} \partial f(x) &= \{x^* \in E^* : f(x) + \langle x^*, y - x \rangle \\ &\leq f(y), \forall y \in E\}, \end{aligned} \quad (3)$$

where the Fenchel conjugate of  $f$  is the function  $f^* : E^* \rightarrow (-\infty, +\infty]$  defined by

$$f^*(x^*) = \sup \{\langle x^*, x \rangle - f(x) : x \in E\}. \quad (4)$$

We know that the Young-Fenchel inequality holds:

$$\langle x^*, x \rangle \leq f(x) + f^*(x^*), \quad \forall x \in E, x^* \in E^*. \quad (5)$$

A function  $f$  on  $E$  is coercive [9] if the sublevel set of  $f$  is bounded; equivalently,

$$\lim_{\|x\| \rightarrow +\infty} f(x) = +\infty. \quad (6)$$

A function  $f$  on  $E$  is said to be strongly coercive [10] if

$$\lim_{\|x\| \rightarrow +\infty} \frac{f(x)}{\|x\|} = +\infty. \quad (7)$$

For any  $x \in \text{int dom } f$  and  $y \in E$ , the right-hand derivative of  $f$  at  $x$  in the direction  $y$  is defined by

$$f^\circ(x, y) := \lim_{t \rightarrow 0^+} \frac{f(x + ty) - f(x)}{t}. \quad (8)$$

The function  $f$  is said to be Gâteaux differentiable at  $x$  if  $\lim_{t \rightarrow 0^+} ((f(x + ty) - f(x))/t)$  exists for any  $y$ . In this case,  $f^\circ(x, y)$  coincides with  $\nabla f(x)$ , the value of the gradient  $\nabla f$  of  $f$  at  $x$ . The function  $f$  is said to be Gâteaux differentiable if it is Gâteaux differentiable for any  $x \in \text{int dom } f$ . The function  $f$  is said to be Fréchet differentiable at  $x$  if this limit is attained uniformly in  $\|y\| = 1$ . Finally,  $f$  is said to be uniformly Fréchet differentiable on a subset  $C$  of  $E$  if the limit is attained uniformly for  $x \in C$  and  $\|y\| = 1$ . It is known that if  $f$  is Gâteaux differentiable (resp., Fréchet differentiable) on  $\text{int dom } f$ , then  $f$  is continuous and its Gâteaux derivative  $\nabla f$  is norm-to-weak\* continuous (resp., continuous) on  $\text{int dom } f$  (see also [11, 12]). We will need the following result.

**Lemma 1** (see [13]). *If  $f : E \rightarrow \mathbb{R}$  is uniformly Fréchet differentiable and bounded on bounded subsets of  $E$ , then  $\nabla f$  is uniformly continuous on bounded subsets of  $E$  from the strong topology of  $E$  to the strong topology of  $E^*$ .*

**Definition 2** (see [14]). The function  $f$  is said to be

- (i) essentially smooth, if  $\partial f$  is both locally bounded and single valued on its domain,
- (ii) essentially strictly convex, if  $(\partial f)^{-1}$  is locally bounded on its domain and  $f$  is strictly convex on every convex subset of  $\text{dom } \partial f$ ,
- (iii) Legendre if it is both essentially smooth and essentially strictly convex.

**Remark 3.** Let  $E$  be a reflexive Banach space. Then we have the following.

- (i)  $f$  is essentially smooth if and only if  $f^*$  is essentially strictly convex (see [14, Theorem 5.4]).
- (ii)  $(\partial f)^{-1} = \partial f^*$  (see [12]).
- (iii)  $f$  is Legendre if and only if  $f^*$  is Legendre (see [14, Corollary 5.5]).
- (iv) If  $f$  is Legendre, then  $\nabla f$  is a bijection satisfying  $\nabla f = (\nabla f^*)^{-1}$ ,  $\text{ran } \nabla f = \text{dom } \nabla f^* = \text{int dom } f^*$ , and  $\text{ran } \nabla f^* = \text{dom } \nabla f = \text{int dom } f$  (see [14, Theorem 5.10]).

Examples of Legendre functions were given in [14, 15]. One important and interesting Legendre function is  $(1/p)\|\cdot\|^p$  ( $1 < p < \infty$ ) when  $E$  is a smooth and strictly convex Banach space. In this case, the gradient  $\nabla f$  of  $f$  is coincident with the generalized duality mapping of  $E$ ; that is,  $\nabla f = J_p$  ( $1 < p < \infty$ ). In particular,  $\nabla f = I$  the identity mapping in Hilbert spaces. In the rest of this paper, we always assume that  $f : E \rightarrow (-\infty, +\infty]$  is Legendre.

Let  $f : E \rightarrow (-\infty, +\infty]$  be a convex and Gâteaux differentiable function. The function  $D_f : \text{dom } f \times \text{int dom } f \rightarrow [0, +\infty)$  defined as

$$D_f(y, x) := f(y) - f(x) - \langle \nabla f(x), y - x \rangle \quad (9)$$

is called the Bregman distance with respect to  $f$  [16].

Recall that the Bregman projection [17] of  $x \in \text{int dom } f$  onto the nonempty closed and convex set  $C \subset \text{dom } f$  is the necessarily unique vector  $P_C^f(x) \in C$  satisfying

$$D_f(P_C^f(x), x) = \inf \{D_f(y, x) : y \in C\}. \quad (10)$$

Concerning the Bregman projection, the following are well known.

**Lemma 4** (see [18]). *Let  $C$  be a nonempty, closed, and convex subset of a reflexive Banach space  $E$ . Let  $f : E \rightarrow \mathbb{R}$  be a Gâteaux differentiable and totally convex function and let  $x \in E$ . Then*

- (a)  $z = P_C^f(x)$  if and only if  $\langle \nabla f(x) - \nabla f(z), y - z \rangle \leq 0$ , for all  $y \in C$ .
- (b)

$$\begin{aligned} D_f(y, P_C^f(x)) + D_f(P_C^f(x), x) &\leq D_f(y, x), \\ \forall x \in E, y \in C. \end{aligned} \quad (11)$$

Let  $f : E \rightarrow (-\infty, +\infty]$  be a convex and Gâteaux differentiable function. The modulus of total convexity of  $f$  at  $x \in \text{int dom } f$  is the function  $\nu_f(x, \cdot) : [0, +\infty) \rightarrow [0, +\infty)$  defined by

$$\nu_f(x, t) := \inf \{D_f(y, x) : y \in \text{dom } f, \|y - x\| = t\}. \quad (12)$$

The function  $f$  is called totally convex at  $x$  if  $\nu_f(x, t) > 0$  whenever  $t > 0$ . The function  $f$  is called totally convex if

it is totally convex at any point  $x \in \text{int dom } f$  and is said to be totally convex on bounded sets if  $\nu_f(B, t) > 0$  for any nonempty bounded subset  $B$  of  $E$  and  $t > 0$ , where the modulus of total convexity of the function  $f$  on the set  $B$  is the function  $\nu_f : \text{int dom } f \times [0, +\infty) \rightarrow [0, +\infty]$  defined by

$$\nu_f(B, t) := \inf \{ \nu_f(x, t) : x \in B \cap \text{dom } f \}. \quad (13)$$

The next lemma will be useful in the proof of our main results.

**Lemma 5** (see [19]). *If  $x \in \text{dom } f$ , then the following statements are equivalent.*

- (i) *The function  $f$  is totally convex at  $x$ .*
- (ii) *For any sequence  $\{y_n\} \subset \text{dom } f$ ,*

$$\lim_{n \rightarrow +\infty} D_f(y_n, x) = 0 \implies \lim_{n \rightarrow +\infty} \|y_n - x\| = 0. \quad (14)$$

Recall that the function  $f$  is called sequentially consistent [18] if, for any two sequences  $\{x_n\}$  and  $\{y_n\}$  in  $E$  such that the first one is bounded,

$$\lim_{n \rightarrow +\infty} D_f(y_n, x_n) = 0 \implies \lim_{n \rightarrow +\infty} \|y_n - x_n\| = 0. \quad (15)$$

**Lemma 6** (see [20]). *The function  $f$  is totally convex on bounded sets if and only if the function  $f$  is sequentially consistent.*

**Lemma 7** (see [21]). *Let  $f : E \rightarrow \mathbb{R}$  be a Gâteaux differentiable and totally convex function. If  $x_0 \in E$  and the sequence  $\{D_f(x_n, x_0)\}$  is bounded, then the sequence  $\{x_n\}$  is bounded too.*

**Lemma 8** (see [21]). *Let  $f : E \rightarrow \mathbb{R}$  be a Gâteaux differentiable and totally convex function,  $x_0 \in E$ , and let  $C$  be a nonempty, closed, and convex subset of  $E$ . Suppose that the sequence  $\{x_n\}$  is bounded and any weak subsequential limit of  $\{x_n\}$  belongs to  $C$ . If  $D_f(x_n, x_0) \leq D_f(P_C^f x_0, x_0)$  for any  $n \in \mathbb{N}$ , then  $\{x_n\}$  converges strongly to  $P_C^f x_0$ .*

Let  $C$  be a convex subset of  $\text{int dom } f$  and let  $T$  be a self-mapping of  $C$ . A point  $p \in C$  is called an asymptotic fixed point of  $T$  (see [22, 23]) if  $C$  contains a sequence  $\{x_n\}$  which converges weakly to  $p$  such that  $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$ . We denote by  $\widehat{F}(T)$  the set of asymptotic fixed points of  $T$ .

*Definition 9.* A mapping  $T$  with a nonempty asymptotic fixed point set  $\widehat{F}(T)$  is said to be

- (i) Bregman strongly nonexpansive (see [24, 25]) with respect to  $\widehat{F}(T)$  if

$$D_f(p, Tx) \leq D_f(p, x), \quad \forall x \in C, p \in \widehat{F}(T), \quad (16)$$

and if, whenever  $\{x_n\} \subset C$  is bounded,  $p \in \widehat{F}(T)$  and

$$\lim_{n \rightarrow \infty} (D_f(p, x_n) - D_f(p, Tx_n)) = 0, \quad (17)$$

it follows that

$$\lim_{n \rightarrow \infty} D_f(x_n, Tx_n) = 0. \quad (18)$$

- (ii) Bregman firmly nonexpansive [26] if, for all  $x, y \in C$ ,

$$\begin{aligned} & \langle \nabla f(Tx) - \nabla f(Ty), Tx - Ty \rangle \\ & \leq \langle \nabla f(x) - \nabla f(y), Tx - Ty \rangle \end{aligned} \quad (19)$$

or, equivalently,

$$\begin{aligned} & D_f(Tx, Ty) + D_f(Ty, Tx) + D_f(Tx, x) + D_f(Ty, y) \\ & \leq D_f(Tx, y) + D_f(Ty, x). \end{aligned} \quad (20)$$

The existence and approximation of Bregman firmly nonexpansive mappings were studied in [26]. It is also known that if  $T$  is Bregman firmly nonexpansive and  $f$  is Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of  $E$ , then  $F(T) = \widehat{F}(T)$  and  $F(T)$  is closed and convex (see [26]). It also follows that every Bregman firmly nonexpansive mapping is Bregman strongly nonexpansive with respect to  $F(T) = \widehat{F}(T)$ .

**Lemma 10** (see [27]). *Let  $E$  be a real reflexive Banach space and  $f : E \rightarrow (-\infty, +\infty]$  a proper lower semicontinuous function; then  $f^* : E^* \rightarrow (-\infty, +\infty]$  is a proper weak\* lower semicontinuous and convex function. Thus, for all  $z \in E$ , we have*

$$D_f\left(z, \nabla f^*\left(\sum_{i=1}^N t_i \nabla f(x_i)\right)\right) \leq \sum_{i=1}^N t_i D_f(z, x_i). \quad (21)$$

In order to solve the equilibrium problem, let us assume that a bifunction  $\Theta : C \times C \rightarrow \mathbb{R}$  satisfies the following conditions [28]:

- (A<sub>1</sub>)  $\Theta(x, x) = 0$ , for all  $x \in C$ .
- (A<sub>2</sub>)  $\Theta$  is monotone; that is,  $\Theta(x, y) + \Theta(y, x) \leq 0$ , for all  $x, y \in C$ .
- (A<sub>3</sub>)  $\limsup_{t \downarrow 0} \Theta(x + t(z - x), y) \leq \Theta(x, y)$  for all  $x, z, y \in C$ .
- (A<sub>4</sub>) The function  $y \mapsto \Theta(x, y)$  is convex and lower semicontinuous.

The resolvent of a bifunction  $\Theta$  [29] is the operator  $\text{Res}_\Theta^f : E \rightarrow 2^C$  defined by

$$\begin{aligned} \text{Res}_\Theta^f(x) &= \{z \in C : \Theta(z, y) + \langle \nabla f(z) - \nabla f(x), y - z \rangle \\ & \geq 0, \forall y \in C\}. \end{aligned} \quad (22)$$

From Lemma 1 in [24], if  $f : E \rightarrow (-\infty, +\infty]$  is a strongly coercive and Gâteaux differentiable function and  $\Theta$  satisfies conditions (A<sub>1</sub>-A<sub>4</sub>), then  $\text{dom}(\text{Res}_\Theta^f) = E$ . We also know the following lemma which gives us some characterizations of the resolvent  $\text{Res}_\Theta^f$ .

**Lemma 11** (see [24]). *Let  $E$  be a real reflexive Banach space and  $C$  a nonempty closed convex subset of  $E$ . Let  $f : E \rightarrow (-\infty, +\infty]$  be a Legendre function. If the bifunction  $\Theta : C \times C \rightarrow \mathbb{R}$  satisfies the conditions  $(A_1)$ – $(A_4)$ , then the followings hold:*

- (i)  $\text{Res}_{\Theta}^f$  is single-valued;
- (ii)  $\text{Res}_{\Theta}^f$  is a Bregman firmly nonexpansive operator;
- (iii)  $F(\text{Res}_{\Theta}^f) = EP(\Theta)$ ;
- (iv)  $EP(\Theta)$  is a closed and convex subset of  $C$ ;
- (v) for all  $x \in E$  and for all  $q \in F(\text{Res}_{\Theta}^f)$ , we have

$$D_f(q, \text{Res}_{\Theta}^f(x)) + D_f(\text{Res}_{\Theta}^f(x), x) \leq D_f(q, x). \quad (23)$$

### 3. Strong Convergence Theorem

In this section, we proved a strong convergence theorem for finding a common element of the set of solutions of an equilibrium problem and a fixed point of Bregman strongly nonexpansive mapping in a real reflexive Banach space by using the shrinking projection method.

**Theorem 12.** *Let  $C$  be a nonempty, closed, and convex subset of a real reflexive Banach space  $E$  and  $f : E \rightarrow \mathbb{R}$  a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of  $E$ . Let  $g$  be a bifunction from  $C \times C$  to  $\mathbb{R}$  satisfying  $(A_1)$ – $(A_4)$  and let  $T$  be a Bregman strongly nonexpansive mapping from  $C$  into itself such that  $F(T) = \hat{F}(T)$  and  $G = F(T) \cap EP(g) \neq \emptyset$ . Let  $\{x_n\}$  be a sequence generated by  $x_0 = x \in C$ ,  $C_0 = C$  and*

$$\begin{aligned} y_n &= \nabla f^*(\alpha_n \nabla f(x_n) + (1 - \alpha_n) \nabla f(Tx_n)), \\ u_n &\in C \text{ such that} \\ g(u_n, y) + \langle \nabla f(u_n) - \nabla f(y_n), y - u_n \rangle &\geq 0, \\ \forall y \in C, \end{aligned} \quad (24)$$

$$C_{n+1} = \{z \in C_n : D_f(z, u_n) \leq D_f(z, x_n)\},$$

$$x_{n+1} = P_{C_{n+1}}^f x$$

for every  $n \in \mathbb{N} \cup \{0\}$ , where  $\{\alpha_n\} \subset [0, 1]$  satisfies  $\liminf_{n \rightarrow \infty} (1 - \alpha_n) > 0$ . Then,  $\{x_n\}$  converges strongly to  $P_{F(T) \cap EP(g)}^f x$ , where  $P_{F(T) \cap EP(g)}^f$  is the Bregman projection of  $E$  onto  $F(T) \cap EP(g)$ .

*Proof.* We divide the proof of Theorem 12 into five steps.

(I) We first prove that  $G$  and  $C_n$  both are closed and convex subset of  $C$  for all  $n \geq 0$ . In fact, it follows from Lemma 11 and by Reich and Sabach [26] that  $EP(g)$  and  $F(T)$  both are closed and convex. Therefore,  $G$  is a closed and convex subset in  $C$ . Furthermore, it is obvious that  $C_0 = C$  is closed and convex. Suppose that  $C_n$  is closed and convex

for some  $n \geq 1$ . Since the inequality  $D_f(z, u_n) \leq D_f(z, x_n)$  is equivalent to

$$\langle \nabla f(x_n), z - x_n \rangle - \langle \nabla f(u_n), z - u_n \rangle \leq f(u_n) - f(x_n). \quad (25)$$

Therefore, we have

$$\begin{aligned} C_{n+1} &= \{z \in C_n : \langle \nabla f(x_n), z - x_n \rangle - \langle \nabla f(u_n), z - u_n \rangle \\ &\leq f(u_n) - f(x_n)\}. \end{aligned} \quad (26)$$

This implies that  $C_{n+1}$  is closed and convex. The desired conclusions are proved. These in turn show that  $P_{F(T) \cap EP(g)}^f$  and  $P_{C_n}^f x$  are well defined.

(II) we prove that  $G := F(T) \cap EP(g) \subset C_n$  for all  $n \geq 0$ .

Indeed, it is obvious that  $G = F(T) \cap EP(g) \subset C_0 = C$ . Suppose that  $G \subset C_n$  for some  $n \in \mathbb{N}$ . Let  $u \in G \subset C_n$ ; since  $u_n = \text{Res}_g^f(y_n)$ , by Lemma 11 and (21), we have

$$\begin{aligned} D_f(u, u_n) &= D_f(u, \text{Res}_g^f(y_n)) \leq D_f(u, y_n) \\ &= D_f(u, \nabla f^*(\alpha_n \nabla f(x_n) + (1 - \alpha_n) \nabla f(Tx_n))) \\ &\leq \alpha_n D_f(u, x_n) + (1 - \alpha_n) D_f(u, Tx_n) \\ &\leq \alpha_n D_f(u, x_n) + (1 - \alpha_n) D_f(u, x_n) \\ &= D_f(u, x_n). \end{aligned} \quad (27)$$

Hence, we have  $u \in C_{n+1}$ . This implies that

$$F(T) \cap EP(g) \subset C_n, \quad \forall n \in \mathbb{N} \cup \{0\}. \quad (28)$$

So,  $\{x_n\}$  is well defined.

(III) We prove that  $\{x_n\}$  is a bounded sequence in  $C$ .

By the definition of  $C_n$ , we have  $x_n = P_{C_n}^f x$  for all  $n \geq 0$ . It follows from Lemma 4(b) that

$$\begin{aligned} D_f(x_n, x) &= D_f(P_{C_n}^f x, x) \leq D_f(u, x) - D_f(u, P_{C_n}^f x) \\ &\leq D_f(u, x), \quad \forall n \geq 0, u \in G. \end{aligned} \quad (29)$$

This implies that  $\{D_f(x_n, x)\}$  is bounded. By Lemma 7,  $\{x_n\}$  is bounded. Since  $f : E \rightarrow \mathbb{R}$  is uniformly Fréchet differentiable and bounded on bounded subsets of  $E$ , by Lemma 1  $\nabla f$  is uniformly continuous and bounded on bounded subsets of  $E$ . This implies that  $\{\nabla f(x_n)\}$  is bounded.

(IV) Now we proved that  $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$ .

From  $x_{n+1} \in C_{n+1} \subset C_n$  and  $x_n = P_{C_n}^f x$ , we have

$$D_f(x_n, x) \leq D_f(x_{n+1}, x), \quad \forall n \in \mathbb{N} \cup \{0\}. \quad (30)$$

Thus,  $\{D_f(x_n, x)\}$  is nondecreasing. So, the limit of  $\{D_f(x_n, x)\}$  exists. Since  $D_f(x_{n+1}, x_n) = D_f(x_{n+1}, P_{C_n}^f x) \leq D_f(x_{n+1}, x) - D_f(P_{C_n}^f x, x) = D_f(x_{n+1}, x) - D_f(x_n, x)$  for all  $n \geq 0$ , we

have  $\lim_{n \rightarrow \infty} D_f(x_{n+1}, x_n) = 0$ . From  $x_{n+1} = P_{C_{n+1}}^f x \in C_{n+1}$ , we have

$$D_f(x_{n+1}, u_n) \leq D_f(x_{n+1}, x_n), \quad \forall n \in \mathbb{N} \cup \{0\}. \quad (31)$$

Therefore, we have

$$\lim_{n \rightarrow \infty} D_f(x_{n+1}, u_n) = 0. \quad (32)$$

From Lemma 5, we have

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = \lim_{n \rightarrow \infty} \|x_{n+1} - u_n\| = 0. \quad (33)$$

So, we have

$$\lim_{n \rightarrow \infty} \|x_n - u_n\| = 0. \quad (34)$$

This means that the sequence  $\{u_n\}$  is bounded. Since  $f$  is uniformly Fréchet differentiable, it follows from Lemma 1 that  $\nabla f$  is uniformly continuous. Therefore, we have

$$\lim_{n \rightarrow \infty} \|\nabla f(x_n) - \nabla f(u_n)\| = 0. \quad (35)$$

Since  $f$  is uniformly Fréchet differentiable on bounded subsets of  $E$ , then  $f$  is uniformly continuous on bounded subsets of  $E$  (see [30, Theorem 1.8]). It follows that

$$\lim_{n \rightarrow \infty} |f(x_n) - f(u_n)| = 0. \quad (36)$$

From the definition of the Bregman distance, we obtain that

$$\begin{aligned} D_f(u, x_n) - D_f(u, u_n) &= [f(u) - f(x_n) - \langle \nabla f(x_n), u - x_n \rangle] \\ &\quad - [f(u) - f(u_n) - \langle \nabla f(u_n), u - u_n \rangle] \\ &= (f(u_n) - f(x_n)) + \langle \nabla f(u_n) - \nabla f(x_n), u - u_n \rangle \\ &\quad + \langle \nabla f(x_n), x_n - u_n \rangle \end{aligned} \quad (37)$$

for any  $u \in G$ .

It follows from (34)–(37) that

$$\lim_{n \rightarrow \infty} (D_f(u, x_n) - D_f(u, u_n)) = 0. \quad (38)$$

On the other hand, from  $u_n = \text{Res}_g^f y_n$  and Lemma 11(v), for any  $u \in G$  we have that

$$\begin{aligned} D_f(u_n, y_n) &= D_f(\text{Res}_g^f y_n, y_n) \\ &\leq D_f(u, y_n) - D_f(u, \text{Res}_g^f y_n) \\ &\leq D_f(u, x_n) - D_f(u, \text{Res}_g^f y_n) \\ &= D_f(u, x_n) - D_f(u, u_n). \end{aligned} \quad (39)$$

So, we have from (38) that

$$\lim_{n \rightarrow \infty} D_f(u_n, y_n) = 0. \quad (40)$$

From Lemma 5, we have

$$\lim_{n \rightarrow \infty} \|u_n - y_n\| = 0. \quad (41)$$

So, from (34) and (41), we have

$$\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0. \quad (42)$$

This means that the sequence  $\{y_n\}$  is bounded. Since  $f$  is uniformly Fréchet differentiable, it follows from Lemma 1 that

$$\lim_{n \rightarrow \infty} \|\nabla f(x_n) - \nabla f(y_n)\| = 0. \quad (43)$$

Since  $f$  is uniformly Fréchet differentiable on bounded subsets of  $E$ , then  $f$  is uniformly continuous on bounded subsets of  $E$  (see [30]). It follows that

$$\lim_{n \rightarrow \infty} |f(x_n) - f(y_n)| = 0. \quad (44)$$

From the definition of the Bregman distance, we obtain that

$$\begin{aligned} D_f(u, y_n) - D_f(u, x_n) &= [f(u) - f(y_n) - \langle \nabla f(y_n), u - y_n \rangle] \\ &\quad - [f(u) - f(x_n) - \langle \nabla f(x_n), u - x_n \rangle] \\ &= (f(x_n) - f(y_n)) - \langle \nabla f(y_n) - \nabla f(x_n), u - y_n \rangle \\ &\quad + \langle \nabla f(x_n), y_n - x_n \rangle \end{aligned} \quad (45)$$

for any  $u \in G$ .

It follows from (42) to (45) that

$$\lim_{n \rightarrow \infty} (D_f(u, y_n) - D_f(u, x_n)) = 0. \quad (46)$$

On the other hand, for any  $u \in G$  we have

$$\begin{aligned} D_f(u, y_n) - D_f(u, x_n) &= D_f(u, \nabla f^*(\alpha_n \nabla f(x_n) + (1 - \alpha_n) \nabla f(Tx_n))) \\ &\quad - D_f(u, x_n) \\ &\leq \alpha_n D_f(u, x_n) + (1 - \alpha_n) D_f(u, Tx_n) - D_f(u, x_n) \\ &= (1 - \alpha_n) (D_f(u, Tx_n) - D_f(u, x_n)). \end{aligned} \quad (47)$$

This together with (46), (16), and  $\lim_{n \rightarrow \infty} \alpha_n < 1$  shows that

$$\lim_{k \rightarrow \infty} (D_f(u, Tx_n) - D_f(u, x_n)) = 0. \quad (48)$$

Since  $T$  is Bregman strongly nonexpansive, it follows from (48) that

$$\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0. \quad (49)$$

(V) Next, we prove that every weak subsequential limit of  $\{x_n\}$  belongs to  $G = F(T) \cap \text{EP}(g)$ .

Since  $\{x_n\}$  is bounded, there exists a subsequence  $\{x_{n_k}\}$  of  $\{x_n\}$  such that  $x_{n_k} \rightarrow x^*$ . Since  $T$  is a Bregman strongly nonexpansive mapping with  $F(T) = \widehat{F}(T)$ , we have  $x^* \in F(T)$ .

From  $x_{n_k} \rightarrow x^*$  and (34), we have  $u_{n_k} \rightarrow x^*$ .

By  $u_n = \text{Res}_g^f y_n$ , we have

$$g(u_n, y) + \langle \nabla f(u_n) - \nabla f(y_n), y - u_n \rangle \geq 0, \quad \forall y \in C. \tag{50}$$

Replacing  $n$  by  $n_k$ , we have from  $(A_2)$  that

$$\langle \nabla f(u_{n_k}) - \nabla f(y_{n_k}), y - u_{n_k} \rangle \geq -g(u_{n_k}, y) \geq g(y, u_{n_k}), \quad \forall y \in C. \tag{51}$$

Since  $g(x, \cdot)$  is convex and lower semicontinuous, it is also weakly lower semicontinuous. So, letting  $k \rightarrow \infty$ , we have from (35), (43), and  $(A_4)$  that

$$g(y, x^*) \leq 0, \quad \forall y \in C. \tag{52}$$

For  $t \in (0, 1]$  and  $y \in C$ , letting  $y_t = ty + (1-t)x^*$ , there are  $y_t \in C$  and  $g(y_t, x^*) \leq 0$ . By condition  $(A_1)$  and  $(A_4)$ , we have

$$0 = g(y_t, y_t) \leq tg(y_t, y) + (1-t)g(y_t, x^*) \leq tg(y_t, y). \tag{53}$$

Dividing both sides of the above equation by  $t$ , we have  $g(y_t, y) \geq 0$ , for all  $y \in C$ . Letting  $t \downarrow 0$ , from condition  $(A_3)$ , we have

$$g(x^*, y) \geq 0, \quad \forall y \in C. \tag{54}$$

Therefore,  $x^* \in \text{EP}(g)$ .

(VI) Now, we prove  $x_n \rightarrow P_{F(T) \cap \text{EP}(g)}^f x$ .

Let  $w = P_{F(T) \cap \text{EP}(g)}^f x$ . From  $w \in F(T) \cap \text{EP}(g) \subset C_{n+1}$ , we have  $D_f(x_{n+1}, x) \leq D_f(w, x)$ . Therefore, Lemma 8 implies that  $\{x_n\}$  converges strongly to  $w = P_{F(T) \cap \text{EP}(g)}^f x$ , as claimed. This completes the proof of Theorem 12.  $\square$

**Corollary 13.** *Let  $C$  be a nonempty, closed, and convex subset of a real reflexive Banach space  $E$  and  $f : E \rightarrow \mathbb{R}$  a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of  $E$ . Let  $g$  be a bifunction from  $C \times C$  to  $\mathbb{R}$  satisfying  $(A_1)$ – $(A_4)$ . Let  $\{x_n\}$  be a sequence generated by  $x_0 = x \in C, C_0 = C$ , and*

$$u_n \in C \text{ such that } g(u_n, y) + \langle \nabla f(u_n) - \nabla f(x_n), y - u_n \rangle \geq 0, \quad \forall y \in C, \tag{55}$$

$$C_{n+1} = \{z \in C_n : D_f(z, u_n) \leq D_f(z, x_n)\},$$

$$x_{n+1} = P_{C_{n+1}}^f x$$

for every  $n \in \mathbb{N} \cup \{0\}$ . Then,  $\{x_n\}$  converges strongly to  $P_{\text{EP}(g)}^f x$ , where  $P_{\text{EP}(g)}^f$  is the Bregman projection of  $E$  onto  $\text{EP}(g)$ .

*Proof.* Putting  $T = I$  in Theorem 12, we obtain Corollary 13.  $\square$

**Corollary 14.** *Let  $C$  be a nonempty, closed, and convex subset of a real reflexive Banach space  $E$  and  $f : E \rightarrow \mathbb{R}$  a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of  $E$ . Let  $T$  be a Bregman strongly nonexpansive mapping from  $C$  into itself such that  $F(T) = \widehat{F}(T)$  and  $G = F(T) \cap \text{EP}(g) \neq \emptyset$ . Let  $\{x_n\}$  be a sequence generated by  $x_0 = x \in C, C_0 = C$ , and*

$$u_n = P_C^f \nabla f^* (\alpha_n \nabla f(x_n) + (1 - \alpha_n) \nabla f(Tx_n)),$$

$$C_{n+1} = \{z \in C_n : D_f(z, u_n) \leq D_f(z, x_n)\}, \tag{56}$$

$$x_{n+1} = P_{C_{n+1}}^f x$$

for every  $n \in \mathbb{N} \cup \{0\}$ , where  $\{\alpha_n\} \subset [0, 1]$  satisfies  $\liminf_{n \rightarrow \infty} (1 - \alpha_n) > 0$ . Then,  $\{x_n\}$  converges strongly to  $P_{F(T)}^f x$ , where  $P_{F(T)}^f$  is the Bregman projection of  $E$  onto  $F(T)$ .

*Proof.* Putting  $g(x, y) = 0$  for all  $x, y \in C$  in Theorem 12, we obtain Corollary 14.  $\square$

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