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Research Article

Oscillation Criteria for Some New Generalized Emden-Fowler Dynamic Equations on Time Scales

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By means of novel analytical techniques, we have established several new oscillation criteria for the generalized Emden-Fowler dynamic equation on a time scale $\mathbb{T}$, that is, $(r(t)|Z^{\Delta}(t)|^{\alpha-1}Z^{\Delta}(t))^{\Delta} + f(t, x(\delta(t))) = 0$, with respect to the case $\int_{t_0}^{\infty} r^{-1/\alpha}(s)\Delta s = \infty$ and the case $\int_{t_0}^{\infty} r^{-1/\alpha}(s)\Delta s < \infty$, where $Z(t) = x(t) + p(t)x(\tau(t))$, α is a constant, $|f(t,u)| \geqslant q(t)|u^{\beta}|$, β is a constant satisfying $\alpha \geqslant \beta > 0$, and r, p, and q are real valued right-dense continuous nonnegative functions defined on $\mathbb{T}$. Noting the parameter value α probably unequal to β, our equation factually includes the existing models as special cases; our results are more general and have wider adaptive range than others' work in the literature.

1. Introduction

In the past two decades, the theory of time scales proposed by Hilger [1] in 1990 has received extensive attention because of its advantage to unify continuous model and discrete model into one case under the scholars' investigation. Numerous authors have considered many aspects of this new theory. Many of those results focus on oscillation and nonoscillation of some equations on time scales. Reader can refer to articles [2–25] and there references cited therein.

In this paper, we consider the oscillatory behavior of the solutions of second-order generalized Emden-Fowler dynamic equation of the form

$$\left(r(t)\left|Z^{\Delta}(t)\right|^{\alpha-1}Z^{\Delta}(t)\right)^{\Delta}+f(t,x(\delta(t)))=0, \quad t\in\mathbb{T},\ t\geqslant t_0, \tag{1}$$

with $Z(t) = x(t) + p(t)x(\tau(t))$, parameter constant α, and conditions (H_1)–(H_6):

(H_1) $\mathbb{T}$ is a time scale which is unbounded above. $[t_0,\infty)_{\mathbb{T}} := [t_0,\infty) \cap \mathbb{T}$, where $t_0 \in \mathbb{T}$ with $t_0 > 0$, $C_{rd}(\mathbb{T},\mathbb{S})$ denotes the collection of all functions $f : \mathbb{T} \to \mathbb{S}$ which are right-dense continuous on $\mathbb{T}$;

(H_2) $r(t) \in C_{rd}(\mathbb{T},(0,\infty))$, $R(t) := \int_{t_0}^{t} r^{-1/\alpha}(s)\Delta s$;

(H_3) $p(t) \in C_{rd}(\mathbb{T},[0,1])$;

(H_4) $\tau(t) \in C_{rd}(\mathbb{T},\mathbb{T})$, $\tau(t) \leqslant t$, for $t \in \mathbb{T}$, $\lim_{t\to\infty}\tau(t) = \infty$, $\delta(t) \in C_{rd}(\mathbb{T},\mathbb{T})$, $\delta(t) \leqslant t$, for $t \in \mathbb{T}$, $\lim_{t\to\infty}\delta(t) = \infty$;

(H_5) $\delta^{\Delta}(t) > 0$ is right-dense continuous on $\mathbb{T}$, and $\delta(\sigma(t)) = \sigma(\delta(t))$ for all $t \in \mathbb{T}$, where $\sigma(t)$ is the forward jump operator on $\mathbb{T}$;

(H_6) $f(t,u) \in C(\mathbb{T}\times\mathbb{R},\mathbb{R})$ is a continuous function such that $uf(t,u) > 0$, for all $u \neq 0$ and there exists a positive right-dense continuous function $q(t)$ defined on $\mathbb{T}$ such that $|f(t,u)| \geqslant q(t)|u^{\beta}|$ for all $t \in \mathbb{T}$ and for all $u \in \mathbb{R}$, where β is a constant satisfying $\alpha \geqslant \beta > 0$.

As a solution of (1), we mean a function $x(t)$ such that $x(t) + p(t)x(\tau(t)) \in C^1_{rd}(t_x,\infty)_{\mathbb{T}}$ and $r(t)|[x(t) + p(t)x(\tau(t))]^{\Delta}|^{\alpha-1}[x(t) + p(t)x(\tau(t))]^{\Delta} \in C^1_{rd}(t_x,\infty)_{\mathbb{T}}$, $t_x \geqslant t_0$ and satisfying (1) for all $t \geqslant t_x$, where $C^1_{rd}(t_x,\infty)_{\mathbb{T}}$ denotes the set of right-dense continuously Δ-differentiable functions on $(t_x,\infty)_{\mathbb{T}}$. In the sequel, we restrict our attention to those solutions of (1) which exist on the half-line $[t_x,\infty)_{\mathbb{T}}$ and satisfy $\sup\{|x(t)| : t > \widetilde{T}\} > 0$ for any $\widetilde{T} \geqslant t_x$. We say that a nontrivial solution of (1) is oscillatory if it has arbitrary large zeros, otherwise we say that it is nonoscillatory. We say that (1) is oscillatory if all its solutions are oscillatory.

Among researchers in the oscillation of functional equations with time scales, Agarwal et al. [2] studied a special case of (1), which is

$$\left(r(t)\left(\left[y(t)+p(t)y(t-\tau_0)\right]^{\Delta}\right)^{\gamma}\right)^{\Delta} + f\left(t, y\left(t-\delta_0\right)\right)=0, \quad t\in\mathbb{T},\ t\geqslant t_0, \tag{2}$$

where

$$\left|f(t,u)\right| \geqslant q(t)|u|^{\gamma}, \qquad \int_{t_0}^{\infty} r^{-1/\gamma}(s)\,\Delta s=\infty, \tag{3}$$

τ_0 and δ_0 are positive constants and $\gamma>0$ is a quotient of odd positive integers. They got some oscillation criteria of (2) for the case when $\gamma>0$ under the condition $r^{\Delta}(t)\geqslant 0$, and the case when $\gamma\geqslant 1$ under the condition $\mu(t)>0$. Subsequently, for the case when $\gamma\geqslant 1$ is an odd positive integer, Saker [7] did not require the conditions $r^{\Delta}(t)\geqslant 0$ and $\mu(t)>0$ and obtained some new oscillation results for (2) under the conditions (3).

Very Recently, in [10–13], Saker et al. have considered the oscillation of several equations with time scales. For example in paper [13], the author is concerned with the quasilinear equation of the form:

$$\left(p(t)\left(\left[y(t)+r(t)y(\tau(t))\right]^{\Delta}\right)^{\gamma}\right)^{\Delta} + f\left(t, y(\delta(t))\right)=0, \tag{4}$$

where $|f(t,u)|\geqslant q(t)|u^{\beta}|$, $\gamma>0$, and $\beta>0$ are ratios of odd positive integers.

However the value range of the equation parameters in our work is wider than those in [2, 7, 10–13] and the equation itself is also different from those in [2, 7, 10–13]. In fact, our approach in constructing the criteria is different from those of Saker and his coauthors' work.

For (2) with $\gamma\geqslant 1$ being a quotient of odd positive integers and without the restrictive conditions $r^{\Delta}(t)\geqslant 0$ and without $\mu(t)>0$, Wu et al. [21] obtained several oscillation criteria for the equation:

$$\left(r(t)\left(\left[y(t)+p(t)y(\tau(t))\right]^{\Delta}\right)^{\gamma}\right)^{\Delta} + f\left(t, y(\delta(t))\right)=0, \quad t\in\mathbb{T},\ t\geqslant t_0, \tag{5}$$

under the conditions (3).

Chen [25] investigated the following second-order Emden-Fowler neutral delay dynamic equation

$$\left(r(t)\left|x^{\Delta}(t)\right|^{\gamma-1}x^{\Delta}(t)\right)^{\Delta} + f\left(t, y(\delta(t))\right)=0, \quad t\in\mathbb{T},\ t\geqslant t_0, \tag{6}$$

with $x(t)=y(t)+p(t)y(\tau(t))$, under the conditions (3). He obtained some oscillation criteria when $\gamma>0$ is a constant and without assuming the conditions $r^{\Delta}(t)\geqslant 0$ and $\mu(t)>0$.

All the above results cannot apply to our model (1) since our model (1) is more general than (2), (6) and those in [10–13], and the function $f(t,u)$ in (1) satisfies (H_6) which makes our model (1) distinguished from all the existing cases. To the best of our knowledge, nothing is known regarding the necessary and sufficient conditions for the qualitative behavior of (1) with $\alpha\neq\beta$ in (H_6) on time scales.

In this paper, even if $\alpha\neq\beta$ in (H_6) and there is no assumptions $r^{\Delta}(t)\geqslant 0$ and $\mu(t)>0$, we have established several new oscillation criteria of (1) for the both cases

$$\lim_{t\to\infty}\int_{t_0}^{t} r^{-1/\alpha}(s)\,\Delta s=\infty, \tag{7}$$

$$\lim_{t\to\infty}\int_{t_0}^{t} r^{-1/\alpha}(s)\,\Delta s<\infty. \tag{8}$$

Factually, we have employed new analytical techniques to present and construct our criteria in Section 3 after reciting two useful lemmas in Section 2. Our results have extended and unified a number of other existing results and handled the cases which are not covered by current criteria. Finally, in Section 4 two examples are demonstrated to illustrate the efficiency of our work with relevant remark.

2. Some Lemmas

Lemma 1 (see [25]). *Suppose that (H_5) holds. Let $x:\mathbb{T}\to\mathbb{R}$. If x^{Δ} exists for all sufficiently large $t\in\mathbb{T}$, then $(x(\delta(t)))^{\Delta}=x^{\Delta}(\delta(t))\delta^{\Delta}(t)$ for all sufficiently large $t\in\mathbb{T}$.*

Lemma 2 (Bohner and Peterson [26, Theorem 1.90]). *Assume that $x(t)$ is Δ-differentiable and eventually positive or eventually negative, then*

$$\left(x^{\alpha}(t)\right)^{\Delta}=\alpha\left\{\int_0^1\left[(1-h)x(t)+hx(\sigma(t))\right]^{\alpha-1}dh\right\}x^{\Delta}(t). \tag{9}$$

Lemma 3 (see [27]). *Let $\Psi(u)=au-bu^{(\lambda+1)/\lambda}$, where a,b,λ are constants, $a\geqslant 0$, $b>0$, $\lambda>0$, and $u\in[0,\infty)$. Then $\Psi(u)$ attains its maximum value on $[0,\infty)$ at $u=u^*:=(a\lambda/b(\lambda+1))^{\lambda}$, and*

$$\max_{u\in[0,\infty)}\Psi(u)=\Psi\left(u^*\right)=\frac{\lambda^{\lambda}}{(\lambda+1)^{\lambda+1}}\frac{a^{\lambda+1}}{b^{\lambda}}. \tag{10}$$

3. Main Results

The case

$$\lim_{t\to\infty}\int_{t_0}^{t} r^{-1/\alpha}(s)\,\Delta s=\infty. \tag{11}$$

Theorem 4. *Assume that* (H_1)–(H_6) *and* (7) *hold. If there exists a function* $\xi(t) \in C^1_{rd}(\mathbb{T}, (0,\infty))$ *such that for any positive number* M,

$$\overline{\lim_{t\to\infty}} \int_{t_0}^{t} \left(\xi(s)\,\overline{p}(s) - Q(s)\right) \Delta s = \infty, \tag{12}$$

where

$$\begin{gathered} \overline{p}(s) = q(s)\left[1 - p(\delta(s))\right]^{\beta}, \\ Q(s) = \frac{\alpha^{\alpha} M (R(\sigma(s)))^{\alpha-\beta} r(\delta(s)) \left(\left(\xi^{\Delta}(s)\right)_{+}\right)^{\alpha+1}}{(\alpha+1)^{\alpha+1} \beta^{\alpha} \xi^{\alpha}(s) \left(\delta^{\Delta}(s)\right)^{\alpha}}, \\ \left(\xi^{\Delta}(s)\right)_{+} := \max\left\{\xi^{\Delta}(s), 0\right\}, \end{gathered} \tag{13}$$

then (1) *is oscillatory.*

Proof. Suppose that (1) has a nonoscillatory solution $x(t)$, then there exists $T_0 \geqslant t_0$ such that $x(t) \neq 0$ for all $t \geqslant T_0$. Without loss of generality, we assume that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\delta(t)) > 0$ for $t \geqslant T_0$, because a similar analysis holds for $x(t) < 0$, $x(\tau(t)) < 0$ and $x(\delta(t)) < 0$. Then the following are deduced from (1), (H_3), and (H_6):

$$\begin{gathered} Z(t) \geqslant x(t) > 0 \quad \text{for } t \geqslant T_0, \\ \left(r(t)\left|Z^{\Delta}(t)\right|^{\alpha-1} Z^{\Delta}(t)\right)^{\Delta} \leqslant 0, \quad t \geqslant T_0. \end{gathered} \tag{14}$$

Therefore $r(t)|Z^{\Delta}(t)|^{\alpha-1}Z^{\Delta}(t)$ is a nonincreasing function and $Z^{\Delta}(t)$ is eventually of one sign.

We claim that

$$Z^{\Delta}(t) > 0 \quad \text{or} \quad Z^{\Delta}(t) = 0, \quad t \geqslant T_0. \tag{15}$$

Otherwise, if there exists a $t_1 \geqslant T_0$ such that $Z^{\Delta}(t) < 0$ for $t \geqslant t_1$, then from (14), for some positive constant K, we have

$$-r(t)\left(-Z^{\Delta}(t)\right)^{\alpha} \leqslant -K, \quad t \geqslant t_1, \tag{16}$$

that is,

$$-Z^{\Delta}(t) \geqslant \left(\frac{K}{r(t)}\right)^{1/\alpha}, \quad t \geqslant t_1, \tag{17}$$

integrating the above inequality from t_1 to t, we have

$$Z(t) \leqslant Z(t_1) - K^{1/\alpha}\left(R(t) - R(t_1)\right). \tag{18}$$

Letting $t \to \infty$, from (7), we get $\lim_{t\to\infty} Z(t) = -\infty$, which contradicts (14). Thus, we have proved (15).

We choose some $T_1 \geqslant T_0$ such that $\delta(t) \geqslant T_0$ for $t \geqslant T_1$. Therefore from (14), (15), and the fact $\delta(t) \leqslant \sigma(t)$, we have that

$$r(\sigma(t))\left(Z^{\Delta}(\sigma(t))\right)^{\alpha} \leqslant r(\delta(t))\left(Z^{\Delta}(\delta(t))\right)^{\alpha}, \quad t \geqslant T_1, \tag{19}$$

which follows that

$$Z^{\Delta}(\delta(t)) \geqslant Z^{\Delta}(\sigma(t))\left(\frac{r(\sigma(t))}{r(\delta(t))}\right)^{1/\alpha}, \quad t \geqslant T_1. \tag{20}$$

On the other hand, from (1), (H_6), and (15), we have

$$\begin{aligned} &\left(r(t)\left(Z^{\Delta}(t)\right)^{\alpha}\right)^{\Delta} + q(t)\left(Z(\delta(t)) - p(\delta(t))\,x(\tau(\delta(t)))\right)^{\beta} \\ &\quad \leqslant 0, \quad t \geqslant T_1. \end{aligned} \tag{21}$$

Noticing (15) and the fact $Z(t) \geqslant x(t)$, we get

$$\left(r(t)\left(Z^{\Delta}(t)\right)^{\alpha}\right)^{\Delta} + \overline{p}(t)\, Z^{\beta}(\delta(t)) \leqslant 0, \quad t \geqslant T_1, \tag{22}$$

where $\overline{p}(t) = q(t)[1 - p(\delta(t))]^{\beta}$.

Define

$$w(t) = \xi(t)\,\frac{r(t)\left(Z^{\Delta}(t)\right)^{\alpha}}{Z^{\beta}(\delta(t))}, \quad \text{for } t \geqslant T_1. \tag{23}$$

Obviously, $w(t) > 0$. By (22), (23) and the product rule and the quotient rule, we obtain

$$\begin{aligned} w^{\Delta}(t) = {} & \frac{\xi(t)}{Z^{\beta}(\delta(t))}\left(r(t)\left(Z^{\Delta}(t)\right)^{\alpha}\right)^{\Delta} + r(\sigma(t))\left(Z^{\Delta}(\sigma(t))\right)^{\alpha} \\ & \times \frac{\xi^{\Delta}(t)\, Z^{\beta}(\delta(t)) - \xi(t)\left(Z^{\beta}(\delta(t))\right)^{\Delta}}{Z^{\beta}(\delta(t))\, Z^{\beta}(\delta(\sigma(t)))} \\ \leqslant {} & -\xi(t)\,\overline{p}(t) + \frac{\xi^{\Delta}(t)}{\xi(\sigma(t))} w(\sigma(t)) \\ & - \frac{r(\sigma(t))\left(Z^{\Delta}(\sigma(t))\right)^{\alpha} \xi(t)\left(Z^{\beta}(\delta(t))\right)^{\Delta}}{Z^{\beta}(\delta(t))\, Z^{\beta}(\delta(\sigma(t)))}. \end{aligned} \tag{24}$$

Now we consider the following two cases.

Case 1. Let $\beta \geqslant 1$. By (15), Lemmas 1 and 2, we have

$$\begin{aligned} &\left(Z^{\beta}(\delta(t))\right)^{\Delta} \\ &\quad = \beta\left\{\int_0^1 \left[(1-h)\,Z(\delta(t)) + hZ(\delta(\sigma(t)))\right]^{\beta-1} \mathrm{d}h\right\} \\ &\qquad \times \left(Z(\delta(t))\right)^{\Delta} \\ &\quad \geqslant \beta\left(Z(\delta(t))\right)^{\beta-1} Z^{\Delta}(\delta(t))\,\delta^{\Delta}(t). \end{aligned} \tag{25}$$

From (H_5), (20), (23)–(25), and the fact that $Z(t)$ is nondecreasing, we obtain

$$\begin{aligned}
&w^{\Delta}(t)\\
&\leqslant -\xi(t)\overline{p}(t)+\frac{\xi^{\Delta}(t)}{\xi(\sigma(t))}w(\sigma(t))\\
&\quad-\frac{r(\sigma(t))\left(Z^{\Delta}(\sigma(t))\right)^{\alpha}\xi(t)\beta(Z(\delta(t)))^{\beta-1}Z^{\Delta}(\delta(t))\delta^{\Delta}(t)}{Z^{\beta}(\delta(t))Z^{\beta}(\delta(\sigma(t)))}\\
&\leqslant -\xi(t)\overline{p}(t)+\frac{\xi^{\Delta}(t)}{\xi(\sigma(t))}w(\sigma(t))\\
&\quad-\frac{r(\sigma(t))\left(Z^{\Delta}(\sigma(t))\right)^{\alpha}\xi(t)\beta Z^{\Delta}(\delta(t))\delta^{\Delta}(t)}{Z^{\beta+1}(\delta(\sigma(t)))}\\
&\leqslant -\xi(t)\overline{p}(t)+\frac{\xi^{\Delta}(t)}{\xi(\sigma(t))}w(\sigma(t))\\
&\quad-\frac{\beta\xi(t)r(\sigma(t))\left(Z^{\Delta}(\sigma(t))\right)^{\alpha+1}\delta^{\Delta}(t)}{Z^{\beta+1}(\delta(\sigma(t)))}\\
&\quad\times\left(\frac{r(\sigma(t))}{r(\delta(t))}\right)^{1/\alpha}\\
&=-\xi(t)\overline{p}(t)+\frac{\xi^{\Delta}(t)}{\xi(\sigma(t))}w(\sigma(t))\\
&\quad-\frac{\beta\xi(t)\delta^{\Delta}(t)}{(\xi(\sigma(t)))^{1+1/\alpha}(Z(\delta(\sigma(t))))^{(\alpha-\beta)/\alpha}(r(\delta(t)))^{1/\alpha}}\\
&\quad\times w^{(\alpha+1)/\alpha}(\sigma(t))\\
&=-\xi(t)\overline{p}(t)+\frac{\xi^{\Delta}(t)}{\xi(\sigma(t))}w(\sigma(t))\\
&\quad-\frac{\beta\xi(t)\delta^{\Delta}(t)}{(\xi(\sigma(t)))^{1+1/\alpha}(Z(\sigma(t)))^{(\alpha-\beta)/\alpha}(r(\delta(t)))^{1/\alpha}}\\
&\quad\times w^{(\alpha+1)/\alpha}(\sigma(t)).
\end{aligned}\tag{26}$$

Case 2. Let $0<\beta<1$. By (15), Lemmas 1 and 2, we get

$$\begin{aligned}
&\left(Z^{\beta}(\delta(t))\right)^{\Delta}\\
&\quad=\beta\left\{\int_0^1\left[(1-h)Z(\delta(t))+hZ(\delta(\sigma(t)))\right]^{\beta-1}\mathrm{d}h\right\}\\
&\qquad\times(Z(\delta(t)))^{\Delta}\\
&\quad\geqslant\beta(Z(\delta(\sigma(t))))^{\beta-1}Z^{\Delta}(\delta(t))\delta^{\Delta}(t).
\end{aligned}\tag{27}$$

From (H_4), (H_5), (20), (23)–(25), and the fact that $Z(t)$ is nondecreasing, we have

$$\begin{aligned}
&w^{\Delta}(t)\\
&\leqslant -\xi(t)\overline{p}(t)+\frac{\xi^{\Delta}(t)}{\xi(\sigma(t))}w(\sigma(t))\\
&\quad-\frac{r(\sigma(t))\left(Z^{\Delta}(\sigma(t))\right)^{\alpha}\xi(t)\beta(Z(\delta(\sigma(t))))^{\beta-1}Z^{\Delta}(\delta(t))\delta^{\Delta}(t)}{Z^{\beta}(\delta(t))Z^{\beta}(\delta(\sigma(t)))}\\
&=-\xi(t)\overline{p}(t)+\frac{\xi^{\Delta}(t)}{\xi(\sigma(t))}w(\sigma(t))\\
&\quad-\frac{r(\sigma(t))\left(Z^{\Delta}(\sigma(t))\right)^{\alpha}\xi(t)\beta Z^{\Delta}(\delta(t))\delta^{\Delta}(t)}{Z^{\beta+1}(\delta(\sigma(t)))}\\
&\leqslant -\xi(t)\overline{p}(t)+\frac{\xi^{\Delta}(t)}{\xi(\sigma(t))}w(\sigma(t))\\
&\quad-\frac{\beta\xi(t)r(\sigma(t))\left(Z^{\Delta}(\sigma(t))\right)^{\alpha+1}\delta^{\Delta}(t)}{Z^{\beta+1}(\delta(\sigma(t)))}\left(\frac{r(\sigma(t))}{r(\delta(t))}\right)^{1/\alpha}\\
&=-\xi(t)\overline{p}(t)+\frac{\xi^{\Delta}(t)}{\xi(\sigma(t))}w(\sigma(t))\\
&\quad-\frac{\beta\xi(t)\delta^{\Delta}(t)}{(\xi(\sigma(t)))^{1+1/\alpha}(Z(\delta(\sigma(t))))^{(\alpha-\beta)/\alpha}(r(\delta(t)))^{1/\alpha}}\\
&\quad\times w^{(\alpha+1)/\alpha}(\sigma(t))\\
&=-\xi(t)\overline{p}(t)+\frac{\xi^{\Delta}(t)}{\xi(\sigma(t))}w(\sigma(t))\\
&\quad-\frac{\beta\xi(t)\delta^{\Delta}(t)}{(\xi(\sigma(t)))^{1+1/\alpha}(Z(\sigma(t)))^{(\alpha-\beta)/\alpha}(r(\delta(t)))^{1/\alpha}}\\
&\quad\times w^{(\alpha+1)/\alpha}(\sigma(t)).
\end{aligned}\tag{28}$$

Therefore, for $\beta>0$, from (26) and (28), we get

$$\begin{aligned}
w^{\Delta}(t)\leqslant&-\xi(t)\overline{p}(t)+\frac{\xi^{\Delta}(t)}{\xi(\sigma(t))}w(\sigma(t))\\
&-\frac{\beta\xi(t)\delta^{\Delta}(t)}{(\xi(\sigma(t)))^{1+1/\alpha}(Z(\sigma(t)))^{(\alpha-\beta)/\alpha}(r(\delta(t)))^{1/\alpha}}\\
&\times w^{(\alpha+1)/\alpha}(\sigma(t)).
\end{aligned}\tag{29}$$

From (14) and (15), there exists a constant $M_1>0$ such that

$$r(t)\left(Z^{\Delta}(t)\right)^{\alpha}\leqslant M_1,\quad t\geqslant T_1,\tag{30}$$

that is

$$Z^{\Delta}(t)\leqslant\left(\frac{M_1}{r(t)}\right)^{1/\alpha},\quad t\geqslant T_1,\tag{31}$$

integrating the above inequality from T_1 to t, we have

$$Z(t) \leqslant Z(T_1) + M_1^{1/\alpha}(R(t) - R(T_1)). \tag{32}$$

Thus, there exist a constant $M_2 > 0$, and $T_2 \geqslant T_1$ such that

$$Z(t) \leqslant M_2 R(t), \quad t \geqslant T_2, \tag{33}$$

so we have

$$\begin{aligned} Z^{(\alpha-\beta)/\alpha}(\sigma(t)) &\leqslant M_2^{(\alpha-\beta)/\alpha}(R(\sigma(t)))^{(\alpha-\beta)/\alpha} \\ &= M_3(R(\sigma(t)))^{(\alpha-\beta)/\alpha}, \quad t \geqslant T_2, \end{aligned} \tag{34}$$

where $M_3 = M_2^{(\alpha-\beta)/\alpha}$.

From (29) and (34), we obtain

$$\begin{aligned} w^{\Delta}(t) \leqslant & -\xi(t)\overline{p}(t) + \frac{\xi^{\Delta}(t)}{\xi(\sigma(t))} w(\sigma(t)) \\ & - \frac{\beta\xi(t)\delta^{\Delta}(t)}{(\xi(\sigma(t)))^{1+1/\alpha} M_3(R(\sigma(t)))^{(\alpha-\beta)/\alpha}(r(\delta(t)))^{1/\alpha}} \\ & \times w^{(\alpha+1)/\alpha}(\sigma(t)), \quad t \geqslant T_2. \end{aligned} \tag{35}$$

Let

$$\Psi(t) = \frac{\beta\xi(t)\delta^{\Delta}(t)}{(\xi(\sigma(t)))^{1+1/\alpha} M_3(R(\sigma(t)))^{(\alpha-\beta)/\alpha}(r(\delta(t)))^{1/\alpha}}; \tag{36}$$

then $\Psi(t) > 0$. So from (35) and (36) we get

$$\begin{aligned} w^{\Delta}(t) \leqslant & -\xi(t)\overline{p}(t) + \frac{\xi^{\Delta}(t)}{\xi(\sigma(t))} w(\sigma(t)) \\ & - \Psi(t) w^{(\alpha+1)/\alpha}(\sigma(t)) \\ \leqslant & -\xi(t)\overline{p}(t) + \frac{(\xi^{\Delta}(t))_+}{\xi(\sigma(t))} w(\sigma(t)) \\ & - \Psi(t) w^{(\alpha+1)/\alpha}(\sigma(t)), \end{aligned} \tag{37}$$

where $(\xi^{\Delta}(t))_+ := \max\{\xi^{\Delta}(t), 0\}$.

Taking $a = (\xi^{\Delta}(t))_+/\xi(\sigma(t))$, $b = \Psi(t)$, by Lemma 3 and (37), we obtain

$$\begin{aligned} w^{\Delta}(t) \leqslant & -\xi(t)\overline{p}(t) + \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}\Psi^{\alpha}(t)} \left(\frac{(\xi^{\Delta}(t))_+}{\xi(\sigma(t))}\right)^{\alpha+1} \\ = & -\left[\xi(t)\overline{p}(t) - \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}\Psi^{\alpha}(t)} \left(\frac{(\xi^{\Delta}(t))_+}{\xi(\sigma(t))}\right)^{\alpha+1}\right] \\ = & -\left[\xi(t)\overline{p}(t) - \frac{\alpha^{\alpha} M_3^{\alpha}(R(\sigma(t)))^{\alpha-\beta} r(\delta(t)) \left((\xi^{\Delta}(t))_+\right)^{\alpha+1}}{(\alpha+1)^{\alpha+1}\beta^{\alpha}\xi^{\alpha}(t)(\delta^{\Delta}(t))^{\alpha}}\right] \\ = & -\left[\xi(t)\overline{p}(t) - \frac{\alpha^{\alpha} M_4(R(\sigma(t)))^{\alpha-\beta} r(\delta(t)) \left((\xi^{\Delta}(t))_+\right)^{\alpha+1}}{(\alpha+1)^{\alpha+1}\beta^{\alpha}\xi^{\alpha}(t)(\delta^{\Delta}(t))^{\alpha}}\right], \end{aligned} \tag{38}$$

where $M_4 = M_3^{\alpha}$.

Integrating the above inequality (38) from T_2 to t, we have

$$\begin{aligned} w(t) \leqslant & w(T_2) \\ & - \int_{T_2}^{t} \Bigg(\xi(s)\overline{p}(s) - \left(\alpha^{\alpha} M_4(R(\sigma(s)))^{\alpha-\beta} r(\delta(s)) \right. \\ & \left. \times \left((\xi^{\Delta}(s))_+\right)^{\alpha+1}\right) \\ & \times \left((\alpha+1)^{\alpha+1}\beta^{\alpha}\xi^{\alpha}(s)\left(\delta^{\Delta}(s)\right)^{\alpha}\right)^{-1} \Bigg) \Delta s \\ \leqslant & w(T_2) + \int_{t_0}^{T_2} \xi(s)\overline{p}(s)\Delta s \end{aligned}$$

$$-\int_{t_0}^{t}\left(\xi(s)\overline{p}(s)-\left(\alpha^{\alpha}M_4(R(\sigma(s)))^{\alpha-\beta}r(\delta(s))\times\left(\left(\xi^{\Delta}(s)\right)_+\right)^{\alpha+1}\right)\times\left((\alpha+1)^{\alpha+1}\beta^{\alpha}\xi^{\alpha}(s)\left(\delta^{\Delta}(s)\right)^{\alpha}\right)^{-1}\right)\Delta s. \quad (39)$$

Since $w(t)>0$ for $t>T_2$, we have

$$\begin{aligned}\int_{t_0}^{t}&\left(\xi(s)\overline{p}(s)-\frac{\alpha^{\alpha}M_4(R(\sigma(s)))^{\alpha-\beta}r(\delta(s))\left(\left(\xi^{\Delta}(s)\right)_+\right)^{\alpha+1}}{(\alpha+1)^{\alpha+1}\beta^{\alpha}\xi^{\alpha}(s)\left(\delta^{\Delta}(s)\right)^{\alpha}}\right)\Delta s\\ &\leqslant w(T_2)+\int_{t_0}^{T_2}\xi(s)\overline{p}(s)\Delta s-w(t)\\ &\leqslant w(T_2)+\int_{t_0}^{T_2}\xi(s)\overline{p}(s)\Delta s,\end{aligned} \quad (40)$$

which contradicts (12). This completes the proof of Theorem 4. □

Next, we use the general weighted functions from the class $\mathcal{F}$ which will be extensively used in the sequel.

Letting $\mathbb{D}\equiv\{(t,s)\in\mathbb{T}\times\mathbb{T}: t\geqslant s\geqslant t_0\}$, we say that a continuous function $H(t,s)\in C_{\mathrm{rd}}(\mathbb{D},\mathbb{R})$ belongs to the class $\mathcal{F}$ if

(i) $H(t,t)=0$ for $t\geqslant t_0$ and $H(t,s)>0$ for $t>s\geqslant t_0$,

(ii) $H(t,s)$ has a nonpositive right-dense continuous Δ-partial derivative $H^{\Delta_s}(t,s)$ with respect to the second variable.

Theorem 5. *Assume that (H_1)–(H_6) and (7) hold. If there exist a function $H(t,s)\in\mathcal{F}$ and a function $\xi(t)\in C^1_{\mathrm{rd}}(\mathbb{T},(0,\infty))$ such that for any positive number M,*

$$\overline{\lim_{t\to\infty}}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[H(t,s)\xi(s)\overline{p}(s)-\widetilde{U}(t,s)\right]\Delta s=\infty, \quad (41)$$

where

$$\overline{p}(s)=q(s)\left[1-p(\delta(s))\right]^{\beta}, \quad (42)$$

$$\widetilde{U}(t,s)=\frac{\alpha^{\alpha}\left(\phi_+(t,s)\right)^{\alpha+1}(\xi(\sigma(s)))^{\alpha+1}M(R(\sigma(s)))^{\alpha-\beta}r(\delta(s))}{(\alpha+1)^{\alpha+1}\beta^{\alpha}(H(t,s))^{\alpha}\xi^{\alpha}(s)\left(\delta^{\Delta}(s)\right)^{\alpha}}, \quad (43)$$

$$\phi_+(t,s):=\max\left\{H^{\Delta_s}(t,s)+\frac{H(t,s)\left(\xi^{\Delta}(s)\right)_+}{\xi(\sigma(s))},0\right\}, \quad (44)$$

$$\left(\xi^{\Delta}(s)\right)_+:=\max\left\{\xi^{\Delta}(s),0\right\}, \quad (45)$$

then (1) *is oscillatory.*

Proof. We proceed as in the proof of Theorem 4 to have (37). From (37) we obtain

$$\xi(t)\overline{p}(t)\leqslant -w^{\Delta}(t)+\frac{\left(\xi^{\Delta}(t)\right)_+}{\xi(\sigma(t))}w(\sigma(t))-\Psi(t)w^{(\alpha+1)/\alpha}(\sigma(t)),\quad t\geqslant T_2. \quad (46)$$

Multiplying (46) (with t replaced by s) by $H(t,s)$, integrating it with respect to s from T_2 to t for $t>T_2$, using integration by parts and (i)-(ii), we get

$$\begin{aligned}\int_{T_2}^{t}&H(t,s)\xi(s)\overline{p}(s)\Delta s\\ &\leqslant-\int_{T_2}^{t}H(t,s)w^{\Delta}(s)\Delta s+\int_{T_2}^{t}\frac{H(t,s)\left(\xi^{\Delta}(s)\right)_+}{\xi(\sigma(s))}w(\sigma(s))\Delta s-\int_{T_2}^{t}H(t,s)\Psi(s)w^{(\alpha+1)/\alpha}(\sigma(s))\Delta s\\ &=H(t,T_2)w(T_2)+\int_{T_2}^{t}H^{\Delta_s}(t,s)w(\sigma(s))\Delta s+\int_{T_2}^{t}\frac{H(t,s)\left(\xi^{\Delta}(s)\right)_+}{\xi(\sigma(s))}w(\sigma(s))\Delta s-\int_{T_2}^{t}H(t,s)\Psi(s)w^{(\alpha+1)/\alpha}(\sigma(s))\Delta s\\ &=H(t,T_2)w(T_2)+\int_{T_2}^{t}\left(H^{\Delta_s}(t,s)+\frac{H(t,s)\left(\xi^{\Delta}(s)\right)_+}{\xi(\sigma(s))}\right)w(\sigma(s))\Delta s-\int_{T_2}^{t}H(t,s)\Psi(s)w^{(\alpha+1)/\alpha}(\sigma(s))\Delta s\\ &=H(t,T_2)w(T_2)+\int_{T_2}^{t}\left[\left(H^{\Delta_s}(t,s)+\frac{H(t,s)\left(\xi^{\Delta}(s)\right)_+}{\xi(\sigma(s))}\right)w(\sigma(s))\right.\end{aligned}$$

$$
\begin{aligned}
& \left. -H(t,s)\Psi(s)w^{(\alpha+1)/\alpha}(\sigma(s)) \right]\Delta s \\
& \leqslant H(t,T_2)w(T_2) \\
& \quad + \int_{T_2}^{t} \left[\phi_+(t,s)w(\sigma(s)) \right. \\
& \quad \left. -H(t,s)\Psi(s)w^{(\alpha+1)/\alpha}(\sigma(s)) \right]\Delta s,
\end{aligned} \tag{47}
$$

where $\phi_+(t,s)$ is defined as in (44).

Taking $a = \phi_+(t,s)$, $b = H(t,s)\Psi(s)$, by Lemma 3 and (47), we obtain

$$
\begin{aligned}
& \int_{T_2}^{t} H(t,s)\xi(s)\overline{p}(s)\Delta s \\
& \leqslant H(t,T_2)w(T_2) \\
& \quad + \int_{T_2}^{t} \Big[\Big(\alpha^{\alpha}(\phi_+(t,s))^{\alpha+1}(\xi(\sigma(s)))^{\alpha+1} \\
& \qquad \times M_3^{\alpha}(R(\sigma(s)))^{\alpha-\beta}r(\delta(s)) \Big) \\
& \qquad \times \Big((\alpha+1)^{\alpha+1}\beta^{\alpha}(H(t,s))^{\alpha} \\
& \qquad \times \xi^{\alpha}(s)\left(\delta^{\Delta}(s)\right)^{\alpha} \Big)^{-1} \Big]\Delta s \\
& \leqslant H(t,T_2)w(T_2) \\
& \quad + \int_{T_2}^{t} \Big[\Big(\alpha^{\alpha}(\phi_+(t,s))^{\alpha+1}(\xi(\sigma(s)))^{\alpha+1} \\
& \qquad \times M_4(R(\sigma(s)))^{\alpha-\beta}r(\delta(s)) \Big) \\
& \qquad \times \Big((\alpha+1)^{\alpha+1}\beta^{\alpha}(H(t,s))^{\alpha} \\
& \qquad \times \xi^{\alpha}(s)\left(\delta^{\Delta}(s)\right)^{\alpha} \Big)^{-1} \Big]\Delta s \\
& \leqslant H(t,t_0)w(T_2) + \int_{T_2}^{t} U(t,s)\Delta s,
\end{aligned} \tag{48}
$$

where $M_4 = M_3^{\alpha}$,

$$
\begin{aligned}
& U(t,s) \\
& = \frac{\alpha^{\alpha}(\phi_+(t,s))^{\alpha+1}(\xi(\sigma(s)))^{\alpha+1}M_4(R(\sigma(s)))^{\alpha-\beta}r(\delta(s))}{(\alpha+1)^{\alpha+1}\beta^{\alpha}(H(t,s))^{\alpha}\xi^{\alpha}(s)\left(\delta^{\Delta}(s)\right)^{\alpha}}.
\end{aligned} \tag{49}
$$

Then it follows that

$$
\frac{1}{H(t,t_0)}\int_{T_2}^{t}\left[H(t,s)\xi(s)\overline{p}(s) - U(t,s)\right]\Delta s \leqslant w(T_2). \tag{50}
$$

Thus we get

$$
\begin{aligned}
& \frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[H(t,s)\xi(s)\overline{p}(s) - U(t,s)\right]\Delta s \\
& = \frac{1}{H(t,t_0)}\left(\int_{t_0}^{T_2} + \int_{T_2}^{t}\right)\left[H(t,s)\xi(s)\overline{p}(s) - U(t,s)\right]\Delta s \\
& \leqslant w(T_2) + \frac{1}{H(t,t_0)}\int_{t_0}^{T_2}\left[H(t,s)\xi(s)\overline{p}(s) - U(t,s)\right]\Delta s \\
& \leqslant w(T_2) + \int_{t_0}^{T_2}\left[\frac{H(t,s)}{H(t,t_0)}\xi(s)\overline{p}(s) - \frac{U(t,s)}{H(t,t_0)}\right]\Delta s \\
& \leqslant w(T_2) + \int_{t_0}^{T_2}\xi(s)\overline{p}(s)\Delta s.
\end{aligned} \tag{51}
$$

Then

$$
\overline{\lim_{t\to\infty}}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[H(t,s)\xi(s)\overline{p}(s) - U(t,s)\right]\Delta s < \infty, \tag{52}
$$

which contradicts (41). This completes the proof of Theorem 5. □

Theorem 6. *Assume that* (H_1)–(H_6) *and* (7) *hold and* $\beta \geqslant 1$. *Furthermore, assume that* $r^{\Delta}(t) \geqslant 0$. *If there exists a function* $\xi(t) \in C_{\mathrm{rd}}^{1}(\mathbb{T},(0,\infty))$ *such that for any positive number* M,

$$
\overline{\lim_{t\to\infty}}\int_{t_0}^{t}\left(\xi(s)\overline{p}(s) - Q(s)\right)\Delta s = \infty, \tag{53}
$$

where

$$
\begin{aligned}
\overline{p}(s) &= q(s)\left[1 - p(\delta(s))\right]^{\beta}, \\
Q(s) &= \frac{\left(\xi^{\Delta}(s)\right)^{2}(r(\sigma(s)))^{(\alpha-\beta)/\alpha}(r(\delta(s)))^{\beta/\alpha}}{4\beta\xi(s)(\delta(s)/2)^{\beta-1}\delta^{\Delta}(s)M^{\alpha-\beta}},
\end{aligned} \tag{54}
$$

then (1) *is oscillatory.*

Proof. We proceed as in the proof of Theorem 4 to have (24). On the other hand, from (22) and (H_3), we deduce

$$
\left(r(t)\left(Z^{\Delta}(t)\right)^{\alpha}\right)^{\Delta} \leqslant 0, \quad t \geqslant T_1, \tag{55}
$$

and from $r^{\Delta}(t) \geqslant 0$ for $t \geqslant t_0$, we can get $Z^{\Delta}(t)$ is nonincreasing. Hence, we have

$$
Z(t) - Z(T_1) = \int_{T_1}^{t} Z^{\Delta}(s)\Delta s \geqslant (t - T_1)Z^{\Delta}(t), \tag{56}
$$

which implies

$$Z(t) \geqslant \frac{t}{2} Z^{\Delta}(t), \quad \text{for } t \geqslant T_2 > 2T_1. \tag{57}$$

Choosing $T_3 \geqslant T_2$ such that $\delta(t) \geqslant T_2$ for $t \geqslant T_3$, we get

$$Z(\delta(t)) \geqslant \frac{\delta(t)}{2} Z^{\Delta}(\delta(t)), \quad \text{for } t \geqslant T_3. \tag{58}$$

From (H_6), (15), (20), (24), (25), (58), and as $Z^{\Delta}(t)$ is nonincreasing, we obtain

$$\begin{aligned}
w^{\Delta}(t) \leqslant & -\xi(t)\overline{p}(t) + \frac{\xi^{\Delta}(t)}{\xi(\sigma(t))} w(\sigma(t)) \\
& - \left(r(\sigma(t)) \left(Z^{\Delta}(\sigma(t))\right)^{\alpha} \xi(t) \beta (Z(\delta(t)))^{\beta-1} \right. \\
& \left. \times Z^{\Delta}(\delta(t)) \delta^{\Delta}(t) \right) \left(Z^{2\beta}(\delta(\sigma(t)))\right)^{-1} \\
\leqslant & -\xi(t)\overline{p}(t) + \frac{\xi^{\Delta}(t)}{\xi(\sigma(t))} w(\sigma(t)) \\
& - \left(r(\sigma(t)) \left(Z^{\Delta}(\sigma(t))\right)^{\alpha} \xi(t) \right. \\
& \left. \times \beta \left((\delta(t)/2) Z^{\Delta}(\delta(t)) \right)^{\beta-1} Z^{\Delta}(\delta(t)) \delta^{\Delta}(t) \right) \\
& \times \left(Z^{2\beta}(\delta(\sigma(t)))\right)^{-1} \\
\leqslant & -\xi(t)\overline{p}(t) + \frac{\xi^{\Delta}(t)}{\xi(\sigma(t))} w(\sigma(t)) \\
& - \left(\beta \xi(t) r(\sigma(t)) \left(Z^{\Delta}(\sigma(t))\right)^{\alpha+\beta} (\delta(t)/2)^{\beta-1} \delta^{\Delta}(t) \right) \\
& \times \left(Z^{2\beta}(\delta(\sigma(t)))\right)^{-1} \left(\frac{r(\sigma(t))}{r(\delta(t))} \right)^{\beta/\alpha} \\
= & -\xi(t)\overline{p}(t) + \frac{\xi^{\Delta}(t)}{\xi(\sigma(t))} w(\sigma(t)) \\
& - \left(\beta \xi(t) (\delta(t)/2)^{\beta-1} \delta^{\Delta}(t) \right) \\
& \times \left(\xi^2(\sigma(t)) (r(\sigma(t)))^{(\alpha-\beta)/\alpha} \left(Z^{\Delta}(\sigma(t))\right)^{\alpha-\beta} \right. \\
& \left. \times (r(\delta(t)))^{\beta/\alpha} \right)^{-1} w^2(\sigma(t)) \\
\leqslant & -\xi(t)\overline{p}(t) + \frac{\xi^{\Delta}(t)}{\xi(\sigma(t))} w(\sigma(t)) \\
& - \left(\beta \xi(t) (\delta(t)/2)^{\beta-1} \delta^{\Delta}(t) \right) \\
& \times \left(\xi^2(\sigma(t)) (r(\sigma(t)))^{(\alpha-\beta)/\alpha} \left(Z^{\Delta}(t)\right)^{\alpha-\beta} \right. \\
& \left. \times (r(\delta(t)))^{\beta/\alpha} \right)^{-1} w^2(\sigma(t)).
\end{aligned} \tag{59}$$

Now, from the fact that $Z^{\Delta}(t)$ is nonnegative and nonincreasing, there exists a $T_4 > T_3$ sufficiently large such that

$$Z^{\Delta}(t) \leqslant \frac{1}{M}, \quad t \geqslant T_4, \tag{60}$$

holds for some positive constant M and therefore

$$\left(Z^{\Delta}(t)\right)^{\alpha-\beta} \leqslant \left(\frac{1}{M}\right)^{\alpha-\beta}, \quad t \geqslant T_4. \tag{61}$$

Combining (59) and (61), we obtain that

$$\begin{aligned}
w^{\Delta}(t) \leqslant & -\xi(t)\overline{p}(t) + \frac{\xi^{\Delta}(t)}{\xi(\sigma(t))} w(\sigma(t)) \\
& - \frac{\beta \xi(t) (\delta(t)/2)^{\beta-1} \delta^{\Delta}(t) M^{\alpha-\beta}}{\xi^2(\sigma(t)) (r(\sigma(t)))^{(\alpha-\beta)/\alpha} (r(\delta(t)))^{\beta/\alpha}} \\
& \times w^2(\sigma(t)), \quad t \geqslant T_4.
\end{aligned} \tag{62}$$

Letting

$$\Phi(t) = \frac{\beta \xi(t) (\delta(t)/2)^{\beta-1} \delta^{\Delta}(t) M^{\alpha-\beta}}{\xi^2(\sigma(t)) (r(\sigma(t)))^{(\alpha-\beta)/\alpha} (r(\delta(t)))^{\beta/\alpha}}, \tag{63}$$

then $\Phi(t) \geqslant 0$. So

$$\begin{aligned}
w^{\Delta}(t) \leqslant & -\xi(t)\overline{p}(t) + \frac{\xi^{\Delta}(t)}{\xi(\sigma(t))} w(\sigma(t)) - \Phi(t) w^2(\sigma(t)) \\
= & -\xi(t)\overline{p}(t) + \frac{1}{4\Phi(t)} \frac{\left(\xi^{\Delta}(t)\right)^2}{\xi^2(\sigma(t))} \\
& - \left[\sqrt{\Phi(t)} w(\sigma(t)) - \frac{1}{2\sqrt{\Phi(t)}} \frac{\xi^{\Delta}(t)}{\xi(\sigma(t))} \right]^2 \\
\leqslant & -\xi(t)\overline{p}(t) + \frac{1}{4\Phi(t)} \frac{\left(\xi^{\Delta}(t)\right)^2}{\xi^2(\sigma(t))} \\
= & -\left[\xi(t)\overline{p}(t) \right. \\
& \left. - \frac{\left(\xi^{\Delta}(t)\right)^2 (r(\sigma(t)))^{(\alpha-\beta)/\alpha} (r(\delta(t)))^{\beta/\alpha}}{4\beta \xi(t) (\delta(t)/2)^{\beta-1} \delta^{\Delta}(t) M^{\alpha-\beta}} \right].
\end{aligned} \tag{64}$$

Integrating the above inequality from T_4 to t, we have

$$\begin{aligned} w(t) &\leqslant w(T_4) \\ &\quad - \int_{T_4}^{t} \Bigg(\xi(s)\overline{p}(s) \\ &\quad - \left(\left(\xi^{\Delta}(s)\right)^2 (r(\sigma(s)))^{(\alpha-\beta)/\alpha} (r(\delta(s)))^{\beta/\alpha} \right) \\ &\quad \times \left(4\beta\xi(s)(\delta(s)/2)^{\beta-1}\delta^{\Delta}(s) M^{\alpha-\beta} \right)^{-1} \Bigg) \Delta s \\ &\leqslant w(T_4) + \int_{t_0}^{T_4} \xi(s)\overline{p}(s)\Delta s \\ &\quad - \int_{t_0}^{t} \Bigg(\xi(s)\overline{p}(s) \\ &\quad - \left(\left(\xi^{\Delta}(s)\right)^2 (r(\sigma(s)))^{(\alpha-\beta)/\alpha} (r(\delta(s)))^{\beta/\alpha} \right) \\ &\quad \times \left(4\beta\xi(s)(\delta(s)/2)^{\beta-1}\delta^{\Delta}(s) M^{\alpha-\beta} \right)^{-1} \Bigg) \Delta s. \end{aligned} \tag{65}$$

Since $w(t) > 0$ for $t > T_4$, we have

$$\begin{aligned} &\int_{t_0}^{t} \left(\xi(s)\overline{p}(s) - \frac{\left(\xi^{\Delta}(s)\right)^2 (r(\sigma(s)))^{(\alpha-\beta)/\alpha} (r(\delta(s)))^{\beta/\alpha}}{4\beta\xi(s)(\delta(s)/2)^{\beta-1}\delta^{\Delta}(s) M^{\alpha-\beta}} \right) \Delta s \\ &\quad \leqslant w(T_4) + \int_{t_0}^{T_4} \xi(s)\overline{p}(s)\Delta s - w(t) \\ &\quad < w(T_4) + \int_{t_0}^{T_4} \xi(s)\overline{p}(s)\Delta s. \end{aligned} \tag{66}$$

which contradicts (53). This completes the proof of Theorem 6. □

Theorem 7. *Assume that (H_1)–(H_6) and (7) hold and $\beta \geqslant 1$. Furthermore, assume that $r^{\Delta}(t) \geqslant 0$. If there exist a function $H(t,s) \in \digamma$ and a function $\xi(t) \in C^1_{rd}(\mathbb{T},(0,\infty))$ such that*

$$H^{\Delta_s}(t,s) + \frac{H(t,s)\xi^{\Delta}(s)}{\xi(\sigma(s))} \leqslant 0, \quad \text{for } t \geqslant s \geqslant t_0, \tag{67}$$

$$\overline{\lim_{t\to\infty}} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)\xi(s)\overline{p}(s)\Delta s = \infty, \tag{68}$$

where

$$\overline{p}(s) = q(s)\left[1 - p(\delta(s))\right]^{\beta}, \tag{69}$$

then (1) *is oscillatory.*

Proof. We proceed as in the proof of Theorem 6 to have (64). From (64) we obtain

$$\begin{aligned} \xi(t)\overline{p}(t) \leqslant &-w^{\Delta}(t) + \frac{\xi^{\Delta}(t)}{\xi(\sigma(t))} w(\sigma(t)) \\ &- \Phi(t) w^2(\sigma(t)), \quad t \geqslant T_4. \end{aligned} \tag{70}$$

Multiplying (70) (with t replaced by s) by $H(t,s)$, integrating it with respect to s from T_4 to t for $t > T_4$, using integration by parts and (i)-(ii), we get

$$\begin{aligned} &\int_{T_4}^{t} H(t,s)\xi(s)\overline{p}(s)\Delta s \\ &\quad \leqslant -\int_{T_4}^{t} H(t,s) w^{\Delta}(s)\Delta s + \int_{T_4}^{t} \frac{H(t,s)\xi^{\Delta}(s)}{\xi(\sigma(s))} w(\sigma(s))\Delta s \\ &\qquad - \int_{T_4}^{t} H(t,s)\Phi(s) w^2(\sigma(s))\Delta s \\ &\quad = H(t,T_4) w(T_4) + \int_{T_4}^{t} H^{\Delta_s}(t,s) w(\sigma(s))\Delta s \\ &\qquad + \int_{T_4}^{t} \frac{H(t,s)\xi^{\Delta}(s)}{\xi(\sigma(s))} w(\sigma(s))\Delta s \\ &\qquad - \int_{T_4}^{t} H(t,s)\Phi(s) w^2(\sigma(s))\Delta s \\ &\quad = H(t,T_4) w(T_4) \\ &\qquad + \int_{T_4}^{t} \left(H^{\Delta_s}(t,s) + \frac{H(t,s)\xi^{\Delta}(s)}{\xi(\sigma(s))} \right) w(\sigma(s))\Delta s \\ &\qquad - \int_{T_4}^{t} H(t,s)\Phi(s) w^2(\sigma(s))\Delta s. \end{aligned} \tag{71}$$

Using (67) in the above inequality (71), we get

$$\int_{T_4}^{t} H(t,s)\xi(s)\overline{p}(s)\Delta s \leqslant H(t,t_0) w(T_4). \tag{72}$$

Then it follows that

$$\frac{1}{H(t,t_0)} \int_{T_4}^{t} H(t,s)\xi(s)\overline{p}(s)\Delta s \leqslant w(T_4). \tag{73}$$

Thus we get

$$\begin{aligned}
&\frac{1}{H(t,t_0)}\int_{t_0}^{t} H(t,s)\,\xi(s)\,\overline{p}(s)\,\Delta s\\
&\quad=\frac{1}{H(t,t_0)}\left(\int_{t_0}^{T_4}+\int_{T_4}^{t}\right)H(t,s)\,\xi(s)\,\overline{p}(s)\,\Delta s\\
&\quad\leqslant w(T_4)+\frac{1}{H(t,t_0)}\int_{t_0}^{T_4} H(t,s)\,\xi(s)\,\overline{p}(s)\,\Delta s\\
&\quad\leqslant w(T_4)+\int_{t_0}^{T_4}\frac{H(t,s)}{H(t,t_0)}\xi(s)\,\overline{p}(s)\,\Delta s\\
&\quad\leqslant w(T_4)+\int_{t_0}^{T_4}\xi(s)\,\overline{p}(s)\,\Delta s.
\end{aligned}\tag{74}$$

Then

$$\overline{\lim_{t\to\infty}}\frac{1}{H(t,t_0)}\int_{t_0}^{t} H(t,s)\,\xi(s)\,\overline{p}(s)\,\Delta s<\infty,\tag{75}$$

which contradicts (68). This completes the proof of Theorem 7. □

Theorem 8. *Assume that* (H_1)–(H_6) *and* (7) *hold and* $\beta\geqslant 1$. *Furthermore, assume that* $r^{\Delta}(t)\geqslant 0$. *If there exist a function* $H(t,s)\in\mathcal{F}$ *and a function* $\xi(t)\in C^1_{\mathrm{rd}}(\mathbb{T},(0,\infty))$ *such that for any positive number* M,

$$\begin{aligned}
&\overline{\lim_{t\to\infty}}\frac{1}{H(t,t_0)}\\
&\quad\times\int_{t_0}^{t}\left[H(t,s)\,\xi(s)\,\overline{p}(s)\right.\\
&\qquad\left.-\frac{\left(H^{\Delta_s}(t,s)+H(t,s)\,\xi^{\Delta}(s)/\xi(\sigma(s))\right)^2}{4H(t,s)\,\Phi(s)}\right]\Delta s=\infty,
\end{aligned}\tag{76}$$

where

$$\begin{aligned}
\overline{p}(s)&=q(s)\left[1-p(\delta(s))\right]^{\beta},\\
\Phi(s)&=\frac{\beta\xi(s)\,(\delta(s)/2)^{\beta-1}\delta^{\Delta}(s)\,M^{\alpha-\beta}}{\xi^2(\sigma(s))\,(r(\sigma(s)))^{(\alpha-\beta)/\alpha}(r(\delta(s)))^{\beta/\alpha}},
\end{aligned}\tag{77}$$

then (1) *is oscillatory.*

Proof. We proceed as those in the proof of Theorem 7 to have (71), that is,

$$\begin{aligned}
&\int_{T_4}^{t} H(t,s)\,\xi(s)\,\overline{p}(s)\,\Delta s\\
&\quad\leqslant H(t,T_4)\,w(T_4)\\
&\qquad+\int_{T_4}^{t}\left(H^{\Delta_s}(t,s)+\frac{H(t,s)\,\xi^{\Delta}(s)}{\xi(\sigma(s))}\right)w(\sigma(s))\,\Delta s\\
&\qquad-\int_{T_4}^{t} H(t,s)\,\Phi(s)\,w^2(\sigma(s))\,\Delta s\\
&\quad=H(t,T_4)\,w(T_4)\\
&\qquad+\int_{T_4}^{t}\frac{\left(H^{\Delta_s}(t,s)+H(t,s)\,\xi^{\Delta}(s)/\xi(\sigma(s))\right)^2}{4H(t,s)\,\Phi(s)}\Delta s\\
&\qquad-\int_{T_4}^{t}\left[\frac{H^{\Delta_s}(t,s)+H(t,s)\,\xi^{\Delta}(s)/\xi(\sigma(s))}{2\sqrt{H(t,s)\,\Phi(s)}}\right.\\
&\qquad\qquad\left.-\sqrt{H(t,s)\,\Phi(s)}w(\sigma(s))\right]^2\Delta s\\
&\quad\leqslant H(t,T_4)\,w(T_4)\\
&\qquad+\int_{T_4}^{t}\frac{\left(H^{\Delta_s}(t,s)+H(t,s)\,\xi^{\Delta}(s)/\xi(\sigma(s))\right)^2}{4H(t,s)\,\Phi(s)}\Delta s\\
&\quad\leqslant H(t,t_0)\,w(T_4)\\
&\qquad+\int_{T_4}^{t}\frac{\left(H^{\Delta_s}(t,s)+H(t,s)\,\xi^{\Delta}(s)/\xi(\sigma(s))\right)^2}{4H(t,s)\,\Phi(s)}\Delta s.
\end{aligned}\tag{78}$$

Then it follows that

$$\begin{aligned}
&\frac{1}{H(t,t_0)}\\
&\quad\times\int_{T_4}^{t}\left[H(t,s)\,\xi(s)\,\overline{p}(s)\right.\\
&\qquad\left.-\frac{\left(H^{\Delta_s}(t,s)+H(t,s)\,\xi^{\Delta}(s)/\xi(\sigma(s))\right)^2}{4H(t,s)\,\Phi(s)}\right]\Delta s\\
&\quad\leqslant w(T_4).
\end{aligned}\tag{79}$$

Thus we get

$$\begin{aligned}
&\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[H(t,s)\,\xi(s)\,\overline{p}(s)\right.\\
&\qquad-\left(H^{\Delta_s}(t,s)+\frac{H(t,s)\,\xi^{\Delta}(s)}{\xi(\sigma(s))}\right)^2
\end{aligned}$$

$$\times \left(4H(t,s)\Phi(s)\right)^{-1}\Bigg]\Delta s$$

$$= \frac{1}{H(t,t_0)} \times \left\{\int_{t_0}^{T_4} + \int_{T_4}^{t}\right\}\Bigg[H(t,s)\xi(s)\overline{p}(s) - \left(H^{\Delta_s}(t,s) + \frac{H(t,s)\xi^{\Delta}(s)}{\xi(\sigma(s))}\right)^2 \times \left(4H(t,s)\Phi(s)\right)^{-1}\Bigg]\Delta s$$

$$\leqslant w(T_4) + \frac{1}{H(t,t_0)} \times \int_{t_0}^{T_4}\Bigg[H(t,s)\xi(s)\overline{p}(s) - \left(H^{\Delta_s}(t,s) + \frac{H(t,s)\xi^{\Delta}(s)}{\xi(\sigma(s))}\right)^2 \times \left(4H(t,s)\Phi(s)\right)^{-1}\Bigg]\Delta s$$

$$\leqslant w(T_4) + \int_{t_0}^{T_4}\Bigg[\frac{H(t,s)}{H(t,t_0)}\xi(s)\overline{p}(s) - \left(H^{\Delta_s}(t,s) + \frac{H(t,s)\xi^{\Delta}(s)}{\xi(\sigma(s))}\right)^2 \times \left(4H(t,s)H(t,t_0)\Phi(s)\right)^{-1}\Bigg]\Delta s$$

$$\leqslant w(T_4) + \int_{t_0}^{T_4}\xi(s)\overline{p}(s)\Delta s. \tag{80}$$

Then

$$\overline{\lim_{t\to\infty}}\frac{1}{H(t,t_0)} \times \int_{t_0}^{t}\Bigg[H(t,s)\xi(s)\overline{p}(s) - \frac{\left(H^{\Delta_s}(t,s) + H(t,s)\xi^{\Delta}(s)/\xi(\sigma(s))\right)^2}{4H(t,s)\Phi(s)}\Bigg]\Delta s < \infty, \tag{81}$$

which contradicts (76). This completes the proof of Theorem 8. □

The case

$$\lim_{t\to\infty}\int_{t_0}^{t} r^{-1/\alpha}(s)\Delta s < \infty. \tag{82}$$

Theorem 9. *Assume that* (H_1)*–*(H_6) *and* (8) *hold and there exists a* $T_* \in [t_0,\infty)_{\mathbb{T}}$ *such that* $p^{\Delta}(t) \geqslant 0$*,* $\tau^{\Delta}(t) \geqslant 0$ *for* $t \geqslant T_*$*, and suppose that there exists a function* $\xi(t) \in C^1_{\mathrm{rd}}(\mathbb{T},(0,\infty))$ *such that* (12) *holds for any positive number* M*, and there exists a function* $\psi(t) \in C^1_{\mathrm{rd}}(\mathbb{T},(0,\infty))$ *satisfying* $\psi(t) \geqslant t$*,* $\psi^{\Delta}(t) > 0$*,* $\delta(t) \leqslant \tau(\psi(t))$ *for* $t \geqslant T_*$ *such that for any positive number* M *and for every* $T_1 \in [T_*,\infty)_{\mathbb{T}}$

$$\overline{\lim_{t\to\infty}}\int_{T_1}^{t}\left[\widetilde{p}(s)V^{\alpha}(\sigma(s)) - G(s)\right]\Delta s = \infty, \tag{83}$$

where

$$\widetilde{p}(s) = q(s)\left(\frac{1}{1+p(\psi(s))}\right)^{\beta},$$

$$V(s) = \int_{\psi(s)}^{\infty} r^{-1/\alpha}(t)\Delta t,$$

$$G(s) = \begin{cases} \dfrac{\alpha^{2\alpha+1}r^{-1/\alpha}(\psi(s))\psi^{\Delta}(s)}{(\alpha+1)^{\alpha+1}\beta^{\alpha}M^{\alpha-\beta}V(\sigma(s))}, & \text{if } 0<\alpha<1,\\[2ex] \dfrac{\alpha^{2\alpha+1}r^{-1/\alpha}(\psi(s))V^{\alpha^2-1}(s)\psi^{\Delta}(s)}{(\alpha+1)^{\alpha+1}\beta^{\alpha}M^{\alpha-\beta}V^{\alpha^2}(\sigma(s))}, & \text{if } \alpha \geqslant 1, \end{cases} \tag{84}$$

then (1) *is oscillatory.*

Proof. Suppose to the contrary that $x(t)$ is an eventually positive solution of (1), then there exists a $T_1 \geqslant T_* \geqslant t_0$ such that $x(t) > 0$, $x(\delta(t)) > 0$, $x(\sigma(t)) > 0$ for all $t \geqslant T_1$, (the case of $x(t)$ is negative and can be considered by the same method). It follows form (H_3) that $Z(t) \geqslant x(t) > 0$ for $t \geqslant T_1$. From (14) it is easy to conclude that there exist two possible cases of the sign of $Z^{\Delta}(t)$.

Case 1. Suppose $Z^{\Delta}(t) \geqslant 0$ for sufficiently large t, then we are back to the case of Theorem 4. Thus the proof of Theorem 4 goes through, and we may get contradiction by (12).

Case 2. Suppose $Z^{\Delta}(t) < 0$ for $t \geqslant T_1$. Define

$$w(t) = \frac{r(t)\left(-Z^{\Delta}(t)\right)^{\alpha-1}Z^{\Delta}(t)}{Z^{\beta}(\psi(t))}, \quad t \geqslant T_1. \tag{85}$$

Then $w(t) < 0$ for $t \geqslant T_1$. From the fact that $Z(t)$ is positive and nonincreasing, we get that

$$Z(\psi(t)) \leqslant \frac{1}{M_0}, \quad t \geqslant T_1, \tag{86}$$

holds for some positive constant M_0.

Noting that $(r(t)(-Z^{\Delta}(t))^{\alpha-1}Z^{\Delta}(t))^{\Delta} \leqslant 0$, $\psi(t) \geqslant t$, so we have

$$Z^{\Delta}(\psi(t)) \leqslant \left(\frac{r(t)}{r(\psi(t))}\right)^{1/\alpha} Z^{\Delta}(t), \tag{87}$$

$$Z^{\Delta}(s) \leqslant \frac{r^{1/\alpha}(t)}{r^{1/\alpha}(s)} Z^{\Delta}(t), \quad s \geqslant t. \tag{88}$$

Integrating the above inequality (88) with respect to s from $\psi(t)$ to v, we have

$$Z(v) \leqslant Z(\psi(t)) + r^{1/\alpha}(t) Z^{\Delta}(t) \int_{\psi(t)}^{v} r^{1/\alpha}(s) \Delta s. \tag{89}$$

Letting $v \to \infty$ in the above inequality, we obtain

$$0 \leqslant Z(\psi(t)) + r^{1/\alpha}(t) Z^{\Delta}(t) V(t). \tag{90}$$

From (86) and (90), we have

$$-\frac{1}{M_0^{\alpha-\beta}} \leqslant w(t) V^{\alpha}(t) \leqslant 0, \quad t \geqslant T_1. \tag{91}$$

If $0 < \beta < 1$. From $Z^{\Delta}(t) < 0$, Lemmas 1 and 2, we have

$$\begin{aligned}
&\left(Z^{\beta}(\psi(t))\right)^{\Delta} \\
&\quad = \beta\left\{\int_0^1 [(1-h)Z(\psi(t)) + hZ(\psi(\sigma(t)))]^{\beta-1} dh\right\} \\
&\qquad \times (Z(\psi(t)))^{\Delta} \\
&\quad \leqslant \beta\left[\int_0^1 Z^{\beta-1}(\psi(t))\, dh\right] Z^{\Delta}(\psi(t)) \psi^{\Delta}(t) \\
&\quad = \beta Z^{\beta-1}(\psi(t)) Z^{\Delta}(\psi(t)) \psi^{\Delta}(t).
\end{aligned} \tag{92}$$

From (1), (H_6), (85), and (92), we get

$$\begin{aligned}
&w^{\Delta}(t) \\
&\quad = \frac{1}{Z^{\beta}(\psi(t))}\left(r(t)\left(-Z^{\Delta}(t)\right)^{\alpha-1} Z^{\Delta}(t)\right)^{\Delta} \\
&\qquad - \left(r(\sigma(t))\left(-Z^{\Delta}(\sigma(t))\right)^{\alpha-1} Z^{\Delta}(\sigma(t)) \left(Z^{\beta}(\psi(t))\right)^{\Delta}\right) \\
&\qquad \times \left(Z^{\beta}(\psi(t)) Z^{\beta}(\psi(\sigma(t)))\right)^{-1} \\
&\quad \leqslant -q(t) \frac{x^{\beta}(\delta(t))}{Z^{\beta}(\psi(t))} \\
&\qquad - \Big(r(\sigma(t))\left(-Z^{\Delta}(\sigma(t))\right)^{\alpha-1} Z^{\Delta}(\sigma(t)) \beta Z^{\beta-1} \\
&\qquad\qquad \times (\psi(t)) Z^{\Delta}(\psi(t)) \psi^{\Delta}(t)\Big) \\
&\qquad \times \left(Z^{\beta}(\psi(t)) Z^{\beta}(\psi(\sigma(t)))\right)^{-1} \\
&\quad \leqslant -q(t) \frac{x^{\beta}(\delta(t))}{Z^{\beta}(\psi(t))} \\
&\qquad - \Big(r(\sigma(t))\left(-Z^{\Delta}(\sigma(t))\right)^{\alpha-1} Z^{\Delta}(\sigma(t)) \\
&\qquad\qquad \times \beta Z^{\Delta}(\psi(t)) \psi^{\Delta}(t)\Big) \left(Z(\psi(t)) Z^{\beta}(\psi(\sigma(t)))\right)^{-1} \\
&\quad \leqslant -q(t) \frac{x^{\beta}(\delta(t))}{Z^{\beta}(\psi(t))} \\
&\qquad - \Big(r(\sigma(t))\left(-Z^{\Delta}(\sigma(t))\right)^{\alpha-1} Z^{\Delta}(\sigma(t)) \\
&\qquad\qquad \times \beta Z^{\Delta}(\psi(t)) \psi^{\Delta}(t)\Big) \left(Z^{\beta+1}(\psi(t))\right)^{-1}.
\end{aligned} \tag{93}$$

If $\beta \geqslant 1$. From $Z^{\Delta}(t) < 0$, Lemmas 1 and 2, we have

$$\begin{aligned}
&\left(Z^{\beta}(\psi(t))\right)^{\Delta} \\
&\quad = \beta\left\{\int_0^1 [(1-h)Z(\psi(t)) + hZ(\psi(\sigma(t)))]^{\beta-1} dh\right\} \\
&\qquad \times (Z(\psi(t)))^{\Delta} \\
&\quad \leqslant \beta\left[\int_0^1 Z^{\beta-1}(\psi(\sigma(t)))\, dh\right] Z^{\Delta}(\psi(t)) \psi^{\Delta}(t) \\
&\quad = \beta Z^{\beta-1}(\psi(\sigma(t))) Z^{\Delta}(\psi(t)) \psi^{\Delta}(t).
\end{aligned} \tag{94}$$

From (1), (H_6), (85) and (94), we get

$$\begin{aligned}
w^{\Delta}(t) = {} & \frac{1}{Z^{\beta}(\psi(t))}\left(r(t)\left(-Z^{\Delta}(t)\right)^{\alpha-1} Z^{\Delta}(t)\right)^{\Delta} \\
& - \Big(r(\sigma(t))\left(-Z^{\Delta}(\sigma(t))\right)^{\alpha-1} \\
& \qquad \times Z^{\Delta}(\sigma(t)) \left(Z^{\beta}(\psi(t))\right)^{\Delta}\Big)
\end{aligned}$$

$$\begin{aligned}
&\times\left(Z^{\beta}(\psi(t))Z^{\beta}(\psi(\sigma(t)))\right)^{-1}\\
&\leqslant -q(t)\frac{x^{\beta}(\delta(t))}{Z^{\beta}(\psi(t))}\\
&\quad-\Big(r(\sigma(t))\left(-Z^{\Delta}(\sigma(t))\right)^{\alpha-1}Z^{\Delta}(\sigma(t))\\
&\quad\times\beta Z^{\Delta}(\psi(t))\psi^{\Delta}(t)\Big)\\
&\quad\times\left(Z^{\beta}(\psi(t))Z(\psi(\sigma(t)))\right)^{-1}\\
&\leqslant -q(t)\frac{x^{\beta}(\delta(t))}{Z^{\beta}(\psi(t))}\\
&\quad-\Big(r(\sigma(t))\left(-Z^{\Delta}(\sigma(t))\right)^{\alpha-1}Z^{\Delta}(\sigma(t))\\
&\quad\times\beta Z^{\Delta}(\psi(t))\psi^{\Delta}(t)\Big)\left(Z^{\beta+1}(\psi(t))\right)^{-1}.
\end{aligned}\tag{95}$$

Therefore, for $\beta > 0$, from (93) and (95), we get

$$\begin{aligned}
&w^{\Delta}(t)\\
&\leqslant -q(t)\frac{x^{\beta}(\delta(t))}{Z^{\beta}(\psi(t))}\\
&\quad-\frac{r(\sigma(t))\left(-Z^{\Delta}(\sigma(t))\right)^{\alpha-1}Z^{\Delta}(\sigma(t))\beta Z^{\Delta}(\psi(t))\psi^{\Delta}(t)}{Z^{\beta+1}(\psi(t))}.
\end{aligned}\tag{96}$$

Noticing that $p^{\Delta}(t) \geqslant 0$ and $\tau^{\Delta}(t) \geqslant 0$, from $Z^{\Delta}(t) = x^{\Delta}(t) + p^{\Delta}(t)x(\tau(t)) + p(\sigma(t))x^{\Delta}(\tau(t))\tau^{\Delta}(t)$, we see that $x^{\Delta}(t) \leqslant 0$ for $t \geqslant T_1$, and from $\delta(t) \leqslant \tau(\psi(t)) \leqslant \psi(t)$ we can get

$$\begin{aligned}
\frac{x^{\beta}(\delta(t))}{Z^{\beta}(\psi(t))} &= \left(\left(\frac{x(\psi(t))}{x(\delta(t))}+p(\psi(t))\frac{x(\tau(\psi(t)))}{x(\delta(t))}\right)^{-1}\right)^{\beta}\\
&\geqslant \left(\frac{1}{1+p(\psi(t))}\right)^{\beta}.
\end{aligned}\tag{97}$$

Thus from (86), (87), (96), (97) and the fact that $(r(t)(-Z^{\Delta}(t))^{\alpha-1}Z^{\Delta}(t))^{\Delta} \leqslant 0$, we have

$$\begin{aligned}
w^{\Delta}(t) &\leqslant -\widetilde{p}(t)\\
&\quad-\frac{r(\sigma(t))\left(-Z^{\Delta}(\sigma(t))\right)^{\alpha-1}Z^{\Delta}(\sigma(t))\beta Z^{\Delta}(t)\psi^{\Delta}(t)}{Z^{\beta+1}(\psi(t))}\\
&\quad\times\left(\frac{r(t)}{r(\psi(t))}\right)^{1/\alpha}\\
&= -\widetilde{p}(t)-\frac{r(t)\left(-Z^{\Delta}(t)\right)^{\alpha-1}Z^{\Delta}(t)\beta Z^{\Delta}(t)\psi^{\Delta}(t)}{Z^{\beta+1}(\psi(t))}\\
&\quad\times\left(\frac{r(t)}{r(\psi(t))}\right)^{1/\alpha}\\
&= -\widetilde{p}(t)-\frac{r(t)\left(-Z^{\Delta}(t)\right)^{\alpha-1}Z^{\Delta}(t)\beta Z^{\Delta}(t)\psi^{\Delta}(t)}{Z^{\beta+1}(\psi(t))}\\
&\quad\times\left(\frac{r(t)}{r(\psi(t))}\right)^{1/\alpha}\\
&\leqslant -\widetilde{p}(t)-\frac{\beta M_0^{(\alpha-\beta)/\alpha}\psi^{\Delta}(t)}{r^{1/\alpha}(\psi(t))}(-w(t))^{(\alpha+1)/\alpha},
\end{aligned}\tag{98}$$

where $\widetilde{p}(t) = q(t)(1/(1+p(\psi(t))))^{\beta}$.

That is

$$w^{\Delta}(t)+\widetilde{p}(t)+\frac{\beta M_0^{(\alpha-\beta)/\alpha}\psi^{\Delta}(t)}{r^{1/\alpha}(\psi(t))}(-w(t))^{(\alpha+1)/\alpha} \leqslant 0, \quad t \geqslant T_1. \tag{99}$$

Multiplying (99) (with t replaced by s) by $V^{\alpha}(\sigma(s))$, integrating it with respect to s from T_1 to t, we have

$$\begin{aligned}
&V^{\alpha}(t)w(t)-V^{\alpha}(T_1)w(T_1)-\int_{T_1}^{t}\left(V^{\alpha}(s)\right)^{\Delta}w(s)\Delta s\\
&\quad+\int_{T_1}^{t}\widetilde{p}(s)V^{\alpha}(\sigma(s))\Delta s\\
&\quad+\int_{T_1}^{t}\frac{\beta M_0^{(\alpha-\beta)/\alpha}V^{\alpha}(\sigma(s))\psi^{\Delta}(s)}{r^{1/\alpha}(\psi(s))}(-w(s))^{(\alpha+1)/\alpha}\Delta s \leqslant 0.
\end{aligned}\tag{100}$$

Next, we consider the following two cases.

Case (i) (let $0 < \alpha < 1$). From Lemma 2 and $V^{\Delta}(t) = -r^{-1/\alpha}(\psi(t))\psi^{\Delta}(t) < 0$, we have

$$\begin{aligned}(V^{\alpha}(t))^{\Delta} &= \alpha\left\{\int_0^1 [(1-h)V(t)+hV(\sigma(t))]^{\alpha-1}\mathrm{d}h\right\}V^{\Delta}(t)\\ &\geqslant \alpha\left[\int_0^1 V^{\alpha-1}(\sigma(t))\,\mathrm{d}h\right]V^{\Delta}(t)\\ &= \alpha V^{\alpha-1}(\sigma(t))V^{\Delta}(t).\end{aligned} \tag{101}$$

From (100) and (101), we get

$$\begin{aligned}&V^{\alpha}(t)w(t) - V^{\alpha}(T_1)w(T_1)\\ &\quad-\int_{T_1}^{t}\alpha V^{\alpha-1}(\sigma(s))V^{\Delta}(s)w(s)\Delta s\\ &\quad+\int_{T_1}^{t}\widetilde{p}(s)V^{\alpha}(\sigma(s))\Delta s\\ &\quad+\int_{T_1}^{t}\frac{\beta M_0^{(\alpha-\beta)/\alpha}V^{\alpha}(\sigma(s))\psi^{\Delta}(s)}{r^{1/\alpha}(\psi(s))}(-w(s))^{(\alpha+1)/\alpha}\Delta s \leqslant 0.\end{aligned} \tag{102}$$

That is

$$\begin{aligned}&V^{\alpha}(t)w(t) + \int_{T_1}^{t}\widetilde{p}(s)V^{\alpha}(\sigma(s))\Delta s\\ &\quad-\int_{T_1}^{t}\Bigg[\alpha V^{\alpha-1}(\sigma(s))\left(-V^{\Delta}(s)\right)(-w(s))\Delta s\\ &\qquad-\frac{\beta M_0^{(\alpha-\beta)/\alpha}V^{\alpha}(\sigma(s))\psi^{\Delta}(s)}{r^{1/\alpha}(\psi(s))}(-w(s))^{(\alpha+1)/\alpha}\Bigg]\Delta s\\ &\leqslant V^{\alpha}(T_1)w(T_1).\end{aligned} \tag{103}$$

Taking $a = \alpha V^{\alpha-1}(\sigma(s))(-V^{\Delta}(s)), b = \beta M_0^{(\alpha-\beta)/\alpha}V^{\alpha}(\sigma(s))\psi^{\Delta}(s)/r^{1/\alpha}(\psi(s))$, by Lemma 3 and (103), we obtain

$$\begin{aligned}&V^{\alpha}(t)w(t) + \int_{T_1}^{t}\widetilde{p}(s)V^{\alpha}(\sigma(s))\Delta s\\ &\quad-\int_{T_1}^{t}\frac{\alpha^{\alpha}r(\psi(s))\left(\alpha V^{\alpha-1}(\sigma(s))\left(-V^{\Delta}(s)\right)\right)^{\alpha+1}}{(\alpha+1)^{\alpha+1}\left(\beta M_0^{(\alpha-\beta)/\alpha}V^{\alpha}(\sigma(s))\psi^{\Delta}(s)\right)^{\alpha}}\Delta s\\ &\leqslant V^{\alpha}(T_1)w(T_1).\end{aligned} \tag{104}$$

That is

$$\begin{aligned}V^{\alpha}(t)w(t) &\leqslant V^{\alpha}(T_1)w(T_1)\\ &\quad-\int_{T_1}^{t}\Bigg[\widetilde{p}(s)V^{\alpha}(\sigma(s))\\ &\qquad-\frac{\alpha^{2\alpha+1}r^{-1/\alpha}(\psi(s))\psi^{\Delta}(s)}{(\alpha+1)^{\alpha+1}\beta^{\alpha}M_0^{\alpha-\beta}V(\sigma(s))}\Bigg]\Delta s.\end{aligned} \tag{105}$$

By (83), we get a contradiction with (91).

Case (ii) (let $\alpha \geqslant 1$). From Lemma 2 and $V^{\Delta}(t) < 0$, we get

$$\begin{aligned}(V^{\alpha}(t))^{\Delta} &= \alpha\left\{\int_0^1 [(1-h)V(t)+hV(\sigma(t))]^{\alpha-1}\mathrm{d}h\right\}V^{\Delta}(t)\\ &\geqslant \alpha\left[\int_0^1 V^{\alpha-1}(t)\,\mathrm{d}h\right]V^{\Delta}(t) = \alpha V^{\alpha-1}(t)V^{\Delta}(t).\end{aligned} \tag{106}$$

From (100) and (106), we obtain

$$\begin{aligned}&V^{\alpha}(t)w(t) - V^{\alpha}(T_1)w(T_1) - \int_{T_1}^{t}\alpha V^{\alpha-1}(s)V^{\Delta}(s)w(s)\Delta s\\ &\quad+\int_{T_1}^{t}\widetilde{p}(s)V^{\alpha}(\sigma(s))\Delta s\\ &\quad+\int_{T_1}^{t}\frac{\beta M_0^{(\alpha-\beta)/\alpha}V^{\alpha}(\sigma(s))\psi^{\Delta}(s)}{r^{1/\alpha}(\psi(s))}(-w(s))^{(\alpha+1)/\alpha}\Delta s \leqslant 0.\end{aligned} \tag{107}$$

That is

$$\begin{aligned}&V^{\alpha}(t)w(t) + \int_{T_1}^{t}\widetilde{p}(s)V^{\alpha}(\sigma(s))\Delta s\\ &\quad-\int_{T_1}^{t}\Bigg[\alpha V^{\alpha-1}(s)\left(-V^{\Delta}(s)\right)(-w(s))\Delta s\\ &\qquad-\frac{\beta M_0^{(\alpha-\beta)/\alpha}V^{\alpha}(\sigma(s))\psi^{\Delta}(s)}{r^{1/\alpha}(\psi(s))}(-w(s))^{(\alpha+1)/\alpha}\Bigg]\Delta s\\ &\leqslant V^{\alpha}(T_1)w(T_1).\end{aligned} \tag{108}$$

Taking $a = \alpha V^{\alpha-1}(s)(-V^{\Delta}(s))$, $b = \beta M_0^{(\alpha-\beta)/\alpha}V^{\alpha}(\sigma(s))\psi^{\Delta}(s)/r^{1/\alpha}(\psi(s))$, by Lemma 3 and (108), we obtain

$$\begin{aligned}&V^{\alpha}(t)w(t) + \int_{T_1}^{t}\widetilde{p}(s)V^{\alpha}(\sigma(s))\Delta s\\ &\quad-\int_{T_1}^{t}\frac{\alpha^{\alpha}r(\psi(s))\left(\alpha V^{\alpha-1}(s)\left(-V^{\Delta}(s)\right)\right)^{\alpha+1}}{(\alpha+1)^{\alpha+1}\left(\beta M_0^{(\alpha-\beta)/\alpha}V^{\alpha}(\sigma(s))\psi^{\Delta}(s)\right)^{\alpha}}\Delta s\\ &\leqslant V^{\alpha}(T_1)w(T_1).\end{aligned} \tag{109}$$

That is

$$\begin{aligned} V^{\alpha}(t)\,w(t) &\leqslant V^{\alpha}(T_1)\,w(T_1) - \int_{T_1}^{t}\Bigg[\widetilde{p}(s)\,V^{\alpha}(\sigma(s)) \\ &\quad - \frac{\alpha^{2\alpha+1}r^{-1/\alpha}(\psi(s))\,V^{\alpha^2-1}(s)\,\psi^{\Delta}(s)}{(\alpha+1)^{\alpha+1}\beta^{\alpha}M_0^{\alpha-\beta}V^{\alpha^2}(\sigma(s))}\Bigg]\Delta s. \end{aligned} \tag{110}$$

By (83), we get a contradiction with (91). This completes the proof of Theorem 9. □

4. Examples

Example 10. Consider the following dynamic equation:

$$\begin{aligned} &\left[\left|\left(x(t)+\frac{1}{1+t^2}x(\delta(t))\right)^{\Delta}\right|^{\alpha-1}\left(x(t)+\frac{1}{1+t^2}x(\delta(t))\right)^{\Delta}\right]^{\Delta} \\ &\quad + \frac{1}{t^2}\left(1+\frac{1}{\delta^2(t)}\right)^{\beta}|x(\delta(t))|^{\beta-1}x(\delta(t)) = 0, \quad t\in\mathbb{T}, \end{aligned} \tag{111}$$

where $\alpha > \beta > 1$ are constants. In (111), $r(t) = 1$, $p(t) = 1/(1+t^2)$, $q(t) = (1/t^2)(1+1/\delta^2(t))^{\beta}$.

If $\mathbb{T} = \overline{q_0^{\mathbb{Z}}} = \{q_0^n : n \in \mathbb{Z}\} \cup \{0\}$, and $\delta(t) = t/q_0$, where $q_0 > 1$ and $q_0 \in \mathbb{R}$, then $\delta^{\Delta}(t) = 1/q_0$. It is easy to get that $\overline{p}(t) = q(t)[1-p(\delta(t))]^{\beta} = 1/t^2$. Choosing $\xi(t) = t$, therefore,

$$\begin{aligned} &\overline{\lim_{t\to\infty}}\int_{t_0}^{t}\Bigg(\xi(s)\,\overline{p}(s) \\ &\quad - \frac{\left(\xi^{\Delta}(s)\right)^2(r(\sigma(s)))^{(\alpha-\beta)/\alpha}(r(\delta(s)))^{\beta/\alpha}}{4\beta\xi(s)\,(\delta(s)/2)^{\beta-1}\delta^{\Delta}(s)\,M^{\alpha-\beta}}\Bigg)\Delta s \\ &= \overline{\lim_{t\to\infty}}\int_{t_0}^{t}\left(\frac{1}{s}-\frac{2^{\beta-1}q_0^{\beta}}{4\beta s^{\beta}M^{(\alpha-\beta)/\alpha}}\right)\Delta s = \infty. \end{aligned} \tag{112}$$

Hence, by Theorem 6, (111) is oscillatory.

Example 11. Consider the following dynamic equation:

$$\begin{aligned} &\left[t^{\alpha}\left|\left(x(t)+\left(1-\frac{1}{1+t^2}\right)x(\delta(t))\right)^{\Delta}\right|^{\alpha-1}\right. \\ &\quad \left.\times\left(x(t)+\left(1-\frac{1}{1+t^2}\right)x(\delta(t))\right)^{\Delta}\right]^{\Delta} \\ &\quad + \frac{1}{t}\left(1+\frac{1}{\delta^2(t)}\right)^{\beta}|x(\delta(t))|^{\beta-1}x(\delta(t)) = 0, \quad t\in\mathbb{T}, \end{aligned} \tag{113}$$

where $\alpha > \beta > 1$. In (113), $r(t) = t^{\alpha}$, $p(t) = 1 - 1/(1+t^2)$, $q(t) = (1/t)(1+\delta^2(t))^{\beta}$.

If $\mathbb{T} = \overline{q_0^{\mathbb{Z}}} = \{q_0^n : n \in \mathbb{Z}\} \cup \{0\}$, and $\delta(t) = t/q_0$, where $q_0 > 1$ and $q_0 \in \mathbb{R}$, then $\delta^{\Delta}(t) = 1/q_0$. It is easy to get that $\overline{p}(t) = q(t)[1-p(\delta(t))]^{\beta} = 1/t$. Choosing $\xi(t) = 1$, $H(t,s) = t - s$, therefore, $(t-s)^{\Delta_s} = -1$,

$$\begin{aligned} &\overline{\lim_{t\to\infty}}\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)\,\xi(s)\,\overline{p}(s)\,\Delta s \\ &= \overline{\lim_{t\to\infty}}\frac{1}{t-t_0}\int_{t_0}^{t}(t-s)\frac{1}{s}\Delta s \\ &= \overline{\lim_{t\to\infty}}\frac{t}{t-t_0}\cdot\frac{1}{t}\int_{t_0}^{t}\frac{t-s}{s}\Delta s \\ &= \infty. \end{aligned} \tag{114}$$

Hence, by Theorem 7, (111) is oscillatory.

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