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Research Article

Growth of Meromorphic Solutions of Some q-Difference Equations

Guowei Zhang

School of Mathematics and Statistics, Anyang Normal University, Anyang, Henan 455000, China

Correspondence should be addressed to Guowei Zhang; zhirobo@gmail.com

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We estimate the growth of the meromorphic solutions of some complex q-difference equations and investigate the convergence exponents of fixed points and zeros of the transcendental solutions of the second order q-difference equation. We also obtain a theorem about the q-difference equation mixing with difference.

1. Introduction and Main Results

In this paper, we mainly use the basic notation of Nevanlinna Theory, such as $T(r,f)$, $N(r,f)$, and $m(r,f)$, and the notation $S(r,f)$ is defined to be any quantity satisfying $S(r,f) = o(T(r,f))$ as $r \to \infty$ possibly outside a set of r of finite linear measure (see [1–3]). In addition, we use the notation $\rho(f)$ to denote the order of growth of the meromorphic function $f(z)$ and $\lambda(f)$ to denote the exponent of convergence of the zeros. We also use the notation $\tau(f)$ to denote the exponent of convergence of fixed points of f. We give the definition of $\tau(f)$ as following.

Definition 1. Let f be a nonconstant meromorphic function. The exponent of convergence of fixed points of f is defined by

$$\tau(f) = \limsup_{r\to\infty} \frac{\log N(r, 1/(f-z))}{\log r}. \tag{1}$$

Recently, a number of papers focused on complex difference equations, such as [4–6] and on difference analogues of Nevanlinna's theory, such as [7, 8]. Correspondingly, there are many papers focused on the q-difference (or c-difference) equations, such as [9–14].

Because of the intimate relations between iteration theory and the functional equations of Schröder, Böttcher and Abel and Bergweiler et al. [10] studied the following functional equation

$$\sum_{j=0}^{n} a_j(z) f(q^j z) = Q(z), \tag{2}$$

where $q (0 < |q| < 1)$ is a complex number and $a_j(z)$, $j = 0, 1, \ldots, n$ and $Q(z)$ are rational functions, and $a_0(z) \not\equiv 0, a_n(z) \equiv 1$. They obtained the following two theorems.

Theorem A. *All meromorphic solutions of* (2) *satisfy* $T(r,f) = O((\log r)^2)$.

Theorem B. *All transcendental meromorphic solutions of* (2) *satisfy* $(\log r)^2 = O(T(r,f))$.

What will happen if the right-hand side of (2) is a rational function in f? That is, for the functional equation

$$\sum_{j=1}^{n} \gamma_j(z) f(q^j z) = R(z, f(z)) = \frac{\sum_{i=0}^{s} \alpha_i(z) f^i(z)}{\sum_{i=0}^{t} \beta_i(z) f^i(z)}, \tag{3}$$

where q $(0 < |q| < 1)$ is a complex number, $\alpha_j(z)$ $(j = 0, 1, \ldots, s)$, $\alpha_s(z) \not\equiv 0$, $\beta_j(z)$ $(j = 0, 1, \ldots, t)$, $\beta_t(z) \not\equiv 0$, $\gamma_j(z)$ $(j = 0, 1, \ldots, n)$, and $\gamma_n = 1$ are coefficients, and $R(z,f)$ is irreducible in f. Gundersen et al. [12] studied the case that $n = 1$ on the left-hand side of (3). Following the results in [12], we continue to study the properties of the solutions of (3) in the case of $n > 1$ on the left-hand side. In fact, we obtain the following theorem.

Theorem 2. *Suppose that f is a nonconstant meromorphic solution of equation of* (3) *and the coefficients are small functions of f. Then, $d = \max\{s, t\} \le n$ and $\rho(f) \le \log(n/d)/(-\log|q|)$.*

In particular, we concern the second-order q-difference equation with rational coefficients, that is, in the case of $n = 2$. From Theorem 2, we know that if f is a nonconstant meromorphic solution, then $d \le 2$. Thus, the second-order q-difference equation is the following form:

$$f\left(q^2 z\right) + \gamma_1(z) f(qz) = \frac{\alpha_0(z) + \alpha_1(z) f(z) + \alpha_2(z) f^2(z)}{\beta_0(z) + \beta_1(z) f(z) + \beta_2(z) f^2(z)}. \tag{4}$$

First of all, we give some remarks.

Remark 3. If $\alpha_2(z)$ and $\beta_2(z)$ are not zero at the same time, by Theorem 2, we derive that the solution of (4) is of order zero.

Remark 4. If $\alpha_2(z) = \beta_2(z) = \beta_1(z) \equiv 0$, by Theorem A, the solutions of (4) is also of order zero.

Remark 5. If $\alpha_2(z) = \beta_2(z) = 0, \beta_1(z) \ne 0$, by Theorem 2, the order of the solutions is less than $\log(2)/(-\log|q|)$. Thus, a question arises: does the equation have a solution which is of order nonzero under this situation? This question is still open.

In [6], Chen and Shon proved some theorems about the properties of solutions of the difference Painlevé I and II equations, such as the exponents of convergence of fixed points and the zeros of transcendental solutions. A natural question arises: how about the exponents of convergence of the fixed points and the zeros of transcendental solutions of the q-difference equation (4)? Do the transcendental solutions have infinitely many fixed points and zeros? The following theorem, in which the coefficients are constants, answers the above questions partly.

Theorem 6. *Suppose that f is a transcendental solution of the equation*

$$f\left(q^2 z\right) + \gamma_1 f(qz) = \frac{\alpha_0 + \alpha_1 f(z) + \alpha_2 f^2(z)}{\beta_0 + \beta_1 f(z) + \beta_2 f^2(z)}, \tag{5}$$

where $|q| < 1$, the coefficients γ_1, α_0, α_1, α_2, β_0, β_1, and β_2 are constants, and at least one of α_2, β_2 is nonzero. Then, $\rho(f) = 0$ and (i) *f has infinitely many fixed points, and* (ii) *f has infinitely many zeros, whenever $\alpha_0 \ne 0$.*

In the rest of the paper, we consider (3) when $|q| > 1$. In [15], Heittokangas et al. considered the essential growth problem for transcendental meromorphic solutions of complex difference equations, which is to find lower bounds for their characteristic functions. Following this idea, Zheng and Chen [14] obtained the following theorem for q-difference equations.

Theorem C. *Suppose that f is a transcendental solution of equation*

$$\sum_{j=1}^{n} a_j(z) f\left(q^j z\right) = R(z, f(z)) = \frac{P(z, f(z))}{Q(z, f(z))}, \tag{6}$$

where $q \in \mathbb{C}$, $|q| > 1$, the coefficients $a_j(z)$ are rational functions, and P, Q are relatively prime polynomials in f over the field of rational functions satisfying $p = \deg_f P$, $t = \deg_f Q$, and $d = p - t \ge 2$. If f has infinitely many poles, then for sufficiently large r, $n(r, f) \ge K d^{\log r/(n \log|q|)}$ holds for some constant $K > 0$. Thus, the lower order of f, which has infinitely many poles, satisfies $\mu(f) \ge \log d/(n \log|q|)$.

Regarding Theorem C, they obtained the lower bound of the order of solutions. Then, how about the upper bound of the order of the solutions? Can the conditions of Theorem C become a little more simple? In fact, we have the following theorem.

Theorem 7. *Suppose that f is a transcendental solution of* (3), *where $|q| > 1$, $n < d = \max\{s, t\}$ and the coefficients are rational functions. Then, $\log(d/n)/(n \log|q|) \le \rho(f) \le \log(d + n - 1)/(\log|q|)$.*

We know that the difference analogues and q-difference analogues of Nevanlinna's theory have been investigated. Consequently, many results on the complex difference equations and q-difference equations have been obtained respectively. Thus, mixing the difference and q-difference equations together is a natural idea. The following Theorem 8 is just a simple application of the above idea, and further investigation is required.

In what follows, we will consider difference products and difference polynomials. By a difference product, we mean a difference monomial, that is, an expression of type

$$\prod_{j}^{s} f\left(z + c_j\right)^{n_j}, \tag{7}$$

where $c_1, \ldots, c_s$ are complex numbers and $n_1, \ldots, n_s$ are natural numbers. A difference polynomial is a finite sum of difference products, that is, an expression of the form

$$P(z, f) = \sum_{\{J\}} b_J(z) \left(\prod_{j \in J} f\left(z + c_j\right) \right), \tag{8}$$

where c_j $(j \in J)$ is a set of distinct complex numbers and the coefficients $b_J(z)$ of difference polynomials are small functions as understood in the usual the Nevanlinna theory; that is, their characteristic is of type $S(r, f)$.

Theorem 8. *Suppose that f is a nonconstant meromorphic solution of the equation*

$$\sum_{i=1}^{n} a_i(z) f\left(q^i z\right) = \sum_{\{J\}} b_J(z) \left(\prod_{j \in J} f\left(z + c_j\right) \right), \tag{9}$$

where $|q| > 1$ and the index set J consists of m elements and the coefficients $a_i(z)$ $(a_n(z) = 1)$ and $b_J(z)$ are small functions of f. If f is of finite order, then $|q| < n + m - 1$.

2. Some Lemmas

The following important result by Valiron and Mohon'ko will be used frequently, one can find the proof in Laine's book [16, page 29].

Lemma 9. *Let f be a meromorphic function. Then, for all irreducible rational function in f,*

$$R(z, f(z)) = \frac{\sum_{j=0}^{p} a_j(z) f(z)^j}{\sum_{j=0}^{q} b_j(z) f(z)^j}, \tag{10}$$

with meromorphic coefficients $a_j(z)$, $b_j(z)$, the characteristic function of $R(z, f(z))$ satisfies

$$T(r, R(z, f(z))) = dT(r, f) + O(\Psi(r)), \tag{11}$$

where $d = \max\{p, q\}$ and

$$\Psi(r) = \max_{i,j}\left\{T\left(r, a_j\right), T\left(r, b_j\right)\right\}. \tag{12}$$

In the particular case when

$$\begin{aligned} T\left(r, a_j\right) &= S(r, f), \quad j = 0, 1, \ldots, p, \\ T\left(r, b_j\right) &= S(r, f), \quad j = 0, 1, \ldots, q, \end{aligned} \tag{13}$$

One has $T(r, R(z, f(z))) = dT(r, f) + S(r, f)$.

The next lemma on the relationship between $T(r, f(qz))$ and $T(|q|r, f(z))$ is due to Bergweiler et al. [10, page 2].

Lemma 10. *One case see that*

$$T(r, f(qz)) = T(|q| r, f) + O(1) \tag{14}$$

holds for any meromorphic function f and any constant q.

Lemma 11 (see [12]). *Let $\Phi : (1, \infty) \to (0, \infty)$ be a monotone increasing function, and let f be a nonconstant meromorphic function. If for some real constant $\alpha \in (0, 1)$, there exist real constants $K_1 > 0$ and $K_2 \geq 1$ such that*

$$T(r, f) \leq K_1 \Phi(\alpha r) + K_2 T(\alpha r, f) + S(\alpha r, f), \tag{15}$$

then

$$\rho(f) \leq \frac{\log K_2}{-\log \alpha} + \limsup_{r \to \infty} \frac{\log \Phi(r)}{\log r}. \tag{16}$$

Lemma 12 (see [9, Theorem 2.2]). *Let $f(z)$ be a nonconstant zero-order meromorphic solution of*

$$P(z, f) = 0, \tag{17}$$

where $P(z, f)$ is a c-difference (or q-difference) equation in $f(z)$. If $P(z, \alpha) \not\equiv 0$, where α is a zero-order meromorphic function such that $T(r, \alpha) = o(T(r, f))$ on a set of logarithmic density 1, and in particular, α is a constant, then

$$m\left(r, \frac{1}{f - \alpha}\right) = o(T(r, f)), \tag{18}$$

on a set of logarithmic density 1.

Lemma 13 (see [7]). *Let f be a meromorphic function of finite order, and let c be a nonzero complex constant. Then one has*

$$m\left(r, \frac{f(z + c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z + c)}\right) = S(r, f). \tag{19}$$

Lemma 14 (see [7]). *Let f be a meromorphic function of finite order ρ, and let c is a nonzero complex constant. Then, for each $\varepsilon > 0$, one has*

$$T(r, f(z + c)) = T(r, f) + O\left(r^{\rho - 1 + \varepsilon}\right) + O(\log r). \tag{20}$$

It is evident that $S(r, f(z + c)) = S(r, f)$ from Lemma 14.

By (7), (8), and Lemmas 13 and 14, Laine and Yang obtained the following lemma in [13].

Lemma 15. *The characteristic function of a difference polynomial $P(z, f)$ in (8) satisfies*

$$T(r, P(z, f)) \leq nT(r, f) + O\left(r^{\rho - 1 + \varepsilon}\right) + S(r, f), \tag{21}$$

provided that f is a meromorphic function of finite order ρ and the index set J consists of n elements.

3. Proof of Theorems

Proof of Theorem 2. Set $\Phi(r) = \max_{i,j,k}\{T(r, \alpha_i(z)), T(r, \beta_j(z)), T(r, \gamma_k(z))\} = S(r, f)$. From (3) and Lemmas 9 and 10 and noting $0 < |q| < 1$, we immediately obtain

$$\begin{aligned} T(r, R(z, f(z))) &= dT(r, f) + O(\Phi(r)) \\ &= T\left(r, \sum_{j=1}^{n} \gamma_j(z) f\left(q^j z\right)\right) \\ &\leq \sum_{i=1}^{n} T\left(|q|^i r, f\right) + O(\Phi(r)) \\ &\leq nT(|q| r, f) + S(r, f) \\ &\leq nT(r, f) + S(r, f), \end{aligned} \tag{22}$$

and so we have $d \leq n$. From (22), we have

$$T(r, f) \leq \frac{n}{d} T(|q| r, f) + S(r, f). \tag{23}$$

Since $S(r, f) = o(T(r, f))$, we have

$$T(r, f) \leq \frac{n}{d(1 - \varepsilon)} T(|q| r, f), \tag{24}$$

for each ε. From (24) and Lemma 11, we have $\rho(f) \leq \log(n/d)/(-\log|q|)$. □

Proof of Theorem 6. Assume that $f(z)$ is a transcendental solution of (5). Since at least one of α_2, β_2 is non-zero, by Remark 3, we obtain that $\rho(f) = 0$.

(I) Set $g(z) = f(z) - z$. Substituting $f(z) = g(z) + z$ into (5), we obtain that

$$g\left(q^2z\right) + \gamma_1 g(qz) + q^2 z + \gamma_1 qz = \frac{\alpha_0 + \alpha_1 g(z) + \alpha_1 z + \alpha_2 (g(z)+z)^2}{\beta_0 + \beta_1 g(z) + \beta_1 z + \beta_2 (g(z)+z)^2}. \tag{25}$$

By (25), we may define

$$\begin{aligned} P_1(z, g(z)) := {} & \left(g\left(q^2z\right) + \gamma_1 g(qz) + q^2 z + \gamma_1 qz\right) \\ & \times \left(\beta_0 + \beta_1 g(z) + \beta_1 z + \beta_2 (g(z)+z)^2\right) \\ & - \alpha_0 + \alpha_1 g(z) + \alpha_1 z + \alpha_2 (g(z)+z)^2. \end{aligned} \tag{26}$$

By (26), we see that

$$\begin{aligned} P_1(z,0) = {} & \left(q^2 + q\gamma_1\right)\beta_2 z^3 + \left[\left(q^2 + q\gamma_1\right)\beta_1 - \alpha_2\right] z^2 \\ & + \left[\left(q^2 + q\gamma_1\right)\beta_0 - \alpha_1\right] z - \alpha_0. \end{aligned} \tag{27}$$

Suppose that $P_1(z,0) \equiv 0$, and we split into two cases. If $q^2 + q\gamma_1 = 0$, then we obtain $\alpha_0 = \alpha_1 = \alpha_2 = 0$. Thus, the right-hand side of (5) is vanishing. This contradicts to our assumption. If $q^2 + q\gamma_1 \neq 0$, we obtain $\alpha_0 = \beta_2 = 0$ and $\alpha_2/\beta_1 = \alpha_1/\beta_0 = q^2 + q\gamma_1$. Then, the right-hand side of (5) becomes $(q^2 + q\gamma_1) f$; this also contradicts to our assumption. Thus, we have $P_1(z,0) \not\equiv 0$. By Lemma 12, we obtain that

$$m\left(r, \frac{1}{g}\right) = o(T(r,f)), \tag{28}$$

on a set of logarithmic density 1. Thus,

$$N\left(r, \frac{1}{f-z}\right) = N\left(r, \frac{1}{g}\right) = (1 - o(1))\, T(r,f), \tag{29}$$

on a set of logarithmic density 1. Hence, by (29), f has infinitely many fixed points and

$$\tau(f) = \lambda(g) = \rho(f). \tag{30}$$

(II) By (5), we derive that

$$\begin{aligned} P_2(z,f) := {} & \left[f\left(q^2z\right) + \gamma_1 f(qz)\right]\left[\beta_0 + \beta_1 f(z) + \beta_2 f^2(z)\right] \\ & - \left[\alpha_0 + \alpha_1 f(z) + \alpha_2 f^2(z)\right]. \end{aligned} \tag{31}$$

By (31) and the assumption $\alpha_0 \neq 0$, we obtain that

$$P_2(z,0) := \alpha_0 \not\equiv 0. \tag{32}$$

By Lemma 12 and (32), we have

$$m\left(r, \frac{1}{f}\right) = o(T(r,f)), \tag{33}$$

on a set of logarithmic density 1. Thus,

$$N\left(r, \frac{1}{f}\right) = (1 - o(1))\, T(r,f), \tag{34}$$

on a set of logarithmic density 1. Hence, by (34), f has infinitely many zeros and

$$\lambda(f) = \rho(f). \tag{35}$$

□

Proof of Theorem 7. From (3), we have

$$f\left(q^n z\right) = \frac{\sum_{i=0}^{s} \alpha_i(z) f^i(z)}{\sum_{i=0}^{t} \beta_i(z) f^i(z)} - \sum_{j=1}^{n-1} \gamma_j(z) f\left(q^j z\right). \tag{36}$$

By the properties of the Nevanlinna characteristic function and Lemma 9, we have

$$\begin{aligned} T\left(r, f\left(q^n z\right)\right) \le {} & dT(r,f) + \sum_{j=1}^{n-1} T\left(r, \gamma_j(z) f\left(q^j z\right)\right) \\ & + O(\log r). \end{aligned} \tag{37}$$

By $|q| > 1$ and Lemma 10, we obtain

$$\begin{aligned} T\left(|q|^n r, f(z)\right) \le {} & dT(r, f(z)) \\ & + \sum_{j=1}^{n-1} T\left(|q|^j r, f(z)\right) + O(\log r) \\ \le {} & (d + n - 1)\, T\left(|q|^{n-1} r, f(z)\right) \\ & + O\left(\log |q|^{n-1} r\right). \end{aligned} \tag{38}$$

Setting $\alpha = 1/|q|$, $R = |q|^n r$, we have

$$T(R, f(z)) \le (d + n - 1)\, T(\alpha R, f(z)) + O(\log \alpha R). \tag{39}$$

Applying Lemma 11 to (39) yields

$$\rho(f) \le \frac{\log(d + n - 1)}{\log|q|}. \tag{40}$$

We now prove the lower bound of the order of the solutions. From (3), by the properties of the Nevanlinna characteristic function and Lemma 9, we have

$$\sum_{j=1}^{n} T\left(r, f\left(q^j z\right)\right) \ge dT(r,f) + O(\log r). \tag{41}$$

By Lemma 10 and noting $|q| > 1$, we obtain

$$T\left(|q|^n r, f\right) \ge kT(r,f) + g(r), \tag{42}$$

where $k = d/n$ and $|g(r)| < K \log r$ for some K and all r greater than some r_0. Hence, for $r > r_0$,

$$\begin{aligned} T\left(|q|^{nj} r, f\right) & \ge kT\left(|q|^{n(j-1)} r, f\right) + g\left(|q|^{n(j-1)} r\right) \\ & \ge \cdots \\ & \ge k^j T(r,f) + E_j(r), \end{aligned} \tag{43}$$

where

$$\left|E_j(r)\right| = \left|k^{j-1}g(r) + k^{j-2}g\left(|q|^n r\right) + \cdots + g\left(|q|^{n(j-1)}r\right)\right|$$
$$\leq Kk^{j-1}\sum_{i=0}^{j-1}\frac{\log|q|^{ni}r}{k^i}$$
$$\leq Kk^{j-1}\left(\log|q|^n\sum_{i=0}^{\infty}\frac{i}{k^i} + \log r\sum_{i=0}^{\infty}\frac{1}{k^i}\right). \quad (44)$$

For r sufficiently large and i, we note that since $k = d/n > 1$, the two series converge, and hence

$$\left|E_j(r)\right| \leq Ck^j \log r, \quad (45)$$

where C is a positive constant. Since f is a transcendental meromorphic function, we can choose r_0 sufficiently large such that for all $r \geq r_0$, by the increasing property of $T(r, f)$, we have $T(r, f) \geq C\log r$ for some constant C. Hence, we get

$$T\left(|q|^{nj}r, f\right) > Ck^j \log r, \quad (46)$$

for some constant C. By the definition of the order of f, we have

$$\rho(f) = \limsup_{j\to\infty,\, r\to\infty}\frac{\log T\left(|q|^{nj}r, f\right)}{\log|q|^{nj}r}$$
$$\geq \frac{\log C + j\log k + \log\log r}{jn\log|q| + \log r}. \quad (47)$$

So we have $\rho(f) \geq \log(d/n)/(n\log|q|)$. Thus, we complete the proof. □

Proof of Theorem 8. Suppose that the order of f is $\rho < \infty$. We rewrite (9) as

$$f(q^n z) = \sum_{\{J\}} b_J(z)\left(\prod_{j\in J} f\left(z + c_j\right)\right) - \sum_{i=1}^{n-1} a_i z f\left(q^i z\right). \quad (48)$$

By the property of the Nevanlinna characteristic function, we have

$$T(r, f(q^n z)) \leq T\left(r, \sum_{\{J\}} b_J(z)\left(\prod_{j\in J} f\left(z + c_j\right)\right)\right)$$
$$+ \sum_{i=1}^{n-1} T\left(r, a_i(z) f\left(q^i z\right)\right) + O(1). \quad (49)$$

By Lemmas 10 and 15, we obtain

$$T\left(|q|^n r, f(z)\right) \leq mT(r, f(z)) + \sum_{i}^{n-1} T\left(|q|^i r, f(z)\right)$$
$$+ O\left(r^{\rho-1+\varepsilon}\right) + S(r, f), \quad (50)$$

for each ε. Since $|q| > 1$, we derive

$$T\left(|q|^n r, f(z)\right) \leq (m + n - 1)\, T\left(|q|^{n-1} r, f(z)\right)$$
$$+ O\left(\left(|q|^{n-1} r\right)^{\rho-1+\varepsilon}\right) + S\left(|q|^{n-1} r, f\right) \quad (51)$$

for each ε. Setting $\alpha = 1/|q|$, $R = |q|^n r$, $\Phi(R) = O((\alpha R)^{\rho-1+\varepsilon})$ and applying Lemma 11 to (51), yield

$$\rho(f) = \rho \leq \frac{\log(m + n - 1)}{\log|q|} + \rho - 1 + \varepsilon, \quad (52)$$

for each ε. Thus, we obtain

$$|q| < m + n - 1. \quad (53)$$

□

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Composition Comments