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Research Article

Solving the Variational Inequality Problem Defined on Intersection of Finite Level Sets

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Consider the variational inequality $\mathrm{VI}(C,F)$ of finding a point $x^* \in C$ satisfying the property $\langle Fx^*, x-x^*\rangle \geq 0$, for all $x \in C$, where C is the intersection of finite level sets of convex functions defined on a real Hilbert space H and $F : H \to H$ is an L-Lipschitzian and η-strongly monotone operator. Relaxed and self-adaptive iterative algorithms are devised for computing the unique solution of $\mathrm{VI}(C,F)$. Since our algorithm avoids calculating the projection P_C (calculating P_C by computing several sequences of projections onto half-spaces containing the original domain C) directly and has no need to know any information of the constants L and η, the implementation of our algorithm is very easy. To prove strong convergence of our algorithms, a new lemma is established, which can be used as a fundamental tool for solving some nonlinear problems.

1. Introduction

The variational inequality problem can mathematically be formulated as the problem of finding a point $x^* \in C$ with the property

$$\langle Fx^*, x-x^*\rangle \geq 0, \quad \forall x \in C, \tag{1}$$

where H is a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, C is a nonempty closed convex subset of H, and $F : C \to H$ is a nonlinear operator. Since its inception by Stampacchia [1] in 1964, the variational inequality problem $\mathrm{VI}(C,F)$ has received much attention due to its applications in a large variety of problems arising in structural analysis, economics, optimization, operations research and engineering sciences; see [1–23] and the references therein. Using the projection technique, one can easily show that $\mathrm{VI}(C,F)$ is equivalent to the fixed-point problem (see, for example, [15]).

Lemma 1. *$x^* \in C$ is a solution of $VI(C,F)$ if and only if $x^* \in C$ satisfies the fixed-point relation:*

$$x^* = P_C(I-\lambda F)x^*, \tag{2}$$

where $\lambda > 0$ is an arbitrary constant, P_C is the orthogonal projection onto C, and I is the identity operator on H.

Recall that an operator $F : C \to H$ is called monotone, if

$$\langle Fx-Fy, x-y\rangle \geq 0 \quad \forall x,y \in C. \tag{3}$$

Moreover, a monotone operator F is called strictly monotone if the equality "=" holds only when $x = y$ in the last relation. It is easy to see that $\mathrm{VI}(C,F)$ (1) has at most one solution if F is strictly monotone.

For variational inequality (1), F is generally assumed to be Lipschitzian and strongly monotone on C; that is, for some constants $L, \eta > 0$, F satisfies the conditions

$$\begin{gathered}\|Fx-Fy\| \leq L\|x-y\|, \quad \forall x,y \in C,\\ \langle Fx-Fy, x-y\rangle \geq \eta\|x-y\|^2, \quad \forall x,y \in C.\end{gathered} \tag{4}$$

In this case, F is also called an L-Lipschitzian and η-strongly monotone operator. It is quite easy to show the simple result as follows.

Lemma 2. *Assume that F satisfies conditions (4) and λ and μ are constants such that $\lambda \in (0,1)$ and $\mu \in (0, 2\eta/L^2)$, respectively. Let $T^\mu = P_C(I-\mu F)$ (or $I-\mu F$) and $T^{\lambda,\mu} = P_C(I-\lambda\mu F)$ (or $I-\lambda\mu F$). Then T^μ and $T^{\lambda,\mu}$ are all contractions* *with coefficients* $1 - \tau$ *and* $1 - \lambda\tau$*, respectively, where* $\tau = (1/2)\mu(2\eta - \mu L^2)$.

Using Banach's contraction mapping principle, the following well-known result can be obtained easily from Lemmas 1 and 2.

Theorem 3. *Assume that F satisfies the conditions (4). Then $VI(C, F)$ has a unique solution. Moreover, for any $0 < \lambda < 2\eta/L^2$, the sequence $\{x_n\}$ with initial guess $x_0 \in C$ and defined recursively by*

$$x_{n+1} = P_C (I - \lambda F) x_n, \quad n \geq 0, \tag{5}$$

converges strongly to the unique solution of $VI(C, F)$.

However, Algorithm (5) has two evident weaknesses. On one hand, Algorithm (5) involves calculating the mapping P_C, while the computation of a projection onto a closed convex subset is generally difficult. If C is the intersection of finite closed convex subsets of H, that is, $C = \bigcap_{i=1}^{m} C_i (\neq \emptyset)$, where C_i $(i = 1, \ldots, m)$ is a closed convex subset of H, then the computation of P_C is much more difficult. On the other hand, the determination of the stepsize λ depends on the constants L and η. This means that in order to implement Algorithm (5), one has first to compute (or estimate) the constants L and η, which is sometimes not an easy work in practice.

In order to overcome the above weaknesses of the algorithm (5), a new relaxed and self-adaptive algorithm is proposed in this paper to solve $\mathrm{VI}(C, F)$, where C is the intersection of finite level sets of convex functions defined on H and $F : H \to H$ is an L-Lipschitzian and η-strongly monotone operator. Our method calculates P_C by computing finite sequences of projections onto half-spaces containing the original set C and selects the stepsizes through a self-adaptive way. The implementation of our algorithm avoids computing P_C directly and has no need to know any information about L and η.

The rest of this paper is organized as follows. Some useful lemmas are listed in the next section; in particular, a new lemma is established in order to prove strong convergence theorems of our algorithms, which can also be used as a fundamental tool for solving some nonlinear problems relating to fixed point. In the last section, a relaxed algorithm (for the case where L and η are known) and a relaxed self-adaptive algorithm (for the case where L and η are not known) are proposed, respectively. The strong convergence theorems of our algorithms are proved.

2. Preliminaries

Throughout the rest of this paper, we denote by H a real Hilbert space and by I the identity operator on H. If $f : H \to \mathbb{R}$ is a differentiable functional, then we denote by ∇f the gradient of f. We will also use the following notations:

(i) $\to$ denotes strong convergence.

(ii) $\rightharpoonup$ denotes weak convergence.

(iii) $\omega_w(x_n) = \{x \mid \exists \{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \rightharpoonup x\}$ denotes the weak ω-limit set of $\{x_n\}$.

Recall a trivial inequality, which is well known and in common use.

Lemma 4. *For all $x, y \in H$, there holds the following relation:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle. \tag{6}$$

Recall that a mapping $T : H \to H$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad x, y \in H. \tag{7}$$

$T : H \to H$ is said to be firmly nonexpansive if, for $x, y \in H$,

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2. \tag{8}$$

The following are characterizations of firmly nonexpansive mappings (see [7] or [24]).

Lemma 5. *Let $T : H \to H$ be an operator. The following statements are equivalent.*

(i) *T is firmly nonexpansive.*

(ii) *$I - T$ is firmly nonexpansive.*

(iii) *$\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty\rangle$, $x, y \in H$.*

We know that the orthogonal projection P_C from H onto a nonempty closed convex subset $C \subset H$ is a typical example of a firmly nonexpansive mapping [7], which is defined by

$$P_C x := \arg\min_{y \in C} \|x - y\|^2, \quad x \in H. \tag{9}$$

It is well known that $P_C x$ is characterized [7] by the inequality (for $x \in H$)

$$P_C x \in C, \quad \langle x - P_C x, y - P_C x\rangle \leq 0, \quad \forall y \in C. \tag{10}$$

It is well known that the following lemma [25] is often used when we analyze the strong convergence of some algorithms for solving some nonlinear problems, such as fixed points of nonlinear mappings, variational inequalities, and split feasibility problems. In fact, this lemma has been regarded as a fundamental tool for solving some nonlinear problems relating to fixed point.

Lemma 6 (see [25]). *Assume (a_n) is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n) a_n + \gamma_n \delta_n, \quad n \geq 0, \tag{11}$$

where (γ_n) is a sequence in $(0, 1)$ and (δ_n) is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$,

(ii) $\limsup_{n \to \infty} \delta_n \leq 0$ *or* $\sum_{n=0}^{\infty} |\gamma_n \delta_n| < \infty$.

Then $\lim_{n \to \infty} a_n = 0$.

In this paper, inspired and encouraged by an idea in [26], we obtain the following lemma. Its key effect on the proofs of our main results will be illustrated in the next section and this may show that this lemma is likely to become a new fundamental tool for solving some nonlinear problems relating to fixed point.

Lemma 7. *Assume (s_n) is a sequence of nonnegative real numbers such that*

$$s_{n+1} \le (1-\gamma_n) s_n + \gamma_n \delta_n + \beta_n, \quad n \ge 0, \tag{12}$$

$$s_{n+1} \le s_n - \eta_n + \alpha_n, \quad n \ge 0, \tag{13}$$

where (γ_n) is a sequence in $(0,1)$, (η_n) is a sequence of nonnegative real numbers and (δ_n), (α_n), and (β_n) are three sequences in $\mathbb{R}$ such that

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$,

(iii) $\lim_{k\to\infty} \eta_{n_k} = 0$ *implies* $\limsup_{k\to\infty} \delta_{n_k} \le 0$ *for any subsequence* $(n_k) \subset (n)$,

(iv) $\limsup_{n\to\infty} (\beta_n/\gamma_n) \le 0$.

Then $\lim_{n\to\infty} s_n = 0$.

Proof. Following and generalizing an idea in [26], we distinguish two cases to prove $s_n \to 0$ as $n \to 0$.

Case 1. (s_n) is eventually decreasing (i.e., there exists $k \ge 0$ such that $s_n > s_{n+1}$ holds for all $n \ge k$). In this case, (s_n) must be convergent, and from (13) it follows that

$$\eta_n \le (s_n - s_{n+1}) + \alpha_n. \tag{14}$$

Noting condition (ii), letting $n \to \infty$ in (14) yields $\eta_n \to 0$ as $n \to \infty$. Using condition (iii), we get that $\limsup_{n\to\infty} \delta_n \le 0$. Noting this together with conditions (i) and (iv), we obtain $s_n \to 0$ by applying Lemma 6 to (12).

Case 2. (s_n) is not eventually decreasing. Hence, we can find an integer n_0 such that $s_{n_0} \le s_{n_0+1}$. Let us now define

$$J_n := \{ n_0 \le k \le n : s_k \le s_{k+1} \}, \quad n > n_0. \tag{15}$$

Obviously, J_n is nonempty and satisfies $J_n \subseteq J_{n+1}$. Let

$$\tau(n) := \max J_n, \quad n > n_0. \tag{16}$$

It is clear that $\tau(n) \to \infty$ as $n \to \infty$ (otherwise, (s_n) is eventually decreasing). It is also clear that $s_{\tau(n)} \le s_{\tau(n)+1}$ for all $n > n_0$. Moreover,

$$s_n \le s_{\tau(n)+1}, \quad \forall n > n_0. \tag{17}$$

In fact, if $\tau_n = n$, then inequity (17) is trivial; if $\tau(n) = n-1$, then $\tau(n)+1 = n$, and (17) is also trivial. If $\tau(n) < n-1$, then there exists an integer $i \ge 2$ such that $\tau(n) + i = n$. Thus we deduce from the definition of $\tau(n)$ that

$$s_{\tau(n)+1} > s_{\tau(n)+2} > \cdots > s_{\tau(n)+i} = s_n, \tag{18}$$

and inequity (17) holds again. Since $s_{\tau(n)} \le s_{\tau(n)+1}$ for all $n > n_0$, it follows from (14) that

$$0 \le \eta_{\tau(n)} \le \alpha_{\tau(n)} \longrightarrow 0, \tag{19}$$

so that $\eta_{\tau(n)} \to 0$ as $n \to \infty$ using condition (ii). Due to the condition (iii), this implies that

$$\limsup_{n\to\infty} \delta_{\tau(n)} \le 0. \tag{20}$$

Noting $s_{\tau(n)} \le s_{\tau(n)+1}$ for all $n > n_0$ again, it follows from (12) that

$$s_{\tau(n)} \le \delta_{\tau(n)} + \frac{\beta_{\tau(n)}}{\gamma_{\tau(n)}}. \tag{21}$$

Combining (20), (21), and condition (iv) yields

$$\limsup_{n\to\infty} s_{\tau(n)} \le 0, \tag{22}$$

and hence $s_{\tau(n)} \to 0$ as $n \to \infty$. This together with (13) implies that

$$s_{\tau(n)+1} \le s_{\tau(n)} - \eta_{\tau(n)} + \alpha_{\tau(n)} \longrightarrow 0, \tag{23}$$

which together with (17), in turn, implies that $s_n \to 0$ as $n \to \infty$. □

The following result is just a special case of Lemma 7, that is, the case where $\beta_n = 0$ for all $n \ge 0$.

Lemma 8. *Assume (s_n) is a sequence of nonnegative real numbers such that*

$$\begin{aligned} s_{n+1} &\le (1-\gamma_n) s_n + \gamma_n \delta_n, \quad n \ge 0, \\ s_{n+1} &\le s_n - \eta_n + \alpha_n, \quad n \ge 0, \end{aligned} \tag{24}$$

where (γ_n) is a sequence in $(0,1)$, (η_n) is a sequence of nonnegative real numbers, and (δ_n) and (α_n) are two sequences in $\mathbb{R}$ such that

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$,

(iii) $\lim_{k\to\infty} \eta_{n_k} = 0$ *implies* $\limsup_{k\to\infty} \delta_{n_k} \le 0$ *for any subsequence* $(n_k) \subset (n)$.

Then $\lim_{n\to\infty} s_n = 0$.

Recall that a function $f : H \to \mathbb{R}$ is called convex if

$$\begin{aligned} f(\lambda x + (1-\lambda) y) &\le \lambda f(x) + (1-\lambda) f(y), \\ &\forall \lambda \in (0,1), \forall x, y \in H. \end{aligned} \tag{25}$$

A differentiable function f is convex if and only if there holds the following relation:

$$f(z) \ge f(x) + \langle \nabla f(x), z - x \rangle, \quad \forall z \in H. \tag{26}$$

Recall that an element $g \in H$ is said to be a subgradient of $f : H \to \mathbb{R}$ at x if

$$f(z) \ge f(x) + \langle g, z - x \rangle, \quad \forall z \in H. \tag{27}$$

A function $f : H \to \mathbb{R}$ is said to be subdifferentiable at x, if it has at least one subgradient at x. The set of subgradients of f at the point x is called the subdifferential of f at x and is denoted by $\partial f(x)$. The last relation above is called the subdifferential inequality of f at x. A function f is called subdifferentiable, if it is subdifferentiable at all $x \in H$. If a function f is differentiable and convex, then its gradient and subgradient coincide.

Recall that a function $f : H \to \mathbb{R}$ is said to be weakly lower semicontinuous (w-lsc) at x if $x_n \rightharpoonup x$ implies

$$f(x) \le \liminf_{n\to\infty} f(x_n). \tag{28}$$

3. Iterative Algorithms

In this section, we consider the iterative algorithms for solving a particular kind of variational inequality (1) in which the closed convex subset C is of the particular structure, that is the intersection of finite level sets of convex functions given as follows:

$$C = \bigcap_{i=1}^{m} \{x \in H : c_i(x) \le 0\}, \tag{29}$$

where m is a positive integer and $c_i : H \to \mathbb{R}$ ($i = 1, \ldots, m$) is a convex function. We always assume that c_i ($i = 1, \ldots, m$) is subdifferentiable on H and ∂c_i ($i = 1, \ldots, m$) is a bounded operator (i.e., bounded on bounded sets). It is worth noting that every convex function defined on a finite-dimensional Hilbert space is subdifferentiable and its subdifferential operator is a bounded operator (see [27, Corollary 7.9]). We also assume that $F : H \to H$ is an L-Lipschitzian and η-strongly monotone operator. It is well known that in this case $\mathrm{VI}(C, F)$ has a unique solution, henceforth, which is denoted by x^*.

Without loss of the generality, we will consider only the case $m = 2$; that is, $C = C^1 \bigcap C^2$, where

$$\begin{aligned} C^1 &= \{x \in H : c_1(x) \le 0\}, \\ C^2 &= \{x \in H : c_2(x) \le 0\}. \end{aligned} \tag{30}$$

All of our results can be extended easily to the general case.

The computation of a projection onto a closed convex subset is generally difficult. To overcome this difficulty, Fukushima [21] suggested a way to calculate the projection onto a level set of a convex function by computing a sequence of projections onto half-spaces containing the original level set. This idea is followed by Yang [28] and López et al. [29], respectively, who introduced the relaxed CQ algorithms for solving the split feasibility problem in finite-dimensional and infinite-dimensional Hilbert spaces, respectively. This idea is also used by Censor et al. [30] in the subgradient extragradient method for solving variational inequalities in a Hilbert space.

We are now in a position to introduce a relaxed algorithm for computing the unique solution x^* of $\mathrm{VI}(C, F)$, where $C = C^1 \bigcap C^2$ and C^i ($i = 1, 2$) is given as in (30). This scheme applies to the case where L and η are easy to be determined.

Algorithm 1. Choose an arbitrary initial guess $x_0 \in H$. The sequence (x_n) is constructed via the formula

$$x_{n+1} = P_{C_n^2} P_{C_n^1} (I - \lambda_n \mu F) x_n, \quad n \geqslant 0, \tag{31}$$

where

$$\begin{aligned} C_n^1 &= \{x \in H : c_1(x_n) \le \langle \xi_n^1, x_n - x\rangle\}, \\ C_n^2 &= \{x \in H : c_2(P_{C_n^1} x_n) \le \langle \xi_n^2, P_{C_n^1} x_n - x\rangle\}, \end{aligned} \tag{32}$$

where $\xi_n^1 \in \partial c_1(x_n)$, $\xi_n^2 \in \partial c_2(P_{C_n^1} x_n)$, the sequence (λ_n) is in $(0, 1)$, and μ is a constant such that $\mu \in (0, 2\eta/L^2)$.

We now analyze strong convergence of Algorithm 1, which also illustrates the application of Lemma 7 (or Lemma 8).

Theorem 9. *Assume that $\lambda_n \to 0$ ($n \to \infty$) and $\sum_{n=1}^{+\infty} \lambda_n = +\infty$. Then the sequence (x_n) generated by Algorithm 1 converges strongly to the unique solution x^* of $VI(C, F)$.*

Proof. Firstly, we verify that (x_n) is bounded. Indeed, it is easy to see from the subdifferential inequality and the definitions of C_n^1 and C_n^2 that $C_n^1 \supset C^1$ and $C_n^2 \supset C^2$ hold for all $n \ge 0$, and hence it follows that $C_n^1 \bigcap C_n^2 \supset C^1 \bigcap C^2 = C$. Since the projection operators $P_{C_n^1}$ and $P_{C_n^2}$ are nonexpansive, we obtain from (31), Lemmas 2 and 4 that

$$\begin{aligned} &\|x_{n+1} - x^*\|^2 \\ &\quad = \|P_{C_n^2} P_{C_n^1} (I - \lambda_n \mu F) x_n - P_{C_n^2} P_{C_n^1} x^*\|^2 \\ &\quad \le \|(I - \lambda_n \mu F) x_n - (I - \lambda_n \mu F) x^* - \lambda_n \mu F x^*\|^2 \\ &\quad \le (1 - \tau\lambda_n) \|x_n - x^*\|^2 \\ &\qquad - 2\lambda_n \mu \langle F x^*, x_n - x^* - \lambda_n \mu F x_n\rangle \\ &\quad \le (1 - \tau\lambda_n) \|x_n - x^*\|^2 - 2\lambda_n \mu \langle F x^*, x_n - x^*\rangle \\ &\qquad + 2\lambda_n^2 \mu^2 \|F x^*\| \|F x_n\|, \end{aligned} \tag{33}$$

$$\begin{aligned} &\|x_{n+1} - x^*\|^2 \\ &\quad = \|P_{C_n^2} P_{C_n^1} (I - \lambda_n \mu F) x_n - P_{C_n^2} P_{C_n^1} (I - \lambda_n \mu F) x^* \\ &\qquad + P_{C_n^2} P_{C_n^1} (I - \lambda_n \mu F) x^* - P_{C_n^2} P_{C_n^1} x^*\|^2 \\ &\quad \le (1 - \tau\lambda_n) \|x_n - x^*\|^2 \\ &\qquad + 2 \langle P_{C_n^2} P_{C_n^1} (I - \lambda_n \mu F) x^* - P_{C_n^2} P_{C_n^1} x^*, x_{n+1} - x^*\rangle \\ &\quad \le (1 - \tau\lambda_n) \|x_n - x^*\|^2 + 2\lambda_n \mu \|F x^*\| \|x_{n+1} - x^*\| \\ &\quad \le (1 - \tau\lambda_n) \|x_n - x^*\|^2 + \frac{1}{4}\tau\lambda_n \|x_{n+1} - x^*\|^2 \\ &\qquad + 4\lambda_n \frac{\mu^2}{\tau} \|F x^*\|^2, \end{aligned} \tag{34}$$

where $\tau = (1/2)\mu(2\eta - \mu L^2)$.

Consequently

$$\|x_{n+1}-x^*\|^2 \le \frac{1-\tau\lambda_n}{1-(1/4)\tau\lambda_n}\|x_n-x^*\|^2 + \frac{(3/4)\tau\lambda_n}{1-(1/4)\tau\lambda_n}\frac{16\mu^2}{3\tau^2}\|Fx^*\|^2. \quad (35)$$

It turns out that

$$\|x_{n+1}-x^*\| \le \max\left\{\|x_n-x^*\|, \frac{4\mu}{\sqrt{3}\tau}\|Fx^*\|\right\}, \quad (36)$$

inductively

$$\|x_n-x^*\| \le \max\left\{\|x_0-x^*\|, \frac{4\mu}{\sqrt{3}\tau}\|Fx^*\|\right\}, \quad (37)$$

and this means that (x_n) is bounded. Obviously, (Fx_n) is also bounded.

Secondly, since a projection is firmly nonexpansive, we obtain

$$\begin{aligned}\left\|P_{C_n^2}P_{C_n^1}x_n - P_{C_n^2}P_{C_n^1}x^*\right\|^2 &\le \left\|P_{C_n^1}x_n - P_{C_n^1}x^*\right\|^2 - \left\|P_{C_n^1}x_n - P_{C_n^2}P_{C_n^1}x_n\right\|^2 \\ &\le \|x_n-x^*\|^2 - \left\|x_n - P_{C_n^1}x_n\right\|^2 - \left\|P_{C_n^1}x_n - P_{C_n^2}P_{C_n^1}x_n\right\|^2;\end{aligned} \quad (38)$$

thus we also have

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \left\|P_{C_n^2}P_{C_n^1}(I-\lambda_n\mu F)x_n - P_{C_n^2}P_{C_n^1}x^*\right\|^2 \\ &= \left\|P_{C_n^2}P_{C_n^1}(I-\lambda_n\mu F)x_n - P_{C_n^2}P_{C_n^1}x_n + P_{C_n^2}P_{C_n^1}x_n - P_{C_n^2}P_{C_n^1}x^*\right\|^2 \\ &\le \left\|P_{C_n^2}P_{C_n^1}x_n - P_{C_n^2}P_{C_n^1}x^*\right\|^2 + 2\lambda_n\mu\|Fx_n\|\cdot\|x_n-x^*\| + \lambda_n^2\mu^2\|Fx_n\|^2 \\ &\le \left\|P_{C_n^2}P_{C_n^1}x_n - P_{C_n^2}P_{C_n^1}x^*\right\|^2 + \lambda_n M,\end{aligned} \quad (39)$$

where M is a positive constant such that $M \ge \sup_n\{2\mu\|Fx_n\|\cdot\|x_n-x^*\| + \lambda_n\mu^2\|Fx_n\|^2\}$. The combination of (38) and (39) leads to

$$\|x_{n+1}-x^*\|^2 \le \|x_n-x^*\|^2 - \left\|x_n-P_{C_n^1}x_n\right\|^2 - \left\|P_{C_n^1}x_n - P_{C_n^2}P_{C_n^1}x_n\right\|^2 + \lambda_n M. \quad (40)$$

Setting

$$\begin{aligned} s_n &= \|x_n-x^*\|^2, \qquad \gamma_n = \tau\lambda_n, \qquad \alpha_n = M\lambda_n, \\ \delta_n &= -\frac{2\mu}{\tau}\langle Fx^*, x_n-x^*\rangle + \frac{2\lambda_n\mu^2}{\tau}\|Fx^*\|\,\|Fx_n\|, \\ \eta_n &= \left\|x_n-P_{C_n^1}x_n\right\|^2 + \left\|P_{C_n^1}x_n - P_{C_n^2}P_{C_n^1}x_n\right\|^2,\end{aligned} \quad (41)$$

then (33) and (40) can be rewritten as the following forms, respectively:

$$\begin{aligned} s_{n+1} &\le (1-\gamma_n)s_n + \gamma_n\delta_n, \\ s_{n+1} &\le s_n - \eta_n + \alpha_n.\end{aligned} \quad (42)$$

Finally, observing that the conditions $\lambda_n \to 0$ and $\sum_{n=1}^{\infty}\lambda_n = \infty$ imply $\alpha_n \to 0$ and $\sum_{n=1}^{\infty}\gamma_n = \infty$, respectively, in order to complete the proof using Lemma 7 (or Lemma 8), it suffices to verify that

$$\lim_{k\to\infty}\eta_{n_k} = 0 \quad (43)$$

implies

$$\limsup_{k\to\infty}\delta_{n_k} \le 0 \quad (44)$$

for any subsequence $(n_k) \subset (n)$. In fact, if $\eta_{n_k} \to 0$ as $k \to \infty$, then $\|x_{n_k} - P_{C_{n_k}^1}x_{n_k}\| \to 0$ and $\|P_{C_{n_k}^1}x_{n_k} - P_{C_{n_k}^2}P_{C_{n_k}^1}x_{n_k}\| \to 0$ hold. Since ∂c_1 and ∂c_2 are bounded on bounded sets, we have two positive constants κ_1 and κ_2 such that $\|\xi_{n_k}^1\| \le \kappa_1$ and $\|\xi_{n_k}^2\| \le \kappa_2$ for all $k \ge 0$ (noting that $(P_{C_{n_k}^1}x_{n_k})$ is also bounded due to the fact that $\|P_{C_{n_k}^1}x_{n_k} - x^*\| = \|P_{C_{n_k}^1}x_{n_k} - P_{C_{n_k}^1}x^*\| \le \|x_{n_k}-x^*\|$). From (32) and the trivial fact that $P_{C_{n_k}^1}x_{n_k} \in C_{n_k}^1$ and $P_{C_{n_k}^2}P_{C_{n_k}^1}x_{n_k} \in C_{n_k}^2$, it follows that

$$c_1(x_{n_k}) \le \left\langle \xi_{n_k}^1, x_{n_k} - P_{C_{n_k}^1}x_{n_k}\right\rangle \le \kappa_1\left\|x_{n_k} - P_{C_{n_k}^1}x_{n_k}\right\|, \quad (45)$$

$$\begin{aligned} c_2\left(P_{C_{n_k}^1}x_{n_k}\right) &\le \langle \xi_{n_k}^2, P_{C_{n_k}^1}x_{n_k} - P_{C_{n_k}^2}P_{C_{n_k}^1}x_{n_k}\rangle \\ &\le \kappa_2\left\|P_{C_{n_k}^1}x_{n_k} - P_{C_{n_k}^2}P_{C_{n_k}^1}x_{n_k}\right\|.\end{aligned} \quad (46)$$

Now if $x' \in \omega_w(x_{n_k})$, and (x_{n_k}) such that $x_{n_k} \rightharpoonup x'$ without loss of the generality, then the w-lsc and (45) imply that

$$c_1(x') \le \liminf_{k\to\infty} c_1(x_{n_k}) \le 0. \quad (47)$$

This means that $x' \in C^1$ holds. On the other hand, noting $\|x_{n_k} - P_{C_{n_k}^1}x_{n_k}\| \to 0$, we can assert that $P_{C_{n_k}^1}x_{n_k} \rightharpoonup x'$ and have from the w-lsc and (46) that

$$c_2(x') \le \liminf_{k\to\infty} c_2\left(P_{C_{n_k}^1}x_{n_k}\right) \le 0. \quad (48)$$

This, in turn, implies that $x' \in C^2$. Moreover, we obtain that $x' \in C^1 \bigcap C^2$ and hence $\omega_w(x_{n_k}) \subset C^1 \bigcap C^2 = C$.

Noting x^* is the unique solution of $\mathrm{VI}(C,F)$, it turns out that

$$\begin{aligned}\limsup_{k\to\infty}\left\{-\frac{2\mu}{\tau}\left\langle Fx^*, x_{n_k}-x^*\right\rangle\right\} &= -\frac{2\mu}{\tau}\liminf_{k\to\infty}\left\langle Fx^*, x_{n_k}-x^*\right\rangle \\ &= -\frac{2\mu}{\tau}\inf_{w\in\omega_w(x_{n_k})}\langle Fx^*, w-x^*\rangle \le 0.\end{aligned} \quad (49)$$

Since $\lambda_n \to 0$ and (Fx_n) is bounded, it is easy to see that $\limsup_{k\to\infty}\delta_{n_k} \le 0$. □

Observing that in Algorithm 1 the determination of the stepsize μ still depends on the constants L and η; this means that in order to implement Algorithm 1, one has first to estimate the constants L and η, which is sometimes not an easy work in practice.

To overcome this difficulty, we furthermore introduce a so-called relaxed and self-adaptive algorithm, that is, a modification of Algorithm 1, in which the stepsize is selected through a self-adaptive way that has no connection with the constants L and η.

Algorithm 2. Choose an arbitrary initial guess $x_0 \in H$ and an arbitrary element $x_1 \in H$ such that $x_1 \neq x_0$. Assume that the nth iterate x_n $(n \ge 1)$ has been constructed. Continue and calculate the $(n+1)$th iterate x_{n+1} via the following formula:

$$x_{n+1} = P_{C_n^2}P_{C_n^1}\left(I - \lambda_n\mu_n F\right)x_n, \quad n \ge 1, \tag{50}$$

where C_n^1 and C_n^2 are given as in (32), the sequence (λ_n) is in $(0,1)$, and the sequence (μ_n) is determined via the following relation:

$$\mu_n = \begin{cases} \dfrac{\langle Fx_n - Fx_{n-1}, x_n - x_{n-1}\rangle}{\|Fx_n - Fx_{n-1}\|^2}, & \text{if } x_n \neq x_{n-1}, \\ \mu_{n-1}, & \text{if } x_n = x_{n-1}, \end{cases} \quad n \ge 1. \tag{51}$$

Firstly, we show that the sequence (x_n) is well defined. Noting strong monotonicity of F, $x_1 \neq x_0$ implies that $Fx_1 \neq Fx_0$ and μ_1 is well defined via the first formula of (51). Consequently, μ_n $(n \ge 2)$ is well defined inductively according to (51) and thus the sequence (x_n) is also well defined.

Next, we estimate (μ_n) roughly. If $x_n \neq x_{n-1}$ (that is, $Fx_n \neq Fx_{n-1}$), set

$$\eta_n = \frac{\langle Fx_n - Fx_{n-1}, x_n - x_{n-1}\rangle}{\|x_n - x_{n-1}\|^2}, \qquad L_n = \frac{\|Fx_n - Fx_{n-1}\|}{\|x_n - x_{n-1}\|}, \quad n \ge 1. \tag{52}$$

Obviously, it turns out that

$$\eta \le \eta_n = \frac{\langle Fx_n - Fx_{n-1}, x_n - x_{n-1}\rangle}{\|x_n - x_{n-1}\|^2} \le \frac{\|Fx_n - Fx_{n-1}\|}{\|x_n - x_{n-1}\|} = L_n \le L. \tag{53}$$

Consequently

$$\frac{\eta}{L^2} \le \mu_n = \frac{\eta_n}{L_n^2} \le \frac{1}{\eta_n} \le \frac{1}{\eta}. \tag{54}$$

By the definition of (μ_n), we can assert that (54) holds for all $n \ge 1$.

Lemma 7 (or Lemma 8) is also important for the proof of the strong convergence of Algorithm 2.

Theorem 10. *Assume that $\lambda_n \to 0$ $(n \to \infty)$ and $\sum_{n=1}^{+\infty}\lambda_n = +\infty$. Then the sequence (x_n) generated by Algorithm 2 converges strongly to the unique solution x^* of $VI(C,F)$.*

Proof. Setting $\gamma_n = \lambda_n\mu_n$ and $\beta_n = (1/2)(2\eta - \gamma_n L^2)$, it concludes observing $\lambda_n \to 0$ and (54) that there exists some positive integer n_0 such that

$$0 < \gamma_n < \frac{\eta}{L^2}, \quad n \ge n_0, \tag{55}$$

and consequently

$$\beta_n \ge \frac{1}{2}\eta, \quad n \ge n_0. \tag{56}$$

Using Lemma 2, we have from (55) that $P_{C_n}(I - \gamma_n F)$ (so is $I - \gamma_n F$) is a contraction with coefficient $1 - \gamma_n\beta_n$. This concludes that, for all $n \ge n_0$,

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \left\|P_{C_n^2}P_{C_n^1}(I - \gamma_n F)x_n - P_{C_n^2}P_{C_n^1}x^*\right\|^2 \\ &\le \left\|(I - \gamma_n F)x_n - (I - \gamma_n F)x^* - \gamma_n Fx^*\right\|^2 \\ &\le (1 - \gamma_n\beta_n)\|x_n - x^*\|^2 \\ &\quad - 2\gamma_n\langle Fx^*, x_n - x^* - \gamma_n Fx_n\rangle, \\ &\le (1 - \gamma_n\beta_n)\|x_n - x^*\|^2 - 2\gamma_n\langle Fx^*, x_n - x^*\rangle \\ &\quad + 2\gamma_n^2\|Fx^*\|\|Fx_n\|, \end{aligned} \tag{57}$$

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \left\|P_{C_n^2}P_{C_n^1}(I - \gamma_n F)x_n - P_{C_n^2}P_{C_n^1}(I - \gamma_n F)x^*\right. \\ &\quad \left. + P_{C_n^2}P_{C_n^1}(I - \gamma_n F)x^* - P_{C_n^2}P_{C_n^1}x^*\right\|^2 \\ &\le (1 - \gamma_n\beta_n)\|x_n - x^*\|^2 \\ &\quad + 2\left\langle P_{C_n^2}P_{C_n^1}(I - \gamma_n F)x^* - P_{C_n^2}P_{C_n^1}x^*, x_{n+1} - x^*\right\rangle \\ &\le (1 - \gamma_n\beta_n)\|x_n - x^*\|^2 + 2\gamma_n\|Fx^*\|\|x_{n+1} - x^*\| \\ &\le (1 - \gamma_n\beta_n)\|x_n - x^*\|^2 \\ &\quad + \frac{\gamma_n\beta_n}{4}\|x_{n+1} - x^*\|^2 + \frac{4\gamma_n}{\beta_n}\|Fx^*\|^2, \end{aligned} \tag{58}$$

and hence

$$\|x_{n+1}-x^*\|^2 \le \frac{1-\gamma_n\beta_n}{1-(1/4)\gamma_n\beta_n}\|x_n-x^*\|^2 + \frac{(3/4)\gamma_n\beta_n}{1-(1/4)\gamma_n\beta_n}\frac{16}{3\beta_n^2}\|Fx^*\|^2. \tag{59}$$

Using (56), it turns out that

$$\|x_{n+1}-x^*\| \le \max\left\{\|x_n-x^*\|, \frac{8}{\sqrt{3}\eta}\|Fx^*\|\right\}, \quad n \ge n_0, \tag{60}$$

inductively

$$\|x_n-x^*\| \le \max\left\{\|x_{n_0}-x^*\|, \frac{8}{\sqrt{3}\eta}\|Fx^*\|\right\}, \quad n \ge n_0, \tag{61}$$

and this means that (x_n) is bounded, so is (Fx_n).

By an argument similar to getting (38)–(40), we have

$$\|x_{n+1}-x^*\|^2 \le \|x_n-x^*\|^2 - \left\|x_n-P_{C_n^1}x_n\right\|^2 - \left\|P_{C_n^1}x_n-P_{C_n^2}P_{C_n^1}x_n\right\|^2 + \gamma_n M, \tag{62}$$

where M is a positive constant. Setting

$$\begin{gathered} s_n = \|x_n-x^*\|^2, \\ \delta_n = -\frac{2}{\beta_n}\langle Fx^*, x_n-x^*\rangle + \frac{2\gamma_n}{\beta_n}\|Fx^*\|\|Fx_n\|, \\ \alpha_n = M\gamma_n, \\ \sigma_n = \left\|x_n-P_{C_n^1}x_n\right\|^2 + \left\|P_{C_n^1}x_n-P_{C_n^2}P_{C_n^1}x_n\right\|^2, \end{gathered} \tag{63}$$

then (57) and (62) can be rewritten as the following forms, respectively:

$$\begin{gathered} s_{n+1} \le (1-\gamma_n\beta_n)s_n + \gamma_n\beta_n\delta_n, \\ s_{n+1} \le s_n - \sigma_n + \alpha_n. \end{gathered} \tag{64}$$

Clearly, $\lambda_n \to 0$ and $\sum_{n=1}^{\infty}\lambda_n = \infty$, together with (54) and (56), imply that $\alpha_n \to 0$ and $\sum_{n=1}^{\infty}\gamma_n\beta_n = \infty$.

By an argument very similar to the proof of Theorem 9, it is not difficult to verify that

$$\lim_{k\to\infty}\sigma_{n_k} = 0 \tag{65}$$

implies

$$\limsup_{k\to\infty}\delta_{n_k} \le 0 \tag{66}$$

for any subsequence $(n_k) \subset (n)$. Thus we can complete the proof by using Lemma 7 (or Lemma 8). □

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References

[1] G. Stampacchia, "Formes bilineaires coercivites sur les ensembles convexes," *Comptes Rendus de l'Académie des Sciences*, vol. 258, pp. 4413–4416, 1964.

[2] C. Baiocchi and A. Capelo, *Variational and Quasivariational Inequalities*, John Wiley & Sons, New York, NY, USA, 1984.

[3] A. Bnouhachem, "A self-adaptive method for solving general mixed variational inequalities," *Journal of Mathematical Analysis and Applications*, vol. 309, no. 1, pp. 136–150, 2005.

[4] H. Brezis, *Operateurs Maximaux Monotone et Semigroupes de Contractions dans les Espace d'Hilbert*, North-Holland, Amsterdam, The Netherlands, 1973.

[5] R. W. Cottle, F. Giannessi, and J. L. Lions, *Variational Inequalities and Complementarity Problems: Theory and Application*, John Wiley & Sons, New York, NY, USA, 1980.

[6] M. Fukushima, "Equivalent differentiable optimization problems and descent methods for asymmetric variational inequality problems," *Mathematical Programming A*, vol. 53, no. 1, pp. 99–110, 1992.

[7] K. Goebel and S. Reich, *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*, vol. 83, Marcel Dekker, New York, NY, USA, 1984.

[8] F. Giannessi, A. Maugeri, and P. M. Pardalos, *Equilibrium Problems: Nonsmooth Optimization and Variational Inequality Models*, Kluwer Academic, Dodrecht, The Netherlands, 2001.

[9] R. Glowinski, J. L. Lions, and R. Tremolier, *Numerical Analysis of Variational Inequalities*, vol. 8, North-Holland, The Netherlands, Amsterdam, 1981.

[10] P. T. Harker and J. S. Pang, "Finite-dimensional variational inequality and nonlinear complementarity problems: a survey of theory, algorithms and applications," *Mathematical Programming B*, vol. 48, no. 2, pp. 161–220, 1990.

[11] B. S. He, "A class of implicit methods for monotone variational inequalities," Reports of the Institute of Mathematics 95-1, Nanjing University, Nanjing, China, 1995.

[12] B. S. He and L. Z. Liao, "Improvements of some projection methods for monotone nonlinear variational inequalities," *Journal of Optimization Theory and Applications*, vol. 112, no. 1, pp. 111–128, 2002.

[13] B. S. He, Z. H. Yang, and X. M. Yuan, "An approximate proximal-extragradient type method for monotone variational inequalities," *Journal of Mathematical Analysis and Applications*, vol. 300, no. 2, pp. 362–374, 2004.

[14] S. He and H. K. Xu, "Variational inequalities governed by boundedly Lipschitzian and strongly monotone operators," *Fixed Point Theory*, vol. 10, no. 2, pp. 245–258, 2009.

[15] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and their Applications*, SIAM, Philadelphia, Pa, USA, 2000.

[16] J. L. Lions and G. Stampacchia, "Variational inequalities," *Communications on Pure and Applied Mathematics*, vol. 20, pp. 493–519, 1967.

[17] H. K. Xu and T. H. Kim, "Convergence of hybrid steepest-descent methods for variational inequalities," *Journal of Optimization Theory and Applications*, vol. 119, no. 1, pp. 185–201, 2003.

[18] H. K. Xu, "Viscosity approximation methods for nonexpansive mappings," *Journal of Mathematical Analysis and Applications*, vol. 298, no. 1, pp. 279–291, 2004.

[19] H. Yang and M. G. H. Bell, "Traffic restraint, road pricing and network equilibrium," *Transportation Research B*, vol. 31, no. 4, pp. 303–314, 1997.

[20] I. Yamada, "The hybrid steepest descent method for the variational inequality problem over the intersection of fixed point sets of nonexpansive mappings," in *Inherently Parallel Algorithms in Feasibility and Optimization and Their Applications*, D. Butnariu, Y. Censor, and S. Reich, Eds., vol. 8, pp. 473–504, North-Holland, Amsterdam, The Netherlands, 2001.

[21] M. Fukushima, "A relaxed projection method for variational inequalities," *Mathematical Programming*, vol. 35, no. 1, pp. 58–70, 1986.

[22] L. C. Ceng, Q. H. Ansari, and J. C. Yao, "Mann-type steepest-descent and modified hybrid steepest-descent methods for variational inequalities in Banach spaces," *Numerical Functional Analysis and Optimization*, vol. 29, no. 9-10, pp. 987–1033, 2008.

[23] L. C. Ceng, M. Teboulle, and J. C. Yao, "Weak convergence of an iterative method for pseudomonotone variational inequalities and fixed-point problems," *Journal of Optimization Theory and Applications*, vol. 146, no. 1, pp. 19–31, 2010.

[24] K. Goebel and W. A. Kirk, *Topics on Metric Fixed Point Theory*, Cambridge University Press, Cambridge, UK, 1990.

[25] H. K. Xu, "Iterative algorithms for nonlinear operators," *Journal of the London Mathematical Society*, vol. 66, no. 1, pp. 240–256, 2002.

[26] P. E. Maingé, "A hybrid extragradient-viscosity method for monotone operators and fixed point problems," *SIAM Journal on Control and Optimization*, vol. 47, no. 3, pp. 1499–1515, 2008.

[27] H. H. Bauschke and J. M. Borwein, "On projection algorithms for solving convex feasibility problems," *SIAM Review*, vol. 38, no. 3, pp. 367–426, 1996.

[28] Q. Yang, "The relaxed CQ algorithm solving the split feasibility problem," *Inverse Problems*, vol. 20, no. 4, pp. 1261–1266, 2004.

[29] G. López, V. Martín-Márquez, F. Wang, and H. K. Xu, "Solving the split feasibility problem without prior knowledge of matrix norms," *Inverse Problems*, vol. 28, no. 8, p. 085004, 18, 2012.

[30] Y. Censor, A. Gibali, and S. Reich, "The subgradient extragradient method for solving variational inequalities in Hilbert space," *Journal of Optimization Theory and Applications*, vol. 148, no. 2, pp. 318–335, 2011.