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Research Article

The $L\omega$-Compactness in $L\omega$-Spaces

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The concepts of $\alpha\omega$-remote neighborhood family, $\gamma\omega$-cover, and $L\omega$-compactness are defined in $L\omega$-spaces. The characterizations of $L\omega$-compactness are systematically discussed. Some important properties of $L\omega$-compactness such as ω-closed heredity, arbitrarily multiplicative property, and preserving invariance under ω-continuous mappings are obtained. Finally, the Alexander ω-subbase lemma and the Tychonoff product theorem with respect to $L\omega$-compactness are given.

1. Introduction

Compactness is one of the most important notions in general topology, fuzzy topology, and L-topology. Many research workers have presented various kinds of compactness [1–19] by means of introducing various operators, such as closure operator, θ-closure operator, δ-closure operator, R-closure operator, S-closure operator, SR-closure operator, and PS-closure operator; because the above operators are all order preserving. That is, they satisfy the following conditions: (i) if $A, B \in L^X$ and $A \leqslant B$, then $\omega(A) \leqslant \omega(B)$; (ii) for any $A \in L^X$, $A \leqslant \omega(A)$, where $\omega : L^X \to L^X$ can take and of the above operators, L^X is the family of all L-sets defined on X and with value in L, L is a fuzzy lattice, and 1_X is the greatest L-set of L^X. We introduced a kind of generalized fuzzy space called $L\omega$-space in [20] in order to unify various elementary concepts in L-topological spaces. In the present paper, we will propose and study a generalized compactness which will be called $L\omega$-compactness in $L\omega$-spaces. The $L\omega$-compactness is a unified form of N-compactness [16, 19], near N-compactness [5], almost N-compactness [6], S-compactness [13], SR-compactness [1], PS-compactness [2], δ-compactness [9], θ-compactness [18], and so forth.

2. Preliminaries

Throughout this paper, L denotes a fuzzy lattice, that is, a completely distributive lattice with order-reserving involution $'$, 0 and 1 denote the least and greatest elements of L, respectively, and M denotes the set that consisting of all nonzero $\vee$-irreducible elements of L. Let X be a nonempty crisp set, L^X the set of all L-fuzzy sets (briefly, L-sets) on X, and $M^*(L^X) = \{x_\alpha : \alpha \in M,\ x \in X\}$ the set of all nonzero $\vee$-irreducible elements (i.e., so-called molecules [17] or points for short) of L^X. The least and the greatest elements of L^X will be denoted by 0_X and 1_X, respectively. For any $\alpha \in M$, $\beta(\alpha)$ is called the greatest minimal set of α [12], and $\beta^*(\alpha) = \beta(\alpha) \cap M$ is said to be the standard minimal set of α [17].

Definition 1 (Chen and Cheng [20]). Let X be a nonempty crisp set.

(i) An operator ω: $L^X \to L^X$ is said to be an ω-operator if (1) for all $A, B \in L^X$ and $A \leqslant B$, $\omega(A) \leqslant \omega(B)$; (2) for all $A \in L^X$, $A \leqslant \omega(A)$.

(ii) An L-set $A \in L^X$ is called an ω-set if $\omega(A) = A$.

(iii) Put $\Omega = \{A \in L^X \mid \omega(A) = A\}$, and call the pair (L^X, ω) an $L\omega$-space.

Definition 2 (Chen and Cheng [20]). Let (L^X, Ω) be an $L\omega$-space, $A \in L^X$, and $x_\alpha \in M^*(L^X)$. If there exists a $Q \in \Omega$ such that $x_\alpha \nleq Q$ and $P \leqslant Q$, then call P an ω-remote neighborhood (briefly, ωR-neighborhood) of x_α. The collection of all ωR-neighborhoods of x_α is denoted by $\omega\eta(x_\alpha)$. If $A \nleq P$ for each $P \in \omega\eta(x_\alpha)$, then x_α is said to be an ω-adherence point of A and the union of all ω-adherence points of A is called the ω-closure of A and denoted by $\omega\,\mathrm{cl}(A)$. If $A = \omega\,\mathrm{cl}(A)$, then call A an ω-closed set and call A' an ω-open set. If P is an ω-closed set and $x_\alpha \nleq P$, then P is said to be an ω-closed remote neighborhood (briefly, ωCR-neighborhood) of x_α and the collection of all ωCR-neighborhoods of x_α is denoted by $\omega\eta^-(x_\alpha)$. Note that $\omega C(L^X)$ and $\omega O(L^X)$ are the family of all ω-closed sets and all ω-open sets in L^X, respectively.

Definition 3 (Chen and Cheng [20]). Let (L^X, Ω) be an $L\omega$-space, $A \in L^X$, and $\omega\operatorname{int}(A) = \vee\{B \in L^X \mid B \leqslant A$ and B is an ω-open set in $L^X\}$. We call $\omega\operatorname{int}(A)$ the ω-interior of A. Obviously, A is ω-open if and only if $A = \omega\operatorname{int}(A)$.

Definition 4 (Huang and Chen [11]). Let (L^X, Ω) be an $L\omega$-space, let N be a molecular net in L^X, and let $x_\alpha \in M^*(L^X)$. If N is eventually not in P for each $P \in \omega\eta^-(x_\alpha)$, then x_α is said to be an ω-limit point of N (or N ω-converges to x_α). If N is frequently not in P for each $P \in \omega\eta^-(x_\alpha)$, then x_α is said to be an ω-cluster point of N (or N ω-accumulates to x_α). The union of all ω-limit points (ω-cluster points) of N is written by ω-lim N (ω-*adN*).

Definition 5 (Huang and Chen [11]). Let (L^X, Ω) be an $L\omega$-space, let I be an ideal in L^X, and let $x_\alpha \in M^*(L^X)$. If $\omega\eta^-(x_\alpha) \subseteq I$, then x_α is called an ω-limit point of I (or $I\omega$-converges to x_α). If $P \vee B \neq 1_X$ for each $P \in \omega\eta^-(x_\alpha)$ and each $B \in I$, then x_α is called an ω-cluster point of I (or I ω-accumulates to x_α). The union of all ω-limit points (ω-cluster points) of I is denoted by ω-lim I (ω-*adI*).

Definition 6 (Chen and Cheng [20]). Let (L^X, Ω) be an $L\omega$-space, $x_\alpha \in M^*(L^X)$, and $\beta, \gamma \in \omega O(L^X)$. Then,

(i) β is said to be an ω-base in (L^X, Ω) if for each $G \in \omega O(L^X)$, there exists a subfamily φ of β such that $G = \vee_{B\in\varphi} B$;

(ii) γ is said to be an ω-subbase in (L^X, Ω) if the collection consisting of all intersections of any finite elements in γ is an ω-base in (L^X, Ω).

Definition 7 (Chen and Cheng [20]). Assume (L^X, Ω_i) to be an $L\omega_i$-space ($i = 1, 2$) and $f : (L^X, \Omega_1) \to (L^Y, \Omega_2)$ an L-valued Zadeh's type function [17]. If $f^{\leftarrow}(B) \in \omega_1 O(L^X)$ for each $B \in \omega_2 O(L^Y)$, then call $f(\omega_1, \omega_2)$-continuous.

3. $L\omega$-Compact Set and Its Characteristics

In this section, we will introduce the concepts of $\alpha\omega$-remote neighborhood family and $\gamma\omega$-cover in an $L\omega$-space first, propose the notion of $L\omega$-compactness by making use of $\alpha\omega$-remote neighborhood family next, and then discuss the characteristics of $L\omega$-compactness.

Definition 8. Suppose (L^X, Ω) be an $L\omega$-space, $A \in L^X$, $\alpha \in M$, and $\Phi \subseteq \omega C(L^X)$. If there exists a $P \in \Phi$ such that $P \in \omega\eta^-(x_\alpha)$ for each molecule x_α in A, then Φ is called an $\alpha\omega$-remote neighborhood family (briefly, $\alpha\omega$-RF) of A, in symbol $\wedge\Phi < A(\alpha\omega)$. If there exists a nonzero $\vee$-irreducible element $\lambda \in \beta^*(\alpha)$ with $\wedge\Phi < A(\lambda\omega)$, then Φ is said to be an $(\alpha\omega)^-$-RF, in symbol $\wedge\Phi \ll A(\alpha\omega)$.

Definition 9. Assume (L^X, Ω) be an $L\omega$-space, $A \in L^X$, $\gamma' \in M$, and $\Gamma \subseteq \omega O(L^X)$. If there is a $B \in \Gamma$ such that $B(x) \nleq \gamma$ for each $x \in \tau_{\gamma'}(A) = \{x \in X \mid A(x) \geqslant \gamma'\}$, then Γ is known as a $\gamma\omega$-cover. If there exists a prime element $t \in \alpha^*(\gamma)$ such that Γ is a $t\omega$-cover of A, then Γ is said to be a $(\gamma\omega)^+$-cover of A, where $\alpha^*(\gamma)$ is the standard maximal set of γ [17].

Definition 10. Assume (L^X, Ω) be an $L\omega$-space and $A \in L^X$. If every $\alpha\omega$-RF Φ of A has a finite subfamily Ψ such that Ψ is an $(\alpha\omega)^-$-RF, where $\alpha \in M$, then call A an $\alpha L\omega$-compact set. If A is an $\alpha L\omega$-compact set for any $\alpha \in M$, then call A an $L\omega$-compact set. Specially, when 1_X is $\alpha L\omega$-compact, we call (L^X, Ω) an $\alpha L\omega$-compact space, and if (L^X, Ω) is $\alpha L\omega$-compact for each $\alpha \in M$, we say that (L^X, Ω) is an $L\omega$-compact space.

Obviously, when ω is the L-closure operator on L^X, the $L\omega$-compactness is just the N-compactness in [19], and while ω takes the θ-closure operator (resp., δ-closure operator, R-closure operator, S-closure operator, PS-closure operator, and SR-closure operator) on L^X, the $L\omega$-compactness is just the θ-compactness (resp., δ-compactness, near N-compactness, S-compactness, PS-compactness, and SR-compactness). Therefore, the $L\omega$-compactness is of the universal significance.

Example 11. Let (L^X, Ω) be an $L\omega$-space and $A \in L^X$. If the support $\sigma_0(A) = \{x \in X \mid A(x) > 0\}$ of A is a finite set, then A is an $L\omega$-compact set.

Proof. Assume that $\sigma_0(A) = \{x_1, x_2, \ldots, x_n\}$ and Φ is an $\alpha\omega$-RF of A. For each $i \in \{1, 2, \ldots, n\}$ we choose an ω-closed set $P_i \in \Phi$ with $\alpha \nleq P_i(x_i)$. Being $\alpha = \sup\beta^*(\alpha)$, there is a $\lambda_i \in \beta^*(\alpha)$ such that $\lambda \nleq P_i(x_i)$. Since $\beta^*(\alpha)$ is an upper directed set, there is a $\lambda \in \beta^*(\alpha)$ with $\lambda \geqslant \lambda_i$ for each $i \in \{1, 2, \ldots, n\}$, and thus $\lambda_i \nleq P_i(x_i)$. Therefore Φ has a finite subfamily $\Psi = \{P_1, P_2, \ldots, P_n\}$ which is an $(\alpha\omega)^-$-RF of A. By Definition 10, A is an $L\omega$-compact set. □

Now we give some characteristics of $L\omega$-compactness as follows.

Theorem 12. *Let (L^X, Ω) be an $L\omega$-space and $A \in L^X$. Then A is an $L\omega$-compact set if and only if the following conditions hold:*

(1) *for each $\alpha \in M$, every $\alpha\omega$-RF Φ of A has a finite subfamily Ψ with $\wedge\Psi < A(\alpha\omega)$;*

(2) *for each $\alpha \in M$, if $\Phi = \{P\}$ is an $\alpha\omega$-RF of A, then Φ is also an $(\alpha\omega)^-$-RF of A.*

Proof. Necessity. Assume that A is $L\omega$-compact and Φ is an $\alpha\omega$-RF of $A(\alpha \in M)$. According to Definition 10, Φ has a finite subfamily Ψ with $\wedge\Psi \ll A(\alpha\omega)$ and so it certainly holds that $\wedge\Psi < A(\alpha\omega)$. Thus (1) is satisfied. If $\Phi = \{P\}$ is an $\alpha\omega$-RF of A, then Φ has a finite Ψ with $\wedge\Psi \ll A(\alpha\omega)$ by the

$L\omega$-compactness of A. Obviously, $\Psi = \Phi$, and hence Φ is an $(\alpha\omega)^{-}$-RF of A. Therefore (2) holds.

Sufficiency. Suppose that conditions (1) and (2) are satisfied, and Φ is an $\alpha\omega$-RF of A ($\alpha \in M$). By (1), there is a finite subfamily Ψ of Φ such that Ψ is an $\alpha\omega$-RF of A. Let $P = \wedge\Psi$. Then $\{P\}$ is an $\alpha\omega$-RF of A. According to (2), $\{P\}$ is also an $\alpha\omega$-RF of A; that is, there exists a $\lambda \in \beta^*(\alpha)$ with $\lambda \nleqslant P(x) = \wedge\{Q(x) \mid Q \in \Psi\}$ for each molecule $x_\lambda \leqslant A$. Since Ψ is finite, we can choose an ω-closed set $Q \in \Psi$ with $\lambda \nleqslant Q(x)$; that is, $Q \in \omega\eta^{-}(x_\lambda)$. This shows that Ψ is an $(\alpha\omega)^{-}$-RF of A. Therefore A is $L\omega$-compact. □

Theorem 13. *Let (L^X, Ω) be an $L\omega$-space and $A \in L^X$. Then A is an $L\omega$-compact set if and only if for each $\gamma' \in M$, every $\gamma\omega$-cover Γ of A has a finite subfamily Ξ such that Ξ is a $(\gamma\omega)^{+}$-cover of A.*

Proof. Necessity. Suppose that A is an $L\omega$-compact set and Γ is any $\gamma\omega$-cover of A ($\gamma' \in M$). Put $\Phi = \Gamma'$. Then $\Phi \subseteq \omega C(L^X)$, and there is an ω-closed set $B' \in \Phi$ with $B(x) \nleqslant \gamma$ for each $x \in \tau_{\gamma'}(A)$; that is, $\gamma' \nleqslant B'(x)$; equivalently, $B' \in \omega\eta^{-}(x_{\gamma'})$. This implies that Φ is a $\gamma'\omega$-RF of A. Thus Φ has a finite subfamily Ψ which is a $(\gamma'\omega)^{-}$-RF of A; that is, there exists $t' \in \beta^*(\gamma')$ such that for each $x \in \tau_{\gamma'}(A)$ we can take an ω-open set $B \in \Psi'$ with $t' \nleqslant B'(x)$. In other words, there are $t \in \alpha^*(\gamma)$ and $B \in \Psi' = \Xi$ with $B(x) \nleqslant t$ for each $x \in \tau_{\gamma'}(A)$. This means that Ξ is a finite subfamily of Γ and a $(\gamma\omega)^{+}$-cover of A.

Sufficiency. Assume that every $\gamma\omega$-cover of A has a finite subfamily which is a $(\gamma\omega)^{+}$-cover of A ($\gamma' \in M$). If Φ is an $\alpha\omega$-RF of A ($\alpha \in M$), then $\Gamma = \Phi'$ is a $\gamma\omega$-cover of A where $\gamma = \alpha'$. Hence Γ has a finite subfamily Ξ which is a $(\gamma\omega)^{+}$-cover of A by the hypothesis. Write $\Psi = \Xi'$. One can easily see that Ψ is a finite subfamily of Φ and is an $(\alpha\omega)^{-}$-RF of A. Therefore A is $L\omega$-compact. □

Theorem 14. *Let (L^X, Ω) be an $L\omega$-space and $A \in L^X$. Then A is $L\omega$-compact if and only if for each $\alpha \in M$ and each $\Phi \subseteq \omega C(L^X)$ having α-finite intersection property for A (i.e., for each finite subfamily Ψ of Φ and each $\lambda \in \beta^*(\alpha)$ there exists a molecule $x_\lambda \leqslant A$ with $x_\lambda \leqslant \wedge\Psi$), there exists a molecule $x_\alpha \leqslant A$ with $x_\alpha \leqslant \wedge\Phi$.*

Proof. Necessity. Grant that A is an $L\omega$-compact set, $\Phi \subseteq \omega C(L^X)$, and Φ has α-finite intersection property for A ($\alpha \in M$). If $x_\alpha \nleqslant \wedge\Phi$ for each $x_\alpha \leqslant A$, then Φ is an $\alpha\omega$-RF of A by the hypothesis of Φ. Hence Φ has a finite subfamily Ψ which is an $(\alpha\omega)^{-}$-RF of A; that is, there is a $\lambda \in \beta^*(\alpha)$ satisfying $x_\lambda \nleqslant \wedge\Psi$ for each $x_\lambda \leqslant A$; in other words, $\vee_{x\in X}(A \wedge (\wedge\Psi))(x) \ngeqslant \lambda$. It contradicts the fact that Φ has α-finite intersection property for A. Hence the necessity is proved.

Sufficiency. Assume that the condition holds and that Φ is an $\alpha\omega$-RF of A. If for any finite subfamily Ψ of Φ, Ψ is not an $(\alpha\omega)^{-}$-RF of A, then for each $\lambda \in \beta^*(\alpha)$ there exists a molecule $x_\lambda \leqslant A$ with $x_\lambda \leqslant \wedge\Psi$; that is, $\vee_{x\in X}(A \wedge (\wedge\Psi))(x) \geqslant \lambda$. This shows that Φ has α-finite intersection property for A. By the assumption we have $x_\alpha \leqslant A$ satisfying $x_\alpha \leqslant \wedge\Psi$. It contradicts that Φ is an $\alpha\omega$-RF of A. Therefore Φ has a finite subfamily Ψ which is an $(\alpha\omega)^{-}$-RF of A, and hence A is $L\omega$-compact. □

Theorem 15. *Let (L^X, Ω) be an $L\omega$-space and $A \in L^X$. Then A is $L\omega$-compact if and only if for each $\alpha \in M$, every α-net in A has an ω-cluster point in A with height α.*

Proof. Necessity. Suppose that A is an $L\omega$-compact set and that $N = \{N(n) \mid n \in D\}$ is an α-net [16] in A. If N does not have any ω-cluster point in A with height α, then there exists a $P[x] \in \omega\eta^{-}(x_\alpha)$ such that N is eventually in $P[X]$ for each $x_\alpha \leqslant A$; that is, there is a $n(x) \in D$ with $N(n) \leqslant P[x]$ whenever $n \geqslant n(x)$. Write $\Phi = \{P[x] \mid x_\alpha \leqslant A\}$. Obviously, Φ is $\alpha\omega$-RF of A. By the $L\omega$-compactness of A, Φ has a finite subfamily $\Psi = \{P[x_i] \mid i = 1, 2, \ldots, m\}$ which is an $(\alpha\omega)^{-}$-RF of A; that is, there is an $i \in \{1, 2, \ldots, m\}$ with $y_r \nleqslant P[X_i]$ for some $r \in \beta^*(\alpha)$ and each $y_r \leqslant A$. Take $P = \wedge_{i=1}^{m} P[x_i]$. Then $y_r \nleqslant P$ for each $y_r \leqslant A$. Since D is a directed set, there is an $n_0 \in D$, such that $n_0 \geqslant n(x_i)$ and $N(n) \leqslant P[x_i]$ ($i = 1, 2, \ldots, m$) whenever $n \geqslant n_o$, and so $N(n) \leqslant P$. This shows that for each $y_r \leqslant A$, $\vee(N(n)) \ngeqslant r$ as long as $n \geqslant n_o$. It contradicts the fact that N is an α-net. Therefore N has at least an ω-cluster point in A with height α.

Sufficiency. Assume that every α-net in A has at least an ω-cluster point with hight α for each $\alpha \in M$, Φ is an $\alpha\omega$-RF of A, and $2^{(\Phi)}$ is the set of all finite subfamilies of Φ. If for each $r \in \beta^*(\alpha)$ and each $\Psi \in 2^{(\Phi)}$, Ψ is not an $r\omega$-RF of A; that is, $x_r \leqslant \wedge\Psi$ for each $x_r \leqslant A$, and hence there exists a molecule $N(r, \Psi) \leqslant A$ satisfying $N(r, \Psi) \leqslant \wedge\Psi$. In $\beta^*(\alpha) \times 2^{(\Phi)}$, we define the relation as follows: $(r_1, \Psi_1) \geqslant (r_2, \Psi_2)$ if and only if $r_1 \geqslant r_2$ and $\Psi_1 \supseteq \Psi_2$, then $\beta^*(\alpha) \times 2^{(\Phi)}$ is a directed set with the relation "$\geqslant$". Let $N = \{N(r, \Psi) \mid (r, \Psi) \in \beta^*(\alpha) \times 2^{(\Phi)}\}$. One can easily see that N is an α-net in A. We assert that N does not have any ω-cluster point in A with hight α. In fact, for each $x_\alpha \leqslant A$, we can choose an ω-closed set $P \in \Phi$ with $P \in \omega\eta^{-}(x_\alpha)$ by the definition of Φ. Taking $r_1 \in \beta^*(\alpha)$ and $\Psi \in 2^{(\Phi)}$, we have $P \in \Psi$ according to $(r, \Psi) \geqslant (r_1, \{P\})$, and hence $N(r, \Psi) \leqslant \wedge\Psi \subseteq P$. This implies that N is eventually in P, and thus x_α is not an ω-cluster point of N. It is in contradiction with the hypothesis of sufficiency. Consequently, A is $L\omega$-compact. □

Definition 16. Let (L^X, Ω) be an $L\omega$-space, let $\mathcal{F}$ be an α-filter in L^X; that is, $\vee_{x\in X}(F \wedge A)(x) \geqslant \alpha$ for each $F \in \mathcal{F}$ and $x_\alpha \in M^*(L^X)$. If $F \nleqslant P$ and for each $P \in \omega\eta^{-}(x_\alpha)$ and each $F \in \mathcal{F}$, then x_α is called an ω-cluster point of $\mathcal{F}$.

Theorem 17. *Let (L^X, Ω) be an $L\omega$-space and $A \in L^X$. Then A is $L\omega$-compact if and only if for each $\alpha \in M$, every α-filter containing A as an element has an ω-cluster point in A with hight α.*

Proof. Necessity. Grant that A is an $L\omega$-compact set and that $\mathcal{F}$ is an α-filter containing A as an element. Then $F \wedge A \in \mathcal{F}$

for each $F \in \mathscr{F}$ and $\vee_{x\in X}(F \wedge A)(x) \geqslant \alpha$, and thus there exists a molecule $N(F, r) \leqslant A$ with hight r for each $r \in \beta^*(\alpha)$. Define $N = \{N(F, r) \leqslant F \wedge A \mid (F, r) \in \mathscr{F} \times \beta^*(\alpha)\}$ and define a relation in $\mathscr{F} \times \beta^*(\alpha)$ as follows:

$$(F_1, r_1) \geqslant (F_2, r_2) \quad \text{iff } F_1 \leqslant F_2,\ r_1 \geqslant r_2. \tag{1}$$

Evidently, $\mathscr{F} \times \beta^*(\alpha)$ is a directed set with the relation "$\geqslant$", and then N is an α-net in A. By the $L\omega$-compactness of A and Theorem 15, N has an ω-cluster point in A with hight α, say x_α. We assert that x_α is also an ω-cluster point of $\mathscr{F}$. In reality, N is frequently not in P for each $P \in \omega\eta^-(x_\alpha)$; that is, for each $F \in \mathscr{F}$ there exist $F_1 \in \mathscr{F}$ with $F_1 \leqslant F$ and some $r \in \beta^*(\alpha)$ satisfying $N(F_1, r) \nleqslant P$. Hence we have $F \nleqslant P$ by virtue of the fact that $N(F_1, r) \leqslant F_1 \leqslant F$. This means that x_α is an ω-cluster point of $\mathscr{F}$. Therefore the necessity is proved.

Sufficiency. Suppose that every α-filter containing A as an element has an ω-cluster point in A with hight α for each $\alpha \in M$ and that Φ is an $\alpha\omega$-RF of A. If for each $\Psi \in 2^{(\Phi)}$, Ψ is not an $(\alpha\omega)^-$-RF of A, then there exists a molecule $x_r \leqslant A$ and $x_r \leqslant \wedge\Psi$ for each $r \in \beta^*(\alpha)$. Put $\mathscr{F} = \{F \in L^X \mid \exists \Psi \in 2^{(\Phi)}$ with $(\wedge\Psi) \wedge A \leqslant F\}$. One can easily verify that $\mathscr{F}$ is an α-filter containing A as an element, and hence $\mathscr{F}$ has an ω-cluster point in A with hight α by the supposition, say x_α. In accordance with Definition 16, we have $F \nleqslant P$ for each $P \in \omega\eta^-(x_\alpha)$ and each $F \in \mathscr{F}$, specially, $\wedge\Psi \nleqslant P$. Since Φ is an $\alpha\omega$-RF of A, there exists an ω-closed set $Q \in \Phi$ with $Q \in \omega\eta^-(x_\alpha)$ for each $x_\alpha \leqslant A$. Obviously, $\{Q\} \in 2^{(\Phi)}$, so $Q \nleqslant Q$, and this is impossible. Hence there must be a $\Psi \in 2^{(\Phi)}$ which is an $(\alpha\omega)^-$-RF of A. This shows that A is $L\omega$-compact. □

Definition 18. Let I be an ideal in L^X. If $\vee_{x\in X}B'(x) \geqslant \alpha$ for each $B \in I$, then I is called an α-ideal ($\alpha \in M$).

Theorem 19. *Let (L^X, Ω) be an $L\omega$-space and $A \in L^X$. Then A is $L\omega$-compact if and only if every α-ideal I whose A is not in I has an ω-cluster point in A with hight α for each $\alpha \in M$.*

Proof. Necessity. Assume that A is an $L\omega$-compact set, I is an α-ideal whose A is not in I, and $N(I) = \{N(I)((b, B)) = b \leqslant A \mid (b, B) \in D(I)\}$ where $D(I) = \{(b, B) \mid b \in M^*(L^X), B \in I$ and $b \nleqslant B\}$. Then $N(I)$ is an α-net in A. Hence $N(I)$ has an ω-cluster point in A with hight α by Theorem 15, say x_α. Obviously, x_α is also an ω-cluster point of I. Consequently, the necessity is proved.

Sufficiency. Grant that every α-ideal whose A is not in it has an ω-cluster point in A with hight α for each $\alpha \in M$ and $\mathscr{F}$ is an α-filter containing A as an element. Let $I = \{F' \in L^X \mid F \in \mathscr{F}\}$. Evidently, I is an α-ideal whose A is not in I. Now we will prove that $\mathscr{F}$ has an ω-cluster point in A with hight α. Actually, by the hypothesis we know that I has an ω-cluster point in A with hight α, say x_α; that is, $F' \vee P \neq 1_X$; equivalently, $F \nleqslant P$, for each $F \in \mathscr{F}$ and each $P \in \omega\eta^-(x_\alpha)$. Therefore x_α is an ω-cluster point of $\mathscr{F}$ in line with Definition 16, and hence A is an $L\omega$-compact set by Theorem 17. This implies that the sufficiency holds. □

4. Some Important Properties of $L\omega$-Compactness

In this section, we still further deliberate the properties of $L\omega$-compactness in an $L\omega$-space.

Theorem 20. *Let (L^X, Ω) be an $L\omega$-space and $A, B \in L^X$. If A is $L\omega$-compact and B is ω-closed, then $A \wedge B$ is $L\omega$-compact.*

Proof. Assume that N is an α-net in $A \wedge B$ ($\alpha \in M$). Then N is also an α-net in A. Since A is ω-compact, N has an ω-cluster point in A with hight α, say x_α. We assert that $x_\alpha \leqslant B$. Actually, since N is an α-net in B and N ω-accumulates x_α, N has an α-subnet T which ω-converges to x_α and so $x_\alpha \leqslant \omega\,\mathrm{cl}(B) = B$. Hence $x_\alpha \leqslant A \wedge B$, and thus $A \wedge B$ is $L\omega$-compact in accordance with Theorem 15. □

This theorem shows that the $L\omega$-compactness is hereditary with respect to ω-closed sets.

Theorem 21. *Let A and B be both $L\omega$-compact sets in (L^X, Ω). Then $A \vee B$ is also an $L\omega$-compact set in (L^X, Ω).*

Proof. Suppose that Φ is an $\alpha\omega$-RF of $A \vee B$ ($\alpha \in M$). Then Φ is an $\alpha\omega$-RF of both A and B. Owing to the $L\omega$-compactness of A, there are $\lambda_1 \in \beta^*(\alpha)$ and $\Psi_1 \in 2^{(\Phi)}$ with $\wedge\Psi_1 < A(\lambda_1\omega)$. Similarly, there exist $\lambda_2 \in \beta^*(\alpha)$ and $\Psi_2 \in 2^{(\Phi)}$ satisfying $\wedge\Psi_2 < A(\lambda_2\omega)$. Take $\lambda = \lambda_1 \wedge \lambda_2$ and $\Psi = \Psi_1 \cup \Psi_2$; then $\lambda \in \beta^*(\alpha)$, $\Psi \in 2^{(\Phi)}$, and $\wedge\Psi < A(\lambda\omega)$; that is, Ψ is an $(\alpha\omega)^-$-RF of $A \vee B$. Consequently, $A \vee B$ is $L\omega$-compact. □

This theorem indicates that the $L\omega$-compactness is finitely additive.

Theorem 22. *Let $L = [0, 1]$, (L^X, Ω) be an $L\omega$-space and let $A \in L^X$ be an $L\omega$-compact set. Then there exists a crisp point $x \in X$ such that $A(x) = \sup\{A(t) \mid t \in X\}$.*

Proof. Let $\alpha = \sup\{A(t) \mid t \in X\}$; then $\alpha \in [0, 1]$. If $\alpha = 0$, then $A = 0_X$ and hence $A(x) = \sup\{A(t) \mid t \in X\}$ holds for each $x \in X$. If $\alpha > 0$, and D is the set of all natural numbers, then we choose $x^n \in X$ with $A(x^n) > \alpha - (1/n)$ and $N = \{x^n_{A(x^n)} \mid n \in D\}$. Obviously, N is an α-net in A, and N has an ω-cluster point x_α in A by virtue of the $L\omega$-compactness of A. Hence $A(x) \geqslant \alpha$ by $x_\alpha \leqslant A$. On the other hand, $A(x) \leqslant \alpha$ by the definition of α. Therefore $A(x) = \alpha = \sup\{A(t) \mid t \in X\}$. □

This theorem implies that an $L\omega$-compact set can reach the maximum at some point in X as a function.

Theorem 23. *Let (L^X, Ω_1) and (L^Y, Ω_2) be an $L\omega_1$-space and an $L\omega_2$-space, respectively, and let $f : L^X \rightarrow L^Y$ be an (ω_1, ω_2)-continuous L-valued Zadeh's type function. If A is an $L\omega_1$-compact set in (L^X, Ω_1), then $f^{\rightarrow}(A)$ is an $L\omega_2$-compact set in (L^Y, Ω_2).*

Proof. Assume that Φ is an $\alpha\omega_2$-RF of $f^{\rightarrow}(A)$ and $y_\alpha \in M^*(L^Y)$ with $y_\alpha \leqslant f^{\rightarrow}(A)(\alpha \in M)$. According to the definition of f, there is a molecule $x_\alpha \in M^*(L^X)$ such that $x_\alpha \leqslant A$ and $f^{\rightarrow}(x_\alpha) = y_\alpha$. Thus there is an ω-closed set $Q \in \Phi$ with $f^{\rightarrow}(x_\alpha) \nleqslant Q$; that is, $x_\alpha \nleqslant f^{\leftarrow}(Q)$. Since f is (ω_1, ω_2)-continuous, $f^{\leftarrow}(Q)$ is ω-closed in (L^X, ω_1), and hence $f^{\leftarrow}(Q) \in \omega_1\eta^-(x_\alpha)$. This means that $f^{\leftarrow}(\Phi) = \{f^{\leftarrow}(Q) \mid Q \in \Phi\}$ is an $\alpha\omega_1$-RF of A. Therefore Φ has a finite subfamily $\Psi = \{Q_1, Q_2, \ldots, Q_n\}$ such that $f^{\leftarrow}(\Psi)$ is an $(\alpha\omega_1)^-$-RF of A. We assert that Ψ is an $(\alpha\omega_2)^-$-RF of $f^{\rightarrow}(A)$. In reality, there exists a $\lambda \in \beta^*(\alpha)$ with $\wedge f^{\leftarrow}(\Psi) < A(\lambda\omega_1)$ by virtue of the fact that $f^{\leftarrow}(\Psi)$ is an $(\alpha\omega_1)^-$-RF of A. Since for each $y_\lambda \leqslant f^{\rightarrow}(A)$ there exists a $x_\lambda \leqslant A$ satisfying $f^{\rightarrow}(x_\lambda) = y_\lambda$, and there exists a $Q \in \Psi$ with $f^{\leftarrow}(Q) \in \omega_1\eta^-(x_\lambda)$, that is, $x_\lambda \nleqslant f^{\leftarrow}(Q)$. Hence $y_\lambda = f^{\rightarrow}(x_\lambda) \nleqslant Q$ by Lemma 3.1 in [19], and so Ψ is an $(\alpha\omega_2)^-$-RF of $f^{\rightarrow}(A)$. Consequently, $f^{\rightarrow}(A)$ is an $L\omega_2$-compact set in (L^Y, Ω_2). □

This theorem means that the $L\omega$-compactness is topological variant under (ω_1, ω_2)-continuous L-valued Zadeh's type functions.

Definition 24. Let (X, Ω) be a crisp ω-space, and let $\mathscr{P}(X)$ be the set of all subsets of X, that is, all crisp sets on X and $A \in L^X$, where $\omega : \mathscr{P}(X) \rightarrow \mathscr{P}(X)$ is a crisp ω-operator which satisfies the following conditions: (1)$\omega(U) \subseteq \omega(V)$ for each $U, V \in \mathscr{P}(X)$ and $U \subseteq V$; (2)$U \subseteq \omega(U)$ for each $U \in \mathscr{P}(X)$.

(i) If $\xi_\alpha(A) = \{x \in X \mid A(x) \leqslant \alpha\} \in \omega C(X)$, where $\omega C(X)$ denotes the set of all crisp ω-closed sets on X and $\alpha \in M$, then A is said to be an L-valued lower semicontinuous function on X.

(ii) Let $\Delta_L(\Omega)$ be the set of all L-valued lower semicontinuous functions on X, and call the pair $(L^X, \Delta_L(\Omega))$ the $L\omega$-space topologically generated by (X, Ω).

Theorem 25. *Let (X, Ω) be a crisp ω-space and let $(L^X, \Delta_L(\Omega))$ be the $L\omega$-space topologically generated by (X, Ω). Then $A \in L^X$ is $L\omega$-compact if and only if $\tau_\alpha(A) = \{x \in X \mid A(x) \geqslant \alpha\}$ is ω-compact for each $\alpha \in M$.*

Proof. Necessity. Provided that $A \in L^X$ is an $L\omega$-compact set in $(L^X, \Delta_L(\Omega))$ and Φ is an ω-open cover of $\tau_\alpha(A)(\alpha \in M)$, let $\Gamma = \{\chi_G \mid G \in \Phi\}$ and $\gamma = \alpha'$, where χ_G is the characteristic function of G. We assert that Γ is a $\gamma\omega$-cover of A. In fact, for each $x \in \tau_{\gamma'}(A)$, there is an ω-open set $G \in \Phi$ with $x \in G$; that is, $\chi_G(x) = 1$. Hence $\chi_G(x) \nleqslant \gamma$ by virtue of the fact that γ is a prime element in L with $\gamma \neq 1$. Thus Φ has a finite subfamily $\{G_1, G_2, \ldots, G_m\}$ such that $\mu = \{\chi_{G_i} \mid i = 1, 2, \ldots, m\} \in 2^{(\Gamma)}$ which is a $(\gamma\omega)^+$-cover of A in line with Theorem 13; that is, there is an $i \in \{1, 2, \ldots, m\}$ such that $\chi_{G_i} \in \mu$ with $\chi_{G_i}(x) \nleqslant \lambda$ for some $\lambda \in \alpha^*(\gamma)$ and each $x \in \tau_\alpha(A)$, and so $x \in G_i$. This implies that $\tau_\alpha(A) \subseteq \cup_{i=1}^m G_i$. Hence $\tau_\alpha(A)$ is an ω-compact set in (X, Ω).

Sufficiency. Grant that $\tau_\alpha(A)$ is an ω-compact set in (X, Ω) for each $\alpha \in M$ and that Γ is a $\gamma\omega$-cover of A where $\gamma = \alpha'$. Then there is an ω-open set $B_x \in \Gamma$ with $B_x(x) \nleqslant \gamma$ for each $x \in \tau_\alpha(A)$, and hence there exists a prime element $t(x) \in \alpha^*(\gamma)$ satisfying $B_x(x) \nleqslant t(x)$. Put $l_{t(x)}(B_x) = \{y \in X \mid B_x(y) \nleqslant t(x)\}$ and $\Phi = \{l_{t(x)}(B_x) \mid x \in \tau_\alpha(A)\}$; then Φ is an ω-open cover of $\tau_\alpha(A)$ according to $x \in l_{t(x)}(B_x)$ and $B_x \in \Delta_L(\Omega)$. Because of the ω-compactness of $\tau_\alpha(A)$, Φ has a finite subfamily $\Psi = \{l_{t(x_i)}(B_{x_i}) \mid i = 1, 2, \ldots, m\}$ which is an ω-open cover of $\tau_\alpha(A)$; that is, there exists an $i \in \{1, 2, \ldots, m\}$ with $x \in l_{t(x_i)}(B_{x_i})$; in other words, $B_{x_i}(x) \nleqslant t(x_i)$ for each $x \in \tau_\alpha(A)$. Take $t = \wedge_{i=1}^m t(x_i)$; evidently, $t \in \alpha^*(\gamma)$ and $B_{x_i}(x) \nleqslant t$. Hence $\mu = \{B_{x_i} \mid i = 1, 2, \ldots, m\}$ is a $(\gamma\omega)^+$-cover of A, and thus A is an $L\omega$-compact set in $(L^X, \Delta_L(\Omega))$ by Theorem 13. □

This theorem indicates that the $L\omega$-compactness is a good extension in the sense of R. Lowen.

Theorem 26. *Let (L^X, Ω) be a stratified ωT_2 and $A \in L^X$. If A is $L\omega$-compact, then A is ω-closed.*

Proof. We only prove that $x_\alpha \leqslant A$ for each $x_\alpha \in M^*(L^X)$ with $x_\alpha \leqslant \omega\,\mathrm{cl}(A)$ by the definition of ω-operator. Actually, if $x_\alpha \leqslant \omega\,\mathrm{cl}(A)$, then there exists a molecular net $N = \{x_{t(n)}^{(n)} \in M^*(L^X) \mid n \in D\}$ in A which ω-converges to x_α in accordance with Theorem 2 in [11]. Write $\lambda = \wedge_{m \in D} \vee_{n \geqslant m} t(n)$; we assert that $\lambda \geqslant \alpha$. In fact, if $\lambda \ngeqslant \alpha$, then there is a $m \in D$ with $\vee_{n \geqslant m} t(n) \ngeqslant \alpha$, and let $d = \vee_{n \geqslant m} t(n)$. Since (L^X, Ω) is stratified, the constant L-set $[d]$ on X is ω-closed and $x_\alpha \nleqslant [d]$, that is, $[d] \in \omega\eta^-(x_\alpha)$. Obviously, N is eventually in $[d]$, and it contradicts the fact that N ω-converges to x_α. Hence $\lambda \geqslant \alpha$; that is, $\vee_{n \geqslant m} t(n) \geqslant \alpha$ for each $m \in D$. For each $r \in \beta^*(\alpha)$ and each $m \in D$ we choose $n(r, m) \in D$ such that $n(r, m) \geqslant m$ and $t(n(r, m)) \geqslant r$, and define the relation "$\geqslant$" in $\beta^*(\alpha) \times D$ as follows:

$$(r_1, m_1) \geqslant (r_2, m_2) \quad \text{iff } r_1 \geqslant r_2,\ m_1 \geqslant m_2. \tag{2}$$

Then $\beta^*(\alpha) \times D$ is a directed set with the relation. Write $S = \{x_{t(n(r,m))}^{n(r,m)} \mid (r, m) \in \beta^*(\alpha) \times D\}$; then $S = N \circ R$, where $R : \beta^*(\alpha) \times D \rightarrow D$ is defined as $R(n(r, m)) = n(r, m)$. Evidently S is a subnet of N and ω-converges to x_α, and S is an α-net in A. Being the $L\omega$-compactness of A, S has an ω-cluster point in A with hight α, say z_α. Since (L^X, Ω) is an ωT_2 space, S ω-converges to x_α and ω-accumulates to z_α, $z = x$ by Theorem 2.7 in [11], and hence $x_\alpha = z_\alpha \leqslant A$. This implies that $\omega\,\mathrm{cl}(A) \leqslant A$; that is, A is an ω-closed set. □

The following example shows that the stratified condition in Theorem 26 can not be omitted.

Example 27. Let $X = \{x\}$ be a single set, $L = [0, 1]$, and let $\omega : L^X \rightarrow L^X$ be the fuzzy closure operator. Define $\omega O(L^X) = \{0_X, x_{1/3}, 1_X\}$, where $A : x \rightarrow [0, 1]$ is defined as $A(x) = x_\alpha$, $\alpha \in [0, 1]$ for $x \in X$. Obviously, (L^X, Ω) is both an $L\omega$-compact space and an N-compact space. According to Example 11 we know that $A = x_{1/3}$ is an $L\omega$-compact set in (L^X, Ω), but A is not ω-closed.

The following theorems imply that the $L\omega$-compactness can strengthen ω-seperation properties.

Theorem 28. *If (L^X, Ω) is both ωT_2 and $L\omega$-compact $L\omega$-space, then (L^X, Ω) is an ω-regular space [11].*

Proof. Let $G \in L^X$ be an ω-closed pseudocrisp set and let x_λ be a molecule which x is not in supp G. By Definition 7.1 in [19], there is an $\alpha \in M$ such that $G(x) > 0$ implies $G(x) \geqslant \alpha$. For each $y_\alpha \in M^*(L^X)$, there are $P_y \in \omega\eta^-(x_\lambda)$ and $Q_y \in \omega\eta^-(y_\alpha)$ satisfying $P_y \vee P_y = 1_X$ by virtue of $x \neq y$ and the ωT_2 separation of (L^X, Ω). Put $\Phi = \{Q_y \mid y_\alpha \leqslant G\}$; then Φ is an $\alpha\omega$-RF of G. Since (L^X, Ω) is an $L\omega$-compact space, G is an $L\omega$-compact set in accordance with Theorem 20, and thus Φ has a finite subfamily $\Psi = \{Q_{y_1}, Q_{y_2}, \ldots, Q_{y_n}\}$ which is an $(\alpha\omega)^-$-RF of G; that is, there is an $r \in \beta^*(\alpha)$ such that for each molecule $z_r \leqslant G$ we have $i \leqslant n$ with $z_r \nleqslant Q_{y_i}$. Let $Q = \wedge_{i=1}^n Q_{y_i}$; then $z_r \nleqslant Q$; that is, $r \nleqslant Q(z)$ for each $z_r \leqslant G$. Since $G(z) > 0$ implies that $G(z) \geqslant \alpha \geqslant r$, $G(z) \nleqslant Q(z)$ for each $z \in \operatorname{supp} G$, and hence $Q \in \omega\eta^-(G)$. Write $P = \vee_{i=1}^n P y_i$; then $P \in \omega\eta^-(x_\lambda)$ and

$$P \vee Q = \left(\vee_{i=1}^n P y_i\right) \vee \left(\wedge_{i=1}^n Q y_i\right) \geqslant \vee_{i=1}^n \left(P y_i \vee Q y_i\right) = 1. \quad (3)$$

Consequently, (L^X, Ω) is an ω-regular space. □

Theorem 29. *Let (L^X, Ω) be an $L\omega$-compact ωT_2 space. Then (L^X, Ω) is an ω-normal space [11].*

Proof. Let both G, H be ω-closed pseudocrisp sets in (L^X, Ω) with $(\operatorname{supp} G) \cap (\operatorname{supp} H) = \phi$. Then there are $\lambda, \mu \in M$ such that $G(x) > 0$ if and only if $G(x) \geqslant \lambda$, and $H(x) > 0$ if and only if $H(x) \geqslant \mu$. According to the proof of Theorem 28, for each molecule $y_\mu \leqslant G$, there is an ω-closed set $P_y \in \omega\eta^-(G)$ satisfying $\lambda \nleqslant P_y(z)$ for each $z \in \operatorname{supp} G$, and there is a $Q_y \in \omega\eta^-(y_\mu)$ such that $P_y \vee Q_y = 1$. One can easily see that $\Phi = \{Q_y \mid y_\mu \leqslant B\}$ is a $\mu\omega$-RF of H. In line with Theorem 20 we know that H is an $L\omega$-compact set, and so Φ has a finite subfamily $\Psi = \{Q_{y_1}, Q_{y_2}, \ldots, Q_{y_n}\}$ such that Ψ is a $(\mu\omega)^-$-RF of H. Put $P = \vee_{i=1}^n P y_i$; $Q = \wedge_{i=1}^n Q_{y_i}$; then $P \in \omega\eta^-(G)$, $Q \in \omega\eta^-(H)$ and $P \vee Q = 1$. Therefore (L^X, Ω) is an ω-normal space. □

5. The Tychonoff Product Theorem

In this section, we will first extend Alexandar's subbase Lemma in general topology and give the Alexandar's ω-subbase lemma and next prove that the Tychonoff product theorem holds in $L\omega$-spaces.

Theorem 30 (Alexandar ω-subbase lemma). *Let (L^X, Ω) be an $L\omega$-space, $A \in L^X$, and let γ be an ω-subbase [20] in L^X. If for each $\alpha\omega$-RF Φ of A where $\Phi \subseteq \gamma' \subseteq \omega C(L^X)$, there is a finite subfamily Ψ of Φ with $\wedge\Psi \ll A(\alpha\omega)(\alpha \in M)$, then A is $L\omega$-compact.*

Proof. Suppose that Φ is an arbitrary $\alpha\omega$-RF of A. We will prove that Φ has a finite subfamily Ψ which is an $(\alpha\omega)^+$-RF of A. In fact, if for each $\Psi \in 2^{(\Phi)}$, $\wedge\Psi \ll A(\alpha\omega)$ does not hold, then $H = \{\Delta \mid \Phi \subseteq \Delta \subseteq \omega C(L^X)$, for all $\Psi \in 2^{(\Delta)}$, $\wedge\Psi \ll A(\alpha\omega)$ does not hold$\} \neq \emptyset$, and H is a partial-ordered set with respect to the upper bound and hence there exists a maximal element Δ_0 in H by Zorn's Lemma. We assert that Δ_0 satisfies the following conditions:

(1) $\wedge\Delta_0 < A(\alpha \geqslant \omega)$;

(2) if $P \in \Delta_0$, then $Q \in \Delta_0$ for each $Q \in \omega C(L^X)$ with $Q \geqslant P$;

(3) if $P, Q \in \omega C(L^X)$ and $P \vee Q \in \Delta_0$, then $P \in \Delta_0$ or $Q \in \Delta_0$.

Actually, since $\wedge\Phi < A(\alpha\omega)$ and $\Phi \subseteq \Delta_0$, condition (1) holds. If $P \in \Delta_0, Q \in \omega C(L^X), Q \geqslant P$, and Q is not in Δ_0, then $\Delta^* = \Delta_0 \cup \{Q\} \in H$ and $\Delta_0 \subset \Delta^*$. It contradicts the fact that Δ_0 is the maximal element in H thus condition (2) holds. Let $P, Q \in \omega C(L^X)$. If P and Q are both not in Δ_0, then $\Delta_0 \cup \{P\}$ and $\Delta_0 \cup \{Q\}$ are both not in H by the maximality of Δ_0, and thus there are $\Psi_1, \Psi_2 \in 2^{(\Delta_0)}$ such that $\wedge(\Psi_1 \cup \{P\}) \ll A(\alpha\omega)$ and $\wedge(\Psi_2 \cup \{Q\}) \ll A(\alpha\omega)$ according to the definition of H; that is, there are $s, t \in \beta^*(\alpha)$ with $\wedge(\Psi_1 \cup \{P\}) < A(s\omega)$ and $\wedge(\Psi_2 \cup \{Q\}) < A(t\omega)$. Since $\beta^*(\alpha)$ is upper directed, we can choose $r \in \beta^*(\alpha)$ with $r \geqslant s \vee t$. Now we prove $\wedge\{\Psi_2 \cup \Psi_2 \cup \{P \vee Q\}\} < A(r\omega)$. In reality, if $\Psi_2 \cup \Psi_2$ does not have any ωR-neighborhood of x_r for each $x_r \leqslant A$, then $\Psi_2 \cup \Psi$ does not have any ωR-neighborhood of x_s and x_t, respectively, and hence $P \in \omega\eta^-(x_s)$ and $Q \in \omega\eta^-(x_t)$. Particularly, $P, Q \in \omega\eta^-(x_r)$ and so $P \vee Q \in \omega\eta^-(x_r)$. This shows that $\wedge(\Psi_2 \cup \Psi_2 \cup \{P \vee Q\}) < A(r\omega)$. Therefore $P \vee Q$ is not in Δ_0 by virtue of the definition of Δ_0 and $\Psi_1, \Psi_2 \in 2^{(\Delta_0)}$. So, condition (3) holds.

From (2) and (3) we have the following result:

(4) If $R \in \Delta_0$, $P_i \in \omega C(L^X)\,(i = 1, 2, \ldots, n)$ and $R \leqslant \vee_{i=1}^n P_i$, then there is an $i \in \{1, 2, \ldots, n\}$ satisfying $P_i \in \Delta_0$.

Consider now $\gamma' \cap \Delta_0$. If $\gamma' \cap \Delta_0$ is an $\alpha\omega$-RF of A, then there is a finite subfamily δ of $\gamma' \cap \Delta_0$ which is an $(\alpha\omega)^-$-RF of A. Evidently, $\delta \in 2^{(\Delta_0)}$; it is in contradiction with $\Delta_0 \in H$. Hence $\gamma' \cap \Delta_0$ is not an $\alpha\omega$-RF of A; that is, there is a molecule x_α in A meeting $x_\alpha \leqslant \wedge(\gamma' \cap \Delta_0)$. We now verify that $x_\alpha \leqslant \wedge\Delta_0$. In fact, if there is $Q \in \Delta_0$ with $x_\alpha \nleqslant Q$, then by Definition 5 in [17] we can take a finite subfamily $\{P_{ij} \mid j \in J_i, i \in I\}$ of γ' satisfying $Q = \wedge_{i \in I} \vee_{j \in J_i} P_{ij}$, where J_i is a finite set for each $i \in I$. Because of $x_\alpha \nleqslant Q$, we can choose $i \in I$ with $x_\alpha \nleqslant \vee_{j \in J_i} P_{ij}$. Since $Q \leqslant \vee_{j \in J_i} P_{ij}$, there is a $j \in J_i$ such that $P_{ij} \in \Delta_0$ by (4). Hence $P_{ij} \in \gamma' \cap \Delta_0$ and $x_\alpha \nleqslant P_{ij}$; it contradicts the fact that $x_\alpha \leqslant \wedge(\gamma' \cap \Delta_0) \leqslant P_{ij}$; thus $x_\alpha \leqslant \wedge\Delta_0$. However, this is in contradiction with (1) again. This implies that Φ has a finite subfamily Ψ with $\wedge\Psi \ll A(\alpha\omega)$. Therefore A is an $L\omega$-compact set in (L^X, Ω). □

Theorem 31. *Let $\{(L^{X_t}, \Omega_t) \mid t \in T\}$ be a collection of $L\omega$-spaces and let (L^X, Ω) be the product space of them. If A_t is an $L\omega$-compact set in (L^{X_t}, Ω_t) for each $t \in T$, then the product $A = \Pi_{t \in T} A_t$ of all $L\omega$-compact sets $A_t (t \in T)$ is an $L\omega$-compact set in (L^X, Ω).*

Proof. Assume that Φ is an $\alpha\omega$-RF of $A(\alpha \in M)$. By Theorem 30 we can grant that every ω-closed set in Φ is of the form $\rho_t^{\leftarrow}(B_t)$ where $B_t \in \omega C(L^{X_t})$ and $\rho_t : L^X \to L^{X_t}$ is a protection because $\{\rho_t^{\leftarrow}(U_t) \mid U_t \in \omega O(L^X), t \in T\}$ is an ω-subbase in (L^X, Ω) [20]. Now we consider the following two cases.

(i) If there exists a $t_0 \in T$ such that no molecule with hight α is contained in A_{t_0}, then by the $L\omega$-compactness of A_{t_0}, there is an $r \in \beta^*(\alpha)$ such that no molecule with hight r is contained in A_{t_0}. In reality, if there exists a molecule with hight r in A_{t_0} for each $r \in \beta^*(\alpha)$, say $N(r)$, then $N = \{N(r) \mid r \in \beta^*(\alpha)\}$ is an α-net in A_{t_0} by the directivity of $\beta^*(\alpha)$. Since A_{t_0} is $L\omega$-compact, N has an ω-cluster point in A_{t_0} with hight α according to Theorem 15. It is in contradiction with the hypothesis of A_{t_0}. Thus it can be seen that there exists an $r \in \beta^*(\alpha)$ with $A_{t_0}(x^{t_0}) \not\geq r$ for each $x^{t_0} \in X_{t_0}$. Hence for each $x \in X$, we have

$$A(x) = (\Pi_{t\in T} A_t)(x) = \wedge_{t\in T} A_t(\rho_t(x)) \leq A_{t_0}(\rho_{t_0}(x)) = A_{t_0}(x^{t_0}), \tag{4}$$

and hence $A(x) \not\geq r$ for each $x \in X$; that is, no molecule with hight r is contained in A. This shows that for each $\Psi \in 2^{(\Phi)}$, Ψ is an $(\alpha\omega)^-$-RF of A.

(ii) Suppose that for each $t \in T$, A_t contains a molecule with hight α, say x_α^t. Since $\Phi \subseteq \{\rho_t^{\leftarrow}(B_t) \mid B_t \in \omega C(L^X), t \in T\}$, we can take $R \subseteq T$ such that $\Phi = \cup_{t\in R}\Phi_t$, where $\Phi_t = \{\rho_t^{\leftarrow}(B_t) \mid B_t \in \mathscr{B}_t \subseteq \omega C(L^X)\}$. Now we prove that there must be $s \in R$ with $\wedge\mathscr{B}_s < A(\alpha\omega)$. In fact, if there is a crisp point $y^t \in X_t$ such that $y^t \leq A_t \wedge (\wedge\mathscr{B}_t)$ for each $t \in R$, then we choose a crisp point z in X as follows: if $t \in R$, $z^t = y^t$; if t is not in R, $z^t = x^t$. Taking any ω-closed set $\rho_t^{\leftarrow}(B_t)$ in Φ, where $t \in R$ and $B_t \in \mathscr{B}_t$, we have

$$\rho_t^{\leftarrow}(B_t)(z) = B_t(z^t) = B_t(y^t) \geq (A_t \wedge (\wedge\mathscr{B}_t))(y^t) \geq \alpha, \tag{5}$$

that is, $z_\alpha \leq \rho_t^{\leftarrow}(B_t)$, and hence $z_\alpha \leq \wedge\Phi$ by the arbitrariness of $\rho_t^{\leftarrow}(B_t) \in \Phi$. On the other hand,

$$A(z) = \wedge_{t\in R} A_t(z^t) = (\wedge_{t\in R} A_t(y^t)) \wedge (\wedge_{t\in R} A_t(x^t)) \geq \alpha. \tag{6}$$

This implies that z_α is a molecule in A; it contradicts the fact that Φ is an $\alpha\omega$-RF of A. Consequently, there is $s \in R$ with $\wedge\mathscr{B}_s < A_s(\alpha)$; thus there is a finite subfamily Γ_s of $\mathscr{B}_s$ with $\Gamma_s < A_s(r\omega)$ for some $r \in \beta^*(\alpha)$. Put $\Psi = \{\rho_s^{\leftarrow}(B_s) \mid B_s \in \Gamma_s\}$; then $\Psi \in 2^{(\Phi)}$. We assert that $\wedge\Psi < A(r\omega)$. Actually, for any molecule e_r in A with hight r we have $A_s(e^s) \geq A(e) \geq r$; that is, e_r^s is a molecule in A_s, where $e = \{e^t\}_{t\in T} \in X$. Hence there exists an ω-closed set $B_s \in \Gamma_s$ meeting $B_s \in \omega\eta^-(e_r^s)$ by virtue of the fact that Γ_s is an $r\omega$-RF of A_s; thus $\rho_s^{\leftarrow}(B_s)(e) = B_s(e^s) \not\geq r$; that is, $\rho_s^{\leftarrow}(B_s) \in \omega\eta^-(e_r)$. This shows that Ψ is an $r\omega$-RF of A. Therefore A is an $L\omega$-compact set in (L^X, Ω). □

Theorem 32 (Tychonoff product theorem). *Let (L^X, Ω) be the product space of a collection of $L\omega$-spaces $\{(L^{X_t}, \Omega_t) \mid t \in T\}$. Then (L^X, Ω) is $L\omega$-compact if and only if for each $t \in T$, (L^{X_t}, Ω_t) is $L\omega$-compact.*

Proof. Necessity. Assume that (L^X, Ω) is an $L\omega$-compact space. Since $\rho_t : (L^X, \Omega) \to (L^{X_t}, \Omega_t)$ is an ω-continuous L-valued Zadeh's type function for each $t \in T$, (L^{X_t}, Ω_t) is an $L\omega$-compact space by Theorem 23. Therefore the necessity holds.

Sufficiency. It follows from Theorem 31. □

The following example shows that the inverse theorem of Theorem 31 does not hold.

Example 33. Let $E = \{e_1, e_2, \ldots\}$ be a countably infinite set, $X_t = E$ for each $t \in T = \{1, 2, \ldots\}$, $L = [0, 1]$, $\Omega_t = [0, 1]^E$ and let ω be a fuzzy closure operator. Then (L^{X_t}, Ω_t) is a discrete $L\omega$-space for each $t \in T$. Define $A_t \in L^{X_t}$ $(t \in T)$ as follows:

$$\text{if } j = 1,\ A_t(e_j) = 1; \text{ if } j \geq 2,\ A_t(e_j) = 1/t.$$

Suppose that (L^X, Ω) is the product space of $\{(L^{X_t}, \Omega_t) \mid t \in T\}$ and $A = \Pi_{t\in T} A_t$. Now we prove that A is an $L\omega$-compact set in (L^X, Ω), but A_t is not an ω-compact set in (L^{X_t}, Ω_t) for each $t \in T$. In reality, for each $x = (x_1, x_2, \ldots) \in X$ we put $x_t = e_{j(t)}^t$, where x_t is a crisp point e_j in X_t; then from the definitions of A_t and fuzzy product set A we know

$$A(x) = (\Pi_{t\in T} A_t)(x) = \wedge_{t\in T} A_t(x_t) = \wedge_{t\in T} A_t(e_{j(t)}^t) = \begin{cases} 0, & \text{if there are infinite elements } t \text{ such that } j(t) \geq 2, \\ \dfrac{1}{t_R}, & \text{if there is a } t_R \in T \text{ such that } j(t_R) \geq 2 \text{ and } j(t) = 1 \text{ whenever } t > t_R. \end{cases} \tag{7}$$

Thus it can be seen that $A \neq 0_X$ and if $A(x) \geq 1/t_R$, then the coordinates $x_t = e_{j(t)}^t = e_1$ of x whenever $t > t_R$. Obviously, points in X satisfying the condition are only finite. Let $\alpha \in M$, that is, $\alpha > 0$, and let Φ be an $\alpha\omega$-RF of A. Choose $t_R \in T$ with $1/t_R < \alpha$. Since there are only finite molecule in A with hight α, denote the finite crisp points as $x^1, x^2, \ldots, x^n$. If $(x^i)_\alpha \leq A$ for each $i \in \{1, 2, \ldots, n\}$, then there is $P_i \in \Phi$ with $P_i(x^i) < \alpha$. Put $s = \max\{P_i(x^i) \mid P_i(x^i) < \alpha, i \leq n\}$; then $s < \alpha$. Taking $s_1 \in (s, \alpha)$ and $r = \max(s_1, 1/t_R)$, we know that A has at most n molecules with hight r, say $(x^i)_r$ $(i \leq n)$. By the definition of Φ, there is a $P_i \in \Phi$ such that $P_i \in \omega\eta^-((x^i)_r)$ for each $(x^i)_r$ in A. Denote $\Psi = \{P_i \in \Phi \mid P_i \in \omega\eta^-((x^i)_r), i \leq n\}$; then $\Psi \in 2^{(\Phi)}$ and Ψ is an $r\omega$-RF of A. This implies that Ψ is an $(\alpha\omega)^-$-RF of A by $r \in \beta^*(\alpha)$. Hence A is $L\omega$-compact in (L^X, Ω). On the other hand, take $D = T$ and $N = \{N(m) \mid m \in D\}$ where $N(m) = (e_m)_{1/t}$ for each $m \in D$ and each $t \in T$; then N is a $(1/t)$-net in A_t. Since (L^{X_t}, Ω_t) is discrete, N does not have any ω-cluster point in A_t with hight $1/t$. Therefore A_t is not $L\omega$-compact in (L^{X_t}, Ω_t) for each $t \in T$ according to Theorem 15.

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