Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2013, Article ID 926078, 8 pages
http://dx.doi.org/10.1155/2013/926078

Research Article

Viscosity Approximation Methods and Strong Convergence Theorems for the Fixed Point of Pseudocontractive and Monotone Mappings in Banach Spaces

Yan Tang

College of Mathematics and Statistics, Chongqing Technology and Business University, Chongqing 400067, China

Correspondence should be addressed to Yan Tang; ttyy7999@163.com

Received 22 January 2013; Revised 26 May 2013; Accepted 27 May 2013

Academic Editor: Filomena Cianciaruso

Suppose that C is a nonempty closed convex subset of a real reflexive Banach space E which has a uniformly Gateaux differentiable norm. A viscosity iterative process is constructed in this paper. A strong convergence theorem is proved for a common element of the set of fixed points of a finite family of pseudocontractive mappings and the set of solutions of a finite family of monotone mappings. And the common element is the unique solution of certain variational inequality. The results presented in this paper extend most of the results that have been proposed for this class of nonlinear mappings.

1. Introduction

Let E be a real Banach space with dual E^*. A normalized duality mapping $J : E \to 2^{E^*}$ is defined by

$$Jx = \left\{ f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2 \right\}, \tag{1}$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing between E and E^*. It is well known that E is smooth if and only if J is single valued, and if E is uniformly smooth, then J is uniformly continuous on bounded subsets of E. Moreover, if E is reflexive and strictly convex Banach space with a strictly convex dual then J^{-1} is single valued, one-to-one surjective, and it is the duality mapping from E^* into E, and then $JJ^{-1} = I_{E^*}$ and $JJ^{-1} = I_E$.

A mapping $A : D(A) \subset E \to E^*$ is said to be monotone if, for each $x, y \in D(A)$, the following inequality holds:

$$\langle x - y,\ Ax - Ay \rangle \geq 0. \tag{2}$$

A mapping $A : D(A) \subset E \to E^*$ is said to be strongly monotone, if there exists a positive real number $\alpha > 0$ such that for all $x, y \in D(A)$

$$\langle x - y,\ Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2. \tag{3}$$

Obviously, the class of monotone mappings includes the class of the α-inverse strongly monotone mappings.

Let C be closed convex subset of Banach space E. A mapping T is said to be pseudocontractive if, for any $x, y \in D(T)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2. \tag{4}$$

A mapping T is said to be κ-strictly pseudo-contractive, if, for any $x, y \in D(T)$, there exist $j(x - y) \in J(x - y)$ and a constant $0 \leq \kappa \leq 1$ such that

$$\langle x - y - (Tx - Ty), j(x - y) \rangle \geq \kappa \|(I - T)x - (I - T)y\|^2. \tag{5}$$

A mapping $T : C \to C$ is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{6}$$

Clearly, the class of pseudo-contractive mappings includes the class of strict pseudo-contractive mappings and non-expansive mappings. We denote by $F(T)$ the set of fixed points of T; that is, $F(T) = \{x \in C : Tx = x\}$.

A mapping $f : C \to C$ is called contractive with a contraction coefficient if there exists a constant $\rho \in (0, 1)$ such that

$$\|f(x) - f(y)\| \leq \rho \|x - y\|, \quad \forall x, y \in C. \tag{7}$$

Let E be a real Banach space with dual E^*. The norm on E is said to be uniformly Gateaux differentiable if for each $y \in S_1(0) := \{x \in E : \|x\| = 1\}$ the limit $\lim_{t\to 0}(\|x + ty\| - \|x\|)/t$ exists uniformly for $x \in S_1(0)$.

For finding an element of the set of fixed points of the non-expansive mappings, Halpern [1] was the first to study the convergence of the scheme in 1967:

$$x_{n+1} = \alpha_{n+1}u + (1 - \alpha_{n+1}) T(x_n). \tag{8}$$

Viscosity approximation methods are very important because they are applied to convex optimization, linear programming, monotone inclusions, and elliptic differential equations. In Hilbert spaces, many authors have studied the fixed points problems of the fixed points for the non-expansive mappings and monotone mappings by the viscosity approximation methods, and obtained a series of good results (see [2–17]).

In 2000, Moudifi [18] introduced the viscosity approximation methods and proved the strong convergence of the following iterative algorithm in Hilbert spaces under some suitable conditions:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T(x_n). \tag{9}$$

Suppose that A is monotone mapping from C into E. The classical variational inequality problem is formulated as finding a point $u \in C$ such that $\langle v - u, Au\rangle \geq 0$, for all $v \in C$. The set of solutions of variational inequality problems is denoted by $VI(C, A)$.

Takahashi and Toyoda [19, 20] introduced the following scheme in Hilbert spaces and studied the weak and strong convergence theorem of the elements of $F(T) \cap VI(C, A)$, respectively, under different conditions:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_{n+1}) TP_C(x_n - \lambda_n x_n), \tag{10}$$

where T is non-expansive mapping and A is α-inverse strong monotone operator.

Recently, Zegeye and Shahzad [21] introduced the following algorithm and obtained the strong convergence theorem but still in Hilbert spaces:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T_{r_n} F_{r_n} x_n, \tag{11}$$

where T_{r_n}, F_{r_n} are nonexpansive mappings.

Our concern now is the following: is it possible to construct a new sequence in Banach spaces which converges strongly to a common element of fixed points of a finite family of pseudocontractive mappings and the solution set of a variational inequality problems for finite family of monotone mappings?

2. Preliminaries

In the sequel, we will use the following lemmas.

Lemma 1 (see, e.g., [5]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1 - \theta_n) a_n + \sigma_n, \quad n \geq 0, \tag{12}$$

where $\{\theta_n\}$ is a sequence in (0,1) and $\{\sigma_n\}$ is a real sequence such that

(i) $\sum_{n=0}^{\infty} \theta_n = \infty$;

(ii) $\limsup_{n\to\infty} \sigma_n/\theta_n \leq 0$ *or* $\sum_{n=0}^{\infty} \sigma_n < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2 (see, e.g., [10]). *Let C be a nonempty, closed, and convex subset of uniformly smooth strictly convex real Banach space E with dual E^*. Let $A : C \to E^*$ be a continuous monotone mapping; define mapping F_r as follows: $x \in E, r \in (0, \infty)$*

$$F_r(x) = \left\{z \in C : \langle y - z, Az\rangle + \frac{1}{r}\langle y - z, \ Jz - Jx\rangle \geq 0, \ \forall y \in C\right\}. \tag{13}$$

Then the following hold:

(i) *F_r is well defined and single valued;*

(ii) *F_r is a firmly non-expansive mapping; that is,* $\langle F_r x - F_r y, \ JF_r x - JF_r y\rangle \leq \langle F_r x - F_r y, Jx - Jy\rangle$;

(iii) $F(F_r) = VI(C, A)$;

(iv) *$VI(C, A)$ is closed and convex.*

Lemma 3 (see, e.g., [22]). *Let C be a nonempty closed convex subset of uniformly smooth strictly convex real Banach space E. Let $T : C \to E$, be a continuous strictly pseudocontractive mapping; define mapping T_r as follows: $x \in E, \ r \in (0, \infty)$*

$$T_r(x) = \left\{z \in C : \langle y - z, Tz\rangle - \frac{1}{r}\langle y - z, (1 + r) Jz - Jx\rangle \leq 0, \ \forall y \in C\right\}. \tag{14}$$

Then the following hold:

(i) *T_r is single valued;*

(ii) *T_r is a firmly nonexpansive mapping; that is,* $\langle T_r x - T_r y, JT_r x - JT_r y\rangle \leq \langle T_r x - T_r y, Jx - Jy\rangle$;

(iii) $F(T_r) = F(T)$;

(iv) *$F(T)$ is closed and convex.*

Lemma 4 (see, e.g., [23]). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequence in a Banach space, and let $\{\beta_n\}$ be a sequence in $[0, 1]$ which satisfies the following condition:*

$$0 < \liminf_{n\to\infty} \beta_n < \limsup_{n\to\infty} \beta_n < 1. \tag{15}$$

Suppose

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) z_n, \quad n \geq 0, \qquad \lim_{n\to\infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0. \tag{16}$$

Then $\lim_{n\to\infty} \|z_n - x_n\| = 0$.

Lemma 5 (see, e.g., [16]). *Let C be a nonempty closed and convex subset of a real smooth Banach space E. A mapping $\Pi_C : E \to C$ is a generalized projection. Let $x \in E$; then $x_0 = \Pi_C x$ if and only if*

$$\langle z - x_0, Jx - Jx_0 \rangle \le 0, \quad \forall z \in C. \tag{17}$$

Lemma 6. *Let E be a real Banach space with dual E^*. $J : E \to 2^{E^*}$ is the generalized duality pairing; then, for all $x, y \in E$,*

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, j(x + y)\rangle, \quad \forall j(x + y) \in J(x + y). \tag{18}$$

3. Main Results

Let C be a nonempty closed convex and bounded subset of a smooth, strictly convex and reflexive real Banach space E with dual E^*. Let $A_i : C \to E^*$, $i = 1, 2, \ldots, m$ be a finite family of continuous monotone mappings, and let $T_i : C \to C$, $i = 1, 2, \ldots, m$, be a finite family of continuous strictly pseudo-contractive mappings. For the rest of this paper, $T_{ir_n}x$ and $F_{ir_n}x$ are mappings defined as follows: for $x \in E$, $r_n \in (0, \infty)$,

$$\begin{aligned} T_{ir_n}(x) &:= \Big\{ z \in C : \langle y - z, T_i z\rangle - \frac{1}{r_n}\langle y - z, (1 + r_n) Jz - Jx\rangle \le 0, \\ &\qquad \forall y \in C \Big\}. \end{aligned} \tag{19}$$

Consider

$$\begin{aligned} F_{ir_n}(x) &:= \Big\{ z \in C : \langle y - z, A_i z\rangle + \frac{1}{r_n}\langle y - z, Jz - Jx\rangle \ge 0, \\ &\qquad \forall y \in C \Big\}. \end{aligned} \tag{20}$$

Denote $F_1 = \cap_{i=1}^m F(T_{ir_n})$, $F_2 = \cap_{i=1}^m F(F_{ir_n})$.

Lemma 7. *Let C be a nonempty closed convex and bounded subset of a smooth Banach space E, and let $\Gamma_i : C \to C$, $i = 1, 2, \ldots, m$, be a finite family of non-expansive mappings such that $\cap_{i=1}^m F(\Gamma_i) \ne \emptyset$. Suppose that $\alpha = \inf\{\alpha_i\} > 0$; then there exists non-expansive mapping $\Gamma : C \to C$ such that $F(\Gamma) = \cap_{i=1}^m F(\Gamma_i)$.*

Proof. Let $\{\alpha_i\}$ be any sequence of positive real numbers satisfying $\sum_{i=1}^m \alpha_i = 1$ and set $\Gamma = \sum_{i=1}^m \alpha_i \Gamma_i$. Since each Γ_i is non-expansive for any $i \in \{1, 2, \ldots, m\}$, we have that Γ is well-defined nonexpansive mapping (see, e.g., [20]), and

$$\begin{aligned} \|\Gamma x - \Gamma y\| &= \Big\| \sum_{i=1}^m \alpha_i \Gamma_i x - \sum_{i=1}^m \alpha_i \Gamma_i y \Big\| \\ &\le \sum_{i=1}^m \alpha_i \|\Gamma_i x - \Gamma_i y\| \le \|x - y\|. \end{aligned} \tag{21}$$

Next, we claim that $F(\Gamma) = \cap_{i=1}^m F(\Gamma_i)$.

Clearly $\cap_{i=1}^m F(\Gamma_i) \subset F(\Gamma)$. Now, we prove that $F(\Gamma) \subset \cap_{i=1}^m F(\Gamma_i)$. Let $x \in F(\Gamma)$ and $p \in \cap_{i=1}^m F(\Gamma_i)$. Then

$$\begin{aligned} 0 = \langle \Gamma x - x, j(x - p)\rangle &= \Big\langle \sum_{i=1}^m \alpha_i \Gamma_i x - x, j(x - p) \Big\rangle \\ &= \sum_{i=1}^m \alpha_i \langle \Gamma_i x - x, j(x - p)\rangle, \end{aligned} \tag{22}$$

and non-expansivity of each Γ_i implies that $\langle \Gamma_i x - x, j(x - p)\rangle = 0$ for each $i = 1, 2, \ldots, m$, which implies $\Gamma_i x = x$ for each $i = 1, 2, \ldots, m$. Therefore, $x \in \cap_{i=1}^m F(\Gamma_i)$, and hence $F(\Gamma) = \cap_{i=1}^m F(\Gamma_i)$. The proof is complete. □

Theorem 8. *Let C be a nonempty closed convex subset of a real reflexive Banach space E which has a uniformly Gateaux differentiable norm. Let $T_i : C \to C$, $i = 1, 2, \ldots, m$, be a finite family of continuous strictly pseudocontractive mappings, let $A_i : C \to E^*$, $i = 1, 2, \ldots, m$, be a finite family of continuous monotone mappings such that $F = F_1 \cap F_2 \ne \emptyset$, and let $f : C \to C$ be a contraction with a contraction coefficient $\rho \in (0, 1)$. T_{ir_n} and F_{ir_n} are defined as (19) and (20), respectively. Let $\{x_n\}$ be a sequence generated by $x_0 \in C$*

$$\begin{aligned} y_n &= \lambda_n x_n + (1 - \lambda_n) \sum_{i=1}^m \mu_i F_{ir_n} x_n, \\ x_{n+1} &= \alpha_n f(x_n) + \beta_n x_n + \gamma_n \sum_{i=1}^m \sigma_i T_{ir_n} y_n, \end{aligned} \tag{23}$$

where $\lambda_n \in [0, 1]$ and $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences of non-negative real numbers in $[0, 1]$, and

(i) $\alpha_n + \beta_n + \gamma_n = 1$, $n \ge 0$, $\sum_{i=1}^m \mu_i = 1$, $\sum_{i=1}^m \sigma_i = 1$, $\mu_i \ge 0$, $\sigma_i \ge 0$;

(ii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty} \beta_n < \limsup_{n\to\infty} \beta_n < 1$;

(iv) $\limsup_{n\to\infty} r_n > 0$, $\sum_{n=1}^\infty |r_{n+1} - r_n| < \infty$.

Then the sequence $\{x_n\}$ converges strongly to an element $\overline{x} = \Pi_F f(\overline{x})$, and also $\overline{x}$ is the unique solution of the variational inequality

$$\langle (f - I)(\overline{x}), j(y - \overline{x})\rangle \le 0, \quad \forall y \in F. \tag{24}$$

Proof. First we prove that $\{x_n\}$ is bounded. Take $p \in F$, because F_{ir_n} is non-expansive; then we have that

$$\begin{aligned} \|y_n - p\| &\le \lambda_n \|x_n - p\| + (1 - \lambda_n) \\ &\quad \times \sum_{i=1}^m \mu_i \|F_{ir_n} x_n - F_{ir_n} p\| \le \|x_n - p\|. \end{aligned} \tag{25}$$

For $n \geq 0$, because T_{ir_n} and F_{ir_n} are nonexpansive and f is contractive, we have from (25) that

$$\begin{aligned} &\|x_{n+1} - p\| \\ &= \left\| \alpha_n (f(x_n) - p) + \beta_n (x_n - p) + \gamma_n \left(\sum_{i=1}^{m} \sigma_i T_{ir_n} y_n - p \right) \right\| \\ &\leq \alpha_n \|f(x_n) - f(p)\| + \alpha_n \|f(p) - p\| \\ &\quad + \beta_n \|x_n - p\| + \gamma_n \|y_n - p\| \\ &\leq \rho\alpha_n \|x_n - p\| + \alpha_n \|f(p) - p\| + (1 - \alpha_n) \|x_n - p\| \\ &\leq [1 - (1-\rho)\alpha_n] \|x_n - p\| + \alpha_n \|f(p) - p\| \\ &\leq \max \left\{ \|x_0 - p\|, \frac{f(p) - p}{1-\rho} \right\}. \end{aligned} \tag{26}$$

Therefore, $\{x_n\}$ is bounded. Consequently, we get that $\{F_{ir_n} x_n\}$, $\{T_{ir_n} y_n\}$ and $\{y_n\}$, $\{f(x_n)\}$ are bounded.

Next, we show that $\|x_{n+1} - x_n\| \to 0$.

Consider

$$\begin{aligned} &\|y_{n+1} - y_n\| \\ &\leq \lambda_{n+1} \|x_{n+1} - x_n\| + (1 - \lambda_{n+1}) \sum_{i=1}^{m} \mu_i \left\| F_{ir_{n+1}} x_{n+1} - F_{ir_n} x_n \right\| \\ &\quad + |\lambda_{n+1} - \lambda_n| \left\| x_n - \sum_{i=1}^{m} \mu_i F_{ir_n} x_n \right\|. \end{aligned} \tag{27}$$

Let $v_{in} = F_{ir_n} x_n$, $v_{i,n+1} = F_{ir_{n+1}} x_{n+1}$; by the definition of mapping F_{ir_n}, we have that

$$\langle y - v_{in}, A_i v_{in} \rangle + \frac{1}{r_n} \langle y - v_{in}, J v_{in} - J x_n \rangle \geq 0, \quad \forall y \in C, \tag{28}$$

$$\langle y - v_{i,n+1}, A_i v_{in+1} \rangle + \frac{1}{r_{n+1}} \langle y - v_{i,n+1}, J v_{i,n+1} - J x_{n+1} \rangle \geq 0, \quad \forall y \in C. \tag{29}$$

Putting $y := v_{i,n+1}$ in (28) and leting $y := v_{in}$ in (29), we have that

$$\begin{aligned} &\langle v_{i,n+1} - v_{in}, A_i v_{in} \rangle + \frac{1}{r_n} \langle v_{i,n+1} - v_{in}, J v_{in} - J x_n \rangle \geq 0, \\ &\langle v_{in} - v_{i,n+1}, A_i v_{i,n+1} \rangle \\ &\quad + \frac{1}{r_{n+1}} \langle v_{in} - v_{i,n+1}, J v_{i,n+1} - J x_{n+1} \rangle \geq 0. \end{aligned} \tag{30}$$

Adding (30), we have that

$$\begin{aligned} &\langle v_{i,n+1} - v_{in}, A_i v_{in} - A_i v_{i,n+1} \rangle \\ &\quad + \left\langle v_{i,n+1} - v_{in}, \frac{J v_{in} - J x_n}{r_n} - \frac{J v_{i,n+1} - J x_{n+1}}{r_{n+1}} \right\rangle \geq 0. \end{aligned} \tag{31}$$

Since $A_i, i \in \{1, 2, \ldots, m\}$ are monotone mappings, which implies that

$$\left\langle v_{i,n+1} - v_{in}, \frac{J v_{in} - J x_n}{r_n} - \frac{J v_{i,n+1} - J x_{n+1}}{r_{n+1}} \right\rangle \geq 0, \tag{32}$$

therefore we have that

$$\begin{aligned} &\left\langle v_{i,n+1} - v_{in}, J v_{in} - J x_n - \frac{r_n (J v_{i,n+1} - J x_{n+1})}{r_{n+1}} \right. \\ &\quad \left. + J v_{i,n+1} - J v_{i,n+1} \right\rangle \geq 0. \end{aligned} \tag{33}$$

That is,

$$\begin{aligned} &\|v_{i,n+1} - v_{in}\|^2 \\ &\leq \left\langle v_{i,n+1} - v_{in}, J x_{n+1} - J x_n + \left(1 - \frac{r_n}{r_{n+1}}\right) \right. \\ &\quad \left. \times (J v_{i,n+1} - J x_{n+1}) \right\rangle \\ &\leq \|v_{i,n+1} - v_{in}\| \left\{ \|x_{n+1} - x_n\| \right. \\ &\quad \left. + \left| 1 - \frac{r_n}{r_{n+1}} \right| \|v_{i,n+1} - x_{n+1}\| \right\}. \end{aligned} \tag{34}$$

Without loss of generality, let b be a real number such that $r_n > b > 0$; then we have that

$$\begin{aligned} \|v_{i,n+1} - v_{in}\| &\leq \|x_{n+1} - x_n\| + \left| 1 - \frac{r_n}{r_{n+1}} \right| \|v_{i,n+1} - x_{n+1}\| \\ &\leq \|x_{n+1} - x_n\| + \frac{1}{b} |r_{n+1} - r_n| K, \end{aligned} \tag{35}$$

where $K = \sup\{\|v_{i,n+1} - x_{n+1}\|\}$.

Then we have from (35) and (27) that

$$\begin{aligned} \|y_{n+1} - y_n\| \leq{}& \|x_{n+1} - x_n\| + \frac{(1 - \lambda_{n+1}) |r_{n+1} - r_n|}{b} K \\ &+ |\lambda_{n+1} - \lambda_n| \left\| x_n - \sum_{i=1}^{m} \mu_i F_{ir_n} x_n \right\|. \end{aligned} \tag{36}$$

On the other hand, let $u_{in} = T_{ir_n} y_n$, $u_{i,n+1} = T_{ir_{n+1}} y_{n+1}$; we have that

$$\langle y - u_{in}, T_i u_{in} \rangle - \frac{1}{r_n} \langle y - u_{in}, (1 + r_n) J u_{in} - J y_n \rangle \leq 0, \quad \forall y \in C, \tag{37}$$

$$\begin{aligned} &\langle y - u_{i,n+1}, T_i u_{i,n+1} \rangle - \frac{1}{r_{n+1}} \langle y - u_{i,n+1}, (1 + r_{n+1}) \\ &\quad \times J u_{i,n+1} - J y_{n+1} \rangle \leq 0, \quad \forall y \in C. \end{aligned} \tag{38}$$

Let $y := u_{i,n+1}$ in (37), and let $y := u_{in}$ in (38); we have that

$$\langle u_{i,n+1} - u_{in}, T_i u_{in}\rangle - \frac{1}{r_n}\langle u_{i,n+1} - u_{in}, (1+r_n) J u_{in} - J y_n\rangle \le 0,$$
$$\langle u_{in} - u_{i,n+1}, T_i u_{i,n+1}\rangle - \frac{1}{r_{n+1}}\langle u_{in} - u_{i,n+1}, (1+r_{n+1}) J u_{i,n+1} - J y_{n+1}\rangle \le 0. \tag{39}$$

Adding (39) and because T_i, $i \in \{1,2,\ldots,m\}$ is pseudo-contractive, we have that

$$\left\langle u_{i,n+1} - u_{in}, \frac{J u_{in} - J y_n}{r_n} - \frac{J u_{i,n+1} - J y_{n+1}}{r_{n+1}} \right\rangle \ge 0. \tag{40}$$

Therefore we have

$$\left\langle u_{i,n+1} - u_{in}, J u_{in} - J y_n - \frac{r_n (J u_{i,n+1} - J y_{n+1})}{r_{n+1}} + J u_{i,n+1} - J u_{i,n+1} \right\rangle \ge 0. \tag{41}$$

Hence we have that

$$\|u_{i,n+1} - u_{in}\| \le \|y_{n+1} - y_n\| + \frac{1}{b}|r_{n+1} - r_n| M, \tag{42}$$

where $M = \sup\{\|u_{in} - y_n\|\}$.

Let $x_{n+1} = \beta_n x_n + (1-\beta_n) z_n$. Hence we have that

$$\begin{aligned} z_{n+1} - z_n &= \frac{\alpha_{n+1}}{1-\beta_{n+1}}(f(x_{n+1}) - f(x_n)) \\ &\quad + \left(\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right) f(x_n) \\ &\quad + \frac{\gamma_{n+1}}{1-\beta_{n+1}}(u_{i,n+1} - u_{in}) \\ &\quad + \left(\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}\right) u_{in}. \end{aligned} \tag{43}$$

Hence we have from (43), (42), and (36) that

$$\begin{aligned} &\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \\ &\le \frac{(\rho-1)\alpha_{n+1}}{1-\beta_{n+1}}\|x_{n+1} - x_n\| + \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right| \\ &\quad \times \{\|f(x_n)\| + \|u_{in}\|\} + \frac{\gamma_{n+1}}{1-\beta_{n+1}}\frac{|r_{n+1} - r_n|}{b} \\ &\quad \times ((1-\lambda_{n+1})K + M) + \frac{\gamma_{n+1}}{1-\beta_{n+1}}|\lambda_{n+1} - \lambda_n| \\ &\quad \times \left\| x_n - \sum_{i=1}^m \mu_i F_{ir_n} x_n \right\|. \end{aligned} \tag{44}$$

Noticing the conditions (ii) and (iv), we have that

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) = 0. \tag{45}$$

Hence we have from Lemma 4 that

$$\limsup_{n\to\infty}\|z_n - x_n\| = 0. \tag{46}$$

Therefore we have that

$$\|x_{n+1} - x_n\| = |1-\beta_n|\,\|z_n - x_n\| \longrightarrow 0. \tag{47}$$

Hence we have from (35), (36), and (42) that

$$\begin{aligned} \|y_{n+1} - y_n\| &\longrightarrow 0, \\ \|u_{i,n+1} - u_{in}\| &\longrightarrow 0, \\ \|v_{i,n+1} - v_{in}\| &\longrightarrow 0. \end{aligned} \tag{48}$$

In addition, since $x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n \sum_{i=1}^m \sigma_i u_{in}$, $y_n = \lambda_n x_n + (1-\lambda_n)\sum_{i=1}^m \mu_i v_{in}$, for all $p \in F$, we have from the monotonicity of A_i, the non-expansivity of T_{ir_n}, and the convexity of $\|\cdot\|^2$ that

$$\begin{aligned} &\|x_{n+1} - p\|^2 \\ &= \left\|\alpha_n f(x_n) + \beta_n x_n + \gamma_n \sum_{i=1}^m \sigma_i u_{in} - p\right\|^2 \\ &\le \|\alpha_n (f(x_n) - p) + \beta_n (x_n - p)\|^2 + \gamma_n \sum_{i=1}^m \sigma_i \|u_{in} - p\|^2 \\ &\le \alpha_n \|f(x_n) - p\|^2 + \beta_n \|x_n - p\|^2 + \gamma_n \|y_n - p\|^2 \\ &\le \alpha_n \|f(x_n) - p\|^2 + \beta_n \|x_n - p\|^2 \\ &\quad + \gamma_n \lambda_n \|x_n - p\|^2 + (1-\lambda_n)\gamma_n \|v_n - p\|^2 \\ &\le \alpha_n \|f(x_n) - p\|^2 + (\beta_n + \gamma_n\lambda_n)\|x_n - p\|^2 \\ &\quad + (1-\lambda_n)\gamma_n (\|x_n - p\|^2 - \|x_n - v_{in}\|^2) \\ &\le \alpha_n \|f(x_n) - p\|^2 + \|x_n - p\|^2 - (1-\lambda_n)\gamma_n \|x_n - v_{in}\|^2. \end{aligned} \tag{49}$$

So we have that

$$\begin{aligned} &(1-\lambda_n)\gamma_n \|x_n - v_{in}\|^2 \\ &\le \alpha_n \|f(x_n) - p\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\ &\le \alpha_n \|f(x_n) - p\|^2 + \|x_n - x_{n+1}\| \times (\|x_n - p\| + \|x_{n+1} - p\|). \end{aligned} \tag{50}$$

Since $\alpha_n \to 0$, so we have from (47) that

$$\|x_n - v_{in}\| \longrightarrow 0. \tag{51}$$

In a similar way, we have that

$$\|x_n - u_{in}\| \longrightarrow 0. \tag{52}$$

Consequently, we have that

$$\begin{aligned} \|y_n - x_n\| &= |1-\lambda_n| \|x_n - v_{in}\| \longrightarrow 0, \\ \|y_n - u_{in}\| &\le \|y_n - x_n\| + \|x_n - u_{in}\| \longrightarrow 0. \end{aligned} \tag{53}$$

Since E is a reflexive real Banach space and the sequence $\{x_n\}$ is bounded, so there exists the subsequence $\{x_{nk}\}$ of $\{x_n\}$ and $w \in C$ such that $x_{nk} \rightharpoonup w$. And because $x_n \to v_{in}$, $n \to \infty$, therefore $v_{ink} \rightharpoonup w$. Next we show that $w \in F$.

Because $v_{in} = F_{ir_n} x_n$, by the definition of mapping F_{ir_n}, we have that

$$\begin{gathered} \langle y - v_{in}, A_i v_{in} \rangle + \frac{1}{r_n} \langle y - v_{in}, J v_{in} - J x_n \rangle \ge 0, \quad \forall y \in C, \\ \langle y - v_{ink}, A_i v_{ink} \rangle + \left\langle y - v_{in}, \frac{J v_{ink} - J x_{nk}}{r_n} \right\rangle \ge 0, \quad \forall y \in C. \end{gathered} \tag{54}$$

Let $v_t = tv + (1-t)w$, $t \in [0,1]$, for all $v \in C$; we have that

$$\begin{aligned} \langle v_t - v_{ink}, A_i v_t \rangle \ge\ & \langle v_t - v_{ink}, A_i v_t \rangle - \langle v_t - v_{ink}, A_i v_{ink} \rangle \\ & - \left\langle v_t - v_{ink}, \frac{J v_{ink} - J x_{nk}}{r_n} \right\rangle \\ =\ & \langle v_t - v_{ink}, \ A_i v_t - A_i v_{ink} \rangle \\ & - \left\langle v_t - v_{ink}, \frac{J v_{ink} - J x_{nk}}{r_n} \right\rangle. \end{aligned} \tag{55}$$

Because $x_{nk} - v_{ink} \to 0$, so $(J v_{ink} - J x_{nk})/r_n \to 0$, and because A_i are monotone, we have that

$$0 \le \lim_{k\to\infty} \langle v_t - v_{ink}, A_i v_t \rangle = \langle v_t - w, A_i v_t \rangle. \tag{56}$$

Consequently we have that

$$\langle v - w, A_i v_t \rangle \ge 0. \tag{57}$$

If $t \to 0$, by the continuity of A_i, we have that $\langle v - w, A_i w \rangle \ge 0$; that is, $w \in VI(C, A_i)$, and then $w \in F_2$.

Similarly, because $u_{in} = T_{ir_n} y_n$, by the definition of mapping T_{ir_n}, we have that

$$\begin{gathered} \langle y - u_{in}, T_i u_{in} \rangle - \frac{1}{r_n} \langle y - u_{in}, (1+r_n) J u_{in} - J y_n \rangle \le 0, \\ \forall y \in C, \\ \langle y - u_{ink}, T_i u_{ink} \rangle - \frac{1}{r_n} \langle y - u_{ink}, (1+r_n) J u_{ink} - J y_{nk} \rangle \le 0, \\ \forall y \in C. \end{gathered} \tag{58}$$

Let $v_t = tv + (1-t)w$, $t \in [0,1]$, for all $v \in C$. Because $\{T_i\}$, $i \in \{1, 2, \dots, m\}$ are pseudocontractive mappings, we have that

$$\begin{aligned} & \langle u_{ink} - v_t, T_i v_t \rangle \\ & \ge \langle u_{ink} - v_t, T_i v_t \rangle + \langle v_t - u_{ink}, T_i u_{ink} \rangle \\ & \quad - \frac{1}{r_n} \langle v_t - u_{ink}, (1+r_n) J u_{ink} - J y_{nk} \rangle \\ & = \langle v_t - u_{ink}, T_i u_{ink} - T_i v_t \rangle \\ & \quad - \left\langle v_t - u_{ink}, \frac{1+r_n}{r_n} J u_{ink} - \frac{1}{r_n} J y_{nk} \right\rangle \\ & \ge -\|v_t - u_{ink}\|^2 - \frac{1}{r_n} \langle v_t - u_{ink}, J u_{ink} - J y_{nk} \rangle \\ & \quad - \langle v_t - u_{ink}, J u_{ink} \rangle \\ & = - \langle v_t - u_{ink}, J v_t \rangle - \frac{1}{r_n} \langle v_t - u_{ink}, J u_{ink} - J y_{nk} \rangle. \end{aligned} \tag{59}$$

Because $y_{nk} - u_{ink} \to 0$, so $J u_{ink} - J y_{nk} \to 0$; we have that

$$\lim_{k\to\infty} \langle u_{ink} - v_t, T_i v_t \rangle \ge \lim_{k\to\infty} \langle u_{ink} - v_t, J v_t \rangle. \tag{60}$$

Consequently we have that

$$\begin{gathered} \langle w - v_t, T_i v_t \rangle \ge \langle w - v_t, J v_t \rangle, \\ \langle v - w, T_i v_t \rangle \le \langle v - w, J v_t \rangle. \end{gathered} \tag{61}$$

If $t \to 0$, by the continuity of T_i, we have that $\langle v - w, T_i w - Jw \rangle \ge 0$, for all $v \in C$; we conclude that $w = T_i w$; that is, $w \in F(T_i)$, and then $w \in F_1$. Consequently $w \in F = F_1 \cap F_2$.

Because $\overline{x} = \Pi_F f(\overline{x})$, we have from Lemma 5 that

$$\begin{aligned} & \limsup_{n\to\infty} \langle f(\overline{x}) - \overline{x}, j(x_{n+1} - \overline{x}) \rangle \\ & \quad = \langle f(\overline{x}) - \overline{x}, j(w - \overline{x}) \rangle \le 0. \end{aligned} \tag{62}$$

Next we show that $x_n \to \overline{x}$. Since $u_{in} = T_{ir_n} y_n$, from formula (23) and Lemma 6 we have that

$$\begin{aligned} & \|x_{n+1} - \overline{x}\|^2 \\ & = \left\| \alpha_n (f(x_n) - \overline{x}) + \beta_n (x_n - \overline{x}) + \gamma_n \left(\sum_{i=1}^m \sigma_i T_{ir_n} y_n - \overline{x} \right) \right\|^2 \\ & \le \left\| \beta_n (x_n - \overline{x}) + \gamma_n \left(\sum_{i=1}^m \sigma_i u_{in} - \overline{x} \right) \right\|^2 \\ & \quad + 2\alpha_n \langle f(x_n) - \overline{x}, \ j(x_{n+1} - \overline{x}) \rangle \end{aligned}$$

$$\begin{aligned}&\leq (1-\alpha_n)^2\|x_n-\overline{x}\|^2 + 2\alpha_n \langle f(x_n) - f(\overline{x}), j(x_{n+1}-\overline{x})\rangle \\ &\quad + 2\alpha_n \langle f(\overline{x}) - \overline{x}, j(x_{n+1}-\overline{x})\rangle \\ &\leq (1-\alpha_n)^2\|x_n-\overline{x}\|^2 + 2\rho\alpha_n \|x_n-\overline{x}\| \|x_{n+1}-\overline{x}\| \\ &\quad + 2\alpha_n \langle f(\overline{x}) - \overline{x}, j(x_{n+1}-\overline{x})\rangle \\ &\leq (1-2\alpha_n(1-\rho))\|x_n-\overline{x}\|^2 + \sigma_n,\end{aligned} \tag{63}$$

where $M_1 = \sup\{\|x_n-\overline{x}\|\}$, $\sigma_n = \alpha_n^2 M_1^2 + 2\rho\alpha_n\|x_{n+1}-x_n\|M_1 + 2\alpha_n\langle f(\overline{x}) - \overline{x}, j(x_{n+1}-\overline{x})\rangle$. Let $\theta_n = 2\alpha_n(1-\rho)$; according to Lemma 1 and formula (62), we have that $\lim_{n\to\infty}\|x_n-\overline{x}\| = 0$; that is, the sequence $\{x_n\}$ converges strongly to $\overline{x} \in F$.

According to formula (62) we conclude that $\overline{x}$ is the solution of the variational inequality (24). Now we show that $\overline{x}$ is the unique solution of the variational inequality (24). Suppose that $\overline{y} \in F$ is another solution of the variational inequality (24). Because $\overline{x}$ is the solution of the variational inequality (24), that is, $\langle (f-I)\overline{x}, j(y-\overline{x})\rangle \leq 0$, for all $y \in F$. And because that $\overline{y} \in F$, then we have

$$\langle (f-I)\overline{x}, j(\overline{y}-\overline{x})\rangle \leq 0. \tag{64}$$

On the other hand, to the solution $\overline{y} \in F$, since $\overline{x} \in F$, so

$$\langle (f-I)\overline{y}, j(\overline{x}-\overline{y})\rangle \leq 0. \tag{65}$$

Adding (65) and (64), we have that

$$\langle \overline{x}-\overline{y} - (f(\overline{x}) - f(\overline{y})), j(\overline{x}-\overline{y})\rangle \leq 0. \tag{66}$$

Hence

$$(1-\rho)\|\overline{x}-\overline{y}\|^2 \leq 0. \tag{67}$$

Because $\rho \in (0,1)$, hence we conclude that $\overline{x} = \overline{y}$, and the uniqueness of the solution is obtained. □

Theorem 9. *Let C be a nonempty closed convex subset of a real reflexive Banach space E which has a uniformly Gateaux differentiable norm. Let $T_i : C \to C$, $i = 1,2,\ldots,m$, be a finite family of continuous strictly pseudocontractive mappings, let $A_i : C \to E^*$, $i = 1,2,\ldots,m$, be a finite family of continuous monotone mappings such that $F = F_1 \cap F_2 \neq \emptyset$, and let $f : C \to C$ be a contraction with a contraction coefficient $\rho \in (0,1)$. T_{ir_n} and F_{ir_n} are defined as (19) and (20), respectively. Let x_n be a sequence generated by $x_0 \in C$*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n \sum_{i=1}^{m} \varepsilon_i T_{ir_n} F_{ir_n} x_n, \tag{68}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences of nonnegative real numbers in $[0,1]$, and

(i) $\alpha_n + \beta_n + \gamma_n = 1$, $n \geq 0$, $\sum_{i=1}^{m} \varepsilon_i = 1$, $\varepsilon_i \geq 0$;

(ii) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty}\beta_n < \limsup_{n\to\infty}\beta_n < 1$;

(iv) $\limsup_{n\to\infty} r_n > 0$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Then the sequence x_n converges strongly to an element $\overline{x} = \Pi_F f(\overline{x})$, and also $\overline{x}$ is the unique solution of the variational inequality

$$\langle (f-I)(\overline{x}), j(y-\overline{x})\rangle \leq 0, \quad \forall y \in F. \tag{69}$$

Proof. Putting $\lambda_n = 0$ in Theorem 8, we can obtain the result. □

If, in Theorems 8 and 9, we let $f :\equiv u \in C$ be a constant mapping, we have the following corollaries.

Corollary 10. *Let C be a nonempty closed convex subset of a real reflexive Banach space E which has a uniformly Gateaux differentiable norm. Let $T_i : C \to C$, $i = 1,2,\ldots,m$, be a finite family of continuous strictly pseudo-contractive mappings, and $A_i : C \to E^*$, $i = 1,2,\ldots,m$, be a finite family of continuous monotone mappings such that $F = F_1 \cap F_2 \neq \emptyset$. T_{ir_n} and F_{ir_n} are defined as (19) and (20), respectively. Let x_n be a sequence generated by $x_0 \in C$*

$$\begin{aligned} y_n &= \lambda_n x_n + (1-\lambda_n) \sum_{i=1}^{m} \mu_i F_{ir_n} x_n, \\ x_{n+1} &= \alpha_n u + \beta_n x_n + \gamma_n \sum_{i=1}^{m} \sigma_i T_{ir_n} y_n, \end{aligned} \tag{70}$$

where $\lambda_n \in [0,1]$ and $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences of nonnegative real numbers in $[0,1]$, and

(i) $\alpha_n + \beta_n + \gamma_n = 1$, $n \geq 0$, $\sum_{i=1}^{m} \mu_i = 1$, $\sum_{i=1}^{m} \sigma_i = 1$, $\mu_i \geq 0$, $\sigma_i \geq 0$;

(ii) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty}\beta_n < \limsup_{n\to\infty}\beta_n < 1$;

(iv) $\limsup_{n\to\infty} r_n > 0$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Then the sequence x_n converges strongly to $\overline{x} = \Pi_F u$, and also $\overline{x}$ is the unique solution of the variational inequality

$$\langle u-\overline{x}, j(y-\overline{x})\rangle \leq 0, \quad \forall y \in F. \tag{71}$$

Corollary 11. *Let C be a nonempty closed convex subset of a real reflexive Banach space E which has a uniformly Gateaux differentiable norm. Let $T_i : C \to C$, $i = 1,2,\ldots,m$, be a finite family of continuous strictly pseudocontractive mappings, and $A_i : C \to E^*$, $i = 1,2,\ldots,m$, be a finite family of continuous monotone mappings such that $F = F_1 \cap F_2 \neq \emptyset$. T_{ir_n} and F_{ir_n} are defined as (19) and (20), respectively. Let x_n be a sequence generated by $x_0 \in C$*

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n \sum_{i=1}^{m} \varepsilon_i T_{ir_n} F_{ir_n} x_n, \tag{72}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences of nonnegative real numbers in $[0,1]$, and

(i) $\alpha_n + \beta_n + \gamma_n = 1$, $n \geq 0$, $\sum_{i=1}^{m} \varepsilon_i = 1$, $\varepsilon_i > 0$;

(ii) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty}\beta_n < \limsup_{n\to\infty}\beta_n < 1$;

(iv) $\limsup_{n\to\infty} r_n > 0$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Then the sequence x_n converges strongly to $\overline{x} = \Pi_F u$, and also $\overline{x}$ is the unique solution of the variational inequality

$$\langle u - \overline{x}, j(y - \overline{x}) \rangle \leq 0, \quad \forall y \in F. \tag{73}$$

Acknowledgments

This work is supported by the National Science Foundation of China (11001287), Natural Science Foundation Project of Chongqing (CSTC, 2012jjA00039).

References

[1] B. Halpern, "Fixed points of nonexpanding maps," *Bulletin of the American Mathematical Society*, vol. 73, pp. 957–961, 1967.

[2] Y. Yao, "A general iterative method for a finite family of nonexpansive mappings," *Nonlinear Analysis, Theory, Methods and Applications*, vol. 66, no. 12, pp. 2676–2687, 2007.

[3] Y. Yao, R. Chen, and J.-C. Yao, "Strong convergence and certain control conditions for modified Mann iteration," *Nonlinear Analysis, Theory, Methods and Applications*, vol. 68, no. 6, pp. 1687–1693, 2008.

[4] S. S. Chang, H. W. Joseph Lee, and C. K. Chan, "On Reich's strong convergence theorem for asymptotically nonexpansive mappings in Banach spaces," *Nonlinear Analysis, Theory, Methods and Applications*, vol. 66, no. 11, pp. 2364–2374, 2007.

[5] H.-K. Xu, "Viscosity approximation methods for nonexpansive mappings," *Journal of Mathematical Analysis and Applications*, vol. 298, no. 1, pp. 279–291, 2004.

[6] S.-Y. Matsushita and W. Takahashi, "Strong convergence theorems for nonexpansive nonself-mappings without boundary conditions," *Nonlinear Analysis, Theory, Methods and Applications*, vol. 68, no. 2, pp. 412–419, 2008.

[7] L. Deng and Q. Liu, "Iterative scheme for nonself generalized asymptotically quasi-nonexpansive mappings," *Applied Mathematics and Computation*, vol. 205, no. 1, pp. 317–324, 2008.

[8] Y. Song, "A new sufficient condition for the strong convergence of Halpern type iterations," *Applied Mathematics and Computation*, vol. 198, no. 2, pp. 721–728, 2008.

[9] N. Shioji and W. Takahashi, "Strong convergence of approximated sequences for nonexpansive mappings in danach spaces," *Proceedings of the American Mathematical Society*, vol. 125, no. 12, pp. 3641–3645, 1997.

[10] W. Takahashi and K. Zembayashi, "Strong and weak convergence theorems for equilibrium problems and relatively nonexpansive mappings in Banach spaces," *Nonlinear Analysis, Theory, Methods and Applications*, vol. 70, no. 1, pp. 45–57, 2009.

[11] H. Zegeye, "An iterative approximation for a common fixed point of two Pseudo-contractive mappings," *ISRN Mathematical Analysis*, vol. 2011, Article ID 621901, 14 pages, 2011.

[12] W. Takahashi, *Non-Linear Functional Analysis-Fixed Point Theory and Its Applications*, Yokohama Publishers Inc, Yokohama, Japan, 2000, (in Japanese).

[13] J. Lou, L.-J. Zhang, and Z. He, "Viscosity approximation methods for asymptotically nonexpansive mappings," *Applied Mathematics and Computation*, vol. 203, no. 1, pp. 171–177, 2008.

[14] L.-C. Ceng, H.-K. Xu, and J.-C. Yao, "The viscosity approximation method for asymptotically nonexpansive mappings in Banach spaces," *Nonlinear Analysis, Theory, Methods and Applications*, vol. 69, no. 4, pp. 1402–1412, 2008.

[15] Y. Song and R. Chen, "Strong convergence theorems on an iterative method for a family of finite nonexpansive mappings," *Applied Mathematics and Computation*, vol. 180, no. 1, pp. 275–287, 2006.

[16] H. Zegeye and N. Shahzad, "Approximating common solution of variational inequality problems for two monotone mappings in Banach spaces," *Optimization Letters*, vol. 5, no. 4, pp. 691–704, 2011.

[17] D.-J. Wen, "Projection methods for a generalized system of nonconvex variational inequalities with different nonlinear operators," *Nonlinear Analysis, Theory, Methods and Applications*, vol. 73, no. 7, pp. 2292–2297, 2010.

[18] A. Moudifi, "Viscosity approximation methods for fixed point problems," *Journal of Mathematical Analysis and Applications*, vol. 241, no. 1, pp. 46–55, 2000.

[19] W. Takahashi and M. Toyoda, "Weak convergence theorems for nonexpansive mappings and monotone mappings," *Journal of Optimization Theory and Applications*, vol. 118, no. 2, pp. 417–428, 2003.

[20] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama, Yokohama, Japan, 2000.

[21] H. Zegeye and N. Shahzad, "Strong convergence of an iterative method for pseudo-contractive and monotone mappings," *Journal of Global Optimization*, vol. 54, no. 1, pp. 173–184, 2011.

[22] H. Zegeye and N. Shahzad, "Strong convergence theorems for a common zero of a countably infinite family of α-inverse strongly accretive mappings," *Nonlinear Analysis, Theory, Methods and Applications*, vol. 71, no. 1-2, pp. 531–538, 2009.

[23] T. Suzuki, "Strong convergence of Krasnoselskii and Mann's type sequences for one-parameter nonexpansive semigroups without Bochner integrals," *Journal of Mathematical Analysis and Applications*, vol. 305, no. 1, pp. 227–239, 2005.