

Research Article

Stability and Hopf Bifurcation Analysis of Coupled Optoelectronic Feedback Loops

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The dynamics of a coupled optoelectronic feedback loops are investigated. Depending on the coupling parameters and the feedback strength, the system exhibits synchronized asymptotically stable equilibrium and Hopf bifurcation. Employing the center manifold theorem and normal form method introduced by Hassard et al. (1981), we give an algorithm for determining the Hopf bifurcation properties.

1. Introduction

In recent research [1–5], it is found that even if several individual systems behave chaotically, in the case where the systems are identical, by proper coupling, the systems can be made to evolve toward a situation of exact isochronal synchronization. Synchronization phenomena are common in coupled semiconductor systems, and they are important examples of oscillators in general, and many works are concerned with coupled semiconductor systems [6–15].

We consider a feedback loop comprises a semiconductor laser that serves as the optical source, a Mach-Zehnder electrooptic modulator, a photoreceiver, an electronic filter, and an amplifier. The dynamics of the feedback loop can be modeled by the delay differential equations [14, 15]:

$$\begin{aligned} \frac{dx_1(t)}{dt} = & -(\gamma_1 + \gamma_2)x_1(t) - \gamma_2 y_1(t) \\ & - \beta \gamma_2 \cos^2 [x_1(t - \tau) + \varphi_0], \end{aligned} \quad (1)$$

$$\frac{dy_1(t)}{dt} = \gamma_1 x_1(t).$$

Here, $x_1(t)$ is the normalized voltage signal applied to the electrooptic modulator, τ is the feedback time delay, γ_1 and γ_2 are the filter low-pass and high-pass corner frequencies, β

is the dimensionless feedback strength, they are all positive constants, and φ_0 is the bias point of the modulator.

Depending on the value of the feedback strength β and delay τ , the loop, which is modeled by system (1), is capable of producing dynamics ranging from periodic oscillations to high-dimensional chaos [1, 14, 15].

We couple two nominally identical optoelectronic feedback loops unidirectionally, that is, the transmitter affects the dynamics of the receiver but not vice versa. Thus, the equations of motion describing the coupled system are given by (1) for the transmitter and

$$\begin{aligned} \frac{dx_2(t)}{dt} = & -(\gamma_1 + \gamma_2)x_2(t) - \gamma_2 y_2(t) \\ & - \beta \gamma_2 \cos^2 [kx_1(t - \tau) + (1 - k)x_2(t - \tau) + \varphi_0], \\ \frac{dy_2(t)}{dt} = & \gamma_1 x_2(t), \end{aligned} \quad (2)$$

for the receiver. In (2), $k > 0$ denotes the coupling strength. We will find that with the variety of k , the dynamical behavior of the coupled system can be different, while the feedback strength β keeps the same value.

The paper is organized as follows. In Section 2, using the method presented in [16], we study the stability, and

the local Hopf bifurcation of the equilibrium of the coupled system (1) and (2) by analyzing the distribution of the roots of the associated characteristic equation. In Section 3, we use the normal form method and the center manifold theory introduced by Hassard et al. [17] to analyze the direction, stability and the period of the bifurcating periodic solutions at critical values of β . In Section 4, some numerical simulations are carried out to illustrate the results obtained from the analysis. In Section 5, we come to some conclusion about the effect caused by the variety of parameters.

2. Stability Analysis

In this section, we consider the linear stability of the nonlinear coupled system

$$\begin{aligned} \frac{dx_1(t)}{dt} &= -(\gamma_1 + \gamma_2)x_1(t) - \gamma_2 y_1(t) \\ &\quad - \beta \gamma_2 \cos^2 [x_1(t - \tau) + \varphi_0], \\ \frac{dy_1(t)}{dt} &= \gamma_1 x_1(t), \\ \frac{dx_2(t)}{dt} &= -(\gamma_1 + \gamma_2)x_2(t) - \gamma_2 y_2(t) \\ &\quad - \beta \gamma_2 \cos^2 [kx_1(t - \tau) + (1 - k)x_2(t - \tau) + \varphi_0], \\ \frac{dy_2(t)}{dt} &= \gamma_1 x_2(t). \end{aligned} \quad (3)$$

It is easy to see that $E(0, -\beta \cos^2 \varphi_0, 0, -\beta \cos^2 \varphi_0)$ is the only equilibrium of system (3). Linearizing system (3) around E and denote $\delta = \sin 2\varphi_0$, we get the linearization system

$$\begin{aligned} \frac{dx_1(t)}{dt} &= -(\gamma_1 + \gamma_2)x_1(t) - \gamma_2 y_1(t) + \beta \delta \gamma_2 x_1(t - \tau), \\ \frac{dy_1(t)}{dt} &= \gamma_1 x_1(t), \\ \frac{dx_2(t)}{dt} &= -(\gamma_1 + \gamma_2)x_2(t) - \gamma_2 y_2(t) \\ &\quad + k\beta \delta \gamma_2 x_1(t - \tau) + (1 - k)\beta \delta \gamma_2 x_2(t - \tau), \\ \frac{dy_2(t)}{dt} &= \gamma_1 x_2(t), \end{aligned} \quad (4)$$

and the characteristic equation of system (4)

$$\begin{aligned} &[\lambda^2 + (\gamma_1 + \gamma_2)\lambda + \gamma_1 \gamma_2 - (1 - k)\beta \delta \gamma_2 \lambda e^{-\lambda \tau}] \\ &\times [\lambda^2 + (\gamma_1 + \gamma_2)\lambda + \gamma_1 \gamma_2 - \beta \delta \gamma_2 \lambda e^{-\lambda \tau}] = 0, \end{aligned} \quad (5)$$

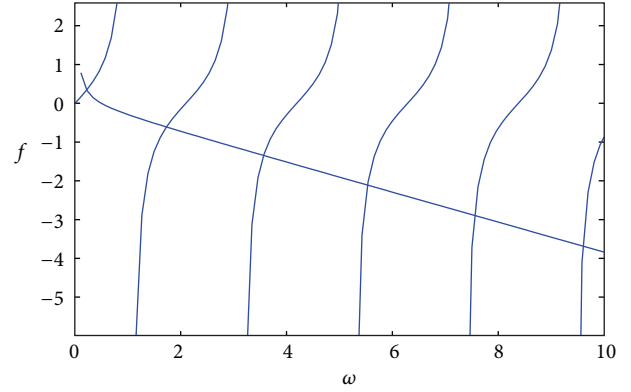


FIGURE 1: The points of intersection of $f_1 = \tan \omega \tau$ and $f_2 = (-\omega^2 + \gamma_1 \gamma_2)/\omega(\gamma_1 + \gamma_2)$, when $\gamma_1 = 0.1$, $\gamma_2 = 2.5$, $\tau = 1.5$.

which is equivalent to

$$\lambda^2 + (\gamma_1 + \gamma_2)\lambda + \gamma_1 \gamma_2 - \beta \delta \gamma_2 \lambda e^{-\lambda \tau} = 0, \quad (6)$$

$$\lambda^2 + (\gamma_1 + \gamma_2)\lambda + \gamma_1 \gamma_2 - (1 - k)\beta \delta \gamma_2 \lambda e^{-\lambda \tau} = 0. \quad (7)$$

Notice that when $\beta = 0$, (5) becomes

$$[\lambda^2 + (\gamma_1 + \gamma_2)\lambda + \gamma_1 \gamma_2]^2 = 0, \quad (8)$$

whose roots are

$$\lambda_{1,2} = -\gamma_1, \quad \lambda_{3,4} = -\gamma_2. \quad (9)$$

So, we have the following lemma.

Lemma 1. *The equilibrium $E_0(0, -\beta \cos^2 \varphi_0, 0, -\beta \cos^2 \varphi_0)$ is asymptotically stable when $\beta = 0$.*

Next, we regard β as the bifurcation parameter to investigate the distribution of roots of (6) and (7).

Let $\lambda = i\omega$ ($\omega > 0$) be a root of (6) and substituting $\lambda = i\omega$ into (6), separating the real and imaginary parts yields

$$-\omega^2 + \gamma_1 \gamma_2 = \beta \delta \gamma_2 \omega \sin \omega \tau, \quad (10)$$

$$\omega(\gamma_1 + \gamma_2) = \beta \delta \gamma_2 \omega \cos \omega \tau.$$

Then, we can get

$$\tan \omega \tau = \frac{-\omega^2 + \gamma_1 \gamma_2}{\omega(\gamma_1 + \gamma_2)}. \quad (11)$$

Hence, (11) has a sequence of roots $\{\omega_j\}_{j \geq 0}$ (see Figure 1), and

$$\omega_j \in \begin{cases} \left(\frac{2j\pi}{\tau}, \frac{2j\pi + \pi/2}{\tau} \right), & \omega_j^2 < \gamma_1 \gamma_2, \\ \left(\frac{2j\pi + 3\pi/2}{\tau}, \frac{2(j+1)\pi}{\tau} \right), & \omega_j^2 > \gamma_1 \gamma_2, \end{cases} \quad (12) \quad j = 0, 1, 2, \dots$$

Define

$$\beta_j = \frac{\gamma_1 + \gamma_2}{\delta\gamma_2 \cos \omega_j \tau}. \tag{13}$$

Then, (ω_j, β_j) is the solution of (10).

From (10), we know that

$$\beta^2 = \frac{1}{\delta^2 \gamma_2^2} \left[\left(\frac{\gamma_1 \gamma_2}{\omega} - \omega \right)^2 + (\gamma_1 + \gamma_2)^2 \right], \tag{14}$$

which gives that

$$\frac{d\beta}{d\omega} = \frac{1}{\beta \delta^2 \gamma_2^2 \omega^3} (\omega^2 + \gamma_1 \gamma_2) (\omega^2 - \gamma_1 \gamma_2). \tag{15}$$

From Figure 1, we know that $\omega_j \rightarrow \infty$ when $j \rightarrow \infty$, which means that $\omega_j^2 > \gamma_1 \gamma_2$; furthermore, β is increasing with respect to ω , when j is sufficiently big.

Reorder the set $\{\beta_j\}$ such that $\beta_0 = \min\{\beta_j\}$ and ω_j is correspondent of β_j ($j = 0, 1, 2, \dots$). Then, we have the following lemma.

Lemma 2. *There exists a sequence values of β denoted by*

$$0 < \beta_0 < \beta_1 < \dots, \tag{16}$$

such that (6) has a pair of imaginary roots $\pm i\omega_j$ when $\beta = \beta_j$ ($j = 0, 1, 2, \dots$), where β_j is defined by (13), and ω_j is the root of (11).

Let

$$\lambda(\tau) = \alpha(\beta) + i\omega(\beta) \tag{17}$$

be the root of (6) satisfying $\alpha(\beta_j) = 0$ and $\omega(\beta_j) = \omega_j$. We have the following conclusion.

Lemma 3. $\alpha'(\beta_j) > 0$.

Proof. Substituting $\lambda(\beta)$ into (6) and taking the derivative with respect to β , it follows that

$$\begin{aligned} 2\lambda \frac{d\lambda}{d\beta} + (\gamma_1 + \gamma_2) \frac{d\lambda}{d\beta} - \delta\gamma_2 \lambda e^{-\lambda\tau} - \beta\delta\gamma_2 e^{-\lambda\tau} \frac{d\lambda}{d\beta} \\ + \tau\beta\delta\gamma_2 \lambda e^{-\lambda\tau} \frac{d\lambda}{d\beta} = 0. \end{aligned} \tag{18}$$

Therefore, noting that $\beta\delta\gamma_2 \lambda e^{-\lambda\tau} = \lambda^2 + (\gamma_1 + \gamma_2)\lambda + \gamma_1 \gamma_2$, we have

$$\frac{d\lambda}{d\beta} = \frac{1}{\beta} \frac{\lambda^3 + (\gamma_1 + \gamma_2)\lambda^2 + \gamma_1 \gamma_2 \lambda}{\lambda^2 - \gamma_1 \gamma_2 + \tau[\lambda^3 + (\gamma_1 + \gamma_2)\lambda^2 + \gamma_1 \gamma_2 \lambda]}, \tag{19}$$

and by a straight computation, we get

$$\begin{aligned} \alpha'(\beta_j) = \frac{\omega_j^2}{\beta_j \Delta} \left[\tau\omega_j^2 (\gamma_1^2 + \gamma_2^2) + (\omega_j^2 + \gamma_1 \gamma_2) (\gamma_1 + \gamma_2) \right. \\ \left. + \tau\omega_j^4 + \tau\gamma_1^2 \gamma_2^2 \right] > 0, \end{aligned} \tag{20}$$

where

$$\begin{aligned} \Delta = \left[(\omega_j^2 + \gamma_1 \gamma_2) + \tau\omega_j^2 (\gamma_1 + \gamma_2) \right]^2 \\ + \left[\tau\omega_j (\omega_j^2 - \gamma_1 \gamma_2) \right]^2. \end{aligned} \tag{21}$$

As to (7), it can be easily found that $-\gamma_1, -\gamma_2$ are two negative roots when $k = 1$, so, next, we only focus on (7) with $k \neq 1$.

Let $\lambda = i\omega(\omega) > 0$ be a root of (7). Using the same method above, we get

$$-\omega^2 + \gamma_1 \gamma_2 = (1 - k) \beta \delta \gamma_2 \omega \sin \omega \tau, \tag{22}$$

$$(\gamma_1 + \gamma_2) \omega = (1 - k) \beta \delta \gamma_2 \omega \cos \omega \tau,$$

$$\tan \omega \tau = \frac{-\omega^2 + \gamma_1 \gamma_2}{\omega (\gamma_1 + \gamma_2)}. \tag{23}$$

Then, when $0 < k < 1$, (23) has a sequence of roots $\{\omega_j\}_{j \geq 0}$, which are the same as those of (11).

When $k > 1$, (23) has a sequence of roots $\{\omega_j\}_{j \geq 0}$, and

$$\begin{aligned} \omega_j \in \left\{ \left(\frac{(2j+1)\pi}{\tau}, \frac{(2j+1)\pi + \pi/2}{\tau} \right), \quad \omega_j^2 < \gamma_1 \gamma_2, \right. \\ \left. \left(\frac{2j\pi + \pi/2}{\tau}, \frac{(2j+1)\pi}{\tau} \right), \quad \omega_j^2 > \gamma_1 \gamma_2, \right. \\ \left. j = 0, 1, 2, \dots \right. \end{aligned} \tag{24}$$

Define

$$\bar{\beta}_j = \frac{\gamma_1 + \gamma_2}{(1 - k) \delta \gamma_2 \cos \omega \tau}. \tag{25}$$

Then, $(\bar{\omega}_j, \bar{\beta}_j)$ is the solution of (22).

Repeat the previous process, we have

$$\frac{d\bar{\beta}}{d\bar{\omega}} = \frac{1}{(1 - k)^2 \beta \delta^2 \gamma_2^2 \omega^3} (\omega^2 + \gamma_1 \gamma_2) (\omega^2 - \gamma_1 \gamma_2). \tag{26}$$

Reorder the set $\{\bar{\beta}_j\}$ such that $\bar{\beta}_0 = \min\{\bar{\beta}_j\}$ and $\bar{\omega}_j$ is correspondent of $\bar{\beta}_j$ ($j = 0, 1, 2, \dots$).

Lemma 4. *There exists a sequence values of $\bar{\beta}$ denoted by*

$$0 < \bar{\beta}_0 < \bar{\beta}_1 < \dots, \tag{27}$$

such that (7) has a pair of imaginary roots $\pm i\bar{\omega}_j$ when $\bar{\beta} = \bar{\beta}_j$ ($j = 0, 1, 2, \dots$), where $\bar{\beta}_j$ is defined by (25), and $\bar{\omega}_j$ is the root of (23).

Let

$$\lambda(\tau) = \alpha(\bar{\beta}) + i\bar{\omega}(\bar{\beta}) \tag{28}$$

be the root of (7) satisfying $\alpha(\bar{\beta}_j) = 0$, $\bar{\omega}(\bar{\beta}_j) = \bar{\omega}_j$. Then, similar to the proof of Lemma 3, we have the following conclusion.

Lemma 5. $\alpha'(\bar{\beta}_j) > 0$.

Compare $\beta_j, \bar{\beta}_j$ and reorder the set $\{\beta_j\}$ and $\{\bar{\beta}_j\}$ and remove the “-” of $\bar{\beta}_j$, such that

$$0 < \beta_0 < \beta_1 < \dots, \tag{29}$$

then from previous lemmas and the Hopf bifurcation theorem for functional differential equations [18], we have the following results on stability and bifurcation to system (3).

Theorem 6. For system(3), the equilibrium E is asymptotically stable when $\beta \in [0, \beta_0)$ and unstable when $\beta \in (\beta_0, +\infty)$; system (3) undergoes a Hopf bifurcation at E when $\beta = \beta_j$, $j = 0, 1, 2, \dots$, where β_j are defined by (13) or (25).

3. The Direction and Stability of the Hopf Bifurcation

In Section 2 we obtained some conditions under which system (3) undergoes the Hopf bifurcation at some critical values of β . In this section, we study the direction, stability, and the period of the bifurcating periodic solutions. The method we used is based on the normal form method and the center manifold theory introduced by Hassard et al. [17].

Move $E(0, -\beta \cos^2 \varphi_0, 0, -\beta \cos^2 \varphi_0)$ to the origin $O(0, 0, 0, 0)$ and denote $\delta = \sin 2\varphi_0$, $\rho = \cos 2\varphi_0$, then system (3) can be written as the form

$$\begin{aligned} \frac{dx_1(t)}{dt} &= -(\gamma_1 + \gamma_2)x_1(t) - \gamma_2\gamma_1(t) + \beta\delta\gamma_2x_1(t - \tau) \\ &\quad + \beta\gamma_2\rho\varphi_0x_1^2(t - \tau) - \frac{2}{3}\beta\delta\gamma_2x_1^3(t - \tau) + O(4), \\ \frac{dy_1(t)}{dt} &= \gamma_1x_1(t), \\ \frac{dx_2(t)}{dt} &= -(\gamma_1 + \gamma_2)x_2(t) - \gamma_2\gamma_2(t) \\ &\quad + \beta\gamma_2 \left[k\delta x_1(t - \tau) + (1 - k)\delta x_2(t - \tau) \right. \\ &\quad \left. + k^2\rho x_1^2(t - \tau) \right. \\ &\quad \left. + 2k(1 - k)\rho x_1(t - \tau)x_2(t - \tau) \right. \\ &\quad \left. + (1 - k)^2\rho x_2^2(t - \tau) - \frac{2}{3}k^3\delta x_1^3(t - \tau) \right. \\ &\quad \left. - 2k^2(1 - k)\delta x_1^2(t - \tau)x_2(t - \tau) \right. \\ &\quad \left. - 2k(1 - k)^2\delta x_1(t - \tau)x_2^2(t - \tau) \right. \\ &\quad \left. - \frac{2}{3}(1 - k)^3\delta x_2^3(t - \tau) \right] + O(4), \\ \frac{dy_2(t)}{dt} &= \gamma_1x_2(t). \end{aligned} \tag{30}$$

Clearly, the phase space is $\mathcal{E} = \mathcal{C}([-\tau, 0], \mathbb{R}^4)$. For convenience, let

$$\beta^* \in \{\beta_j\} \cup \{\bar{\beta}_j\}, \tag{31}$$

and $\beta = \beta^* + \mu$, $\mu \in \mathbb{R}$. From the analysis above we know that $\mu = 0$ is the Hopf bifurcation value for system(30). Let $i\omega^*$ be the root of the characteristic equation associate with the linearization of system (30) when $\beta = \beta^*$. For $\phi = (\phi_1, \phi_2, \phi_3, \phi_4) \in \mathcal{E}$, let

$$L_\mu(\phi) = B\phi(0) + C\phi(-\tau), \tag{32}$$

where

$$\begin{aligned} B &= \begin{pmatrix} -(\gamma_1 + \gamma_2) & -\gamma_2 & 0 & 0 \\ \gamma_1 & 0 & 0 & 0 \\ 0 & 0 & -(\gamma_1 + \gamma_2) & -\gamma_2 \\ 0 & 0 & \gamma_1 & 0 \end{pmatrix}, \\ C &= \begin{pmatrix} \beta\delta\gamma_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ k\beta\delta\gamma_2 & 0 & (1 - k)\beta\delta\gamma_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{33}$$

By the Rieze representation theorem, there exists a 4×4 matrix, $\eta(\theta, \mu)$ ($-\tau \leq \theta \leq 0$), whose elements are of bounded variation functions such that

$$L_\mu(\phi) = \int_{-\tau}^0 d\eta(\theta, \mu)\phi(\theta), \quad \phi \in \mathcal{E}. \tag{34}$$

In fact, we can choose

$$\eta(\theta, \mu) = \begin{cases} B, & \theta = 0, \\ 0, & \theta \in (-\tau, 0) \\ -C, & \theta = -\tau. \end{cases} \tag{35}$$

Then, (30) is satisfied.

For $\phi \in \mathcal{E}$, define the operator $A(\mu)$ as

$$A(\mu)\phi(\theta) = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & \theta \in [-\tau, 0), \\ \int_{-\tau}^0 d\eta(t, \mu)\phi(t), & \theta = 0, \end{cases} \tag{36}$$

and $R(\mu)\phi$ as

$$R(\mu)\phi(\theta) = \begin{cases} 0, & \theta \in [-\tau, 0), \\ f(\mu, \phi), & \theta = 0, \end{cases} \tag{37}$$

where

$$f(\mu, \phi) = \beta^* \gamma_2 \begin{pmatrix} \rho\phi_1^2(-\tau) - \frac{2}{3}\delta\phi_1^3(-\tau) + O(4) \\ 0 \\ k^2\rho\phi_1^2(-\tau) + k(1-k)\rho\phi_1(-\tau)\phi_3(-\tau) \\ +(1-k)^2\rho\phi_3^2(-\tau) - \frac{2}{3}k^3\delta\phi_1^3(-\tau) - 2k^2(1-k)\delta\phi_1^2(-\tau)\phi_3(t-\tau) \\ -2k(1-k)^2\delta\phi_1(-\tau)\phi_3^2(-\tau) - \frac{2}{3}(1-k)^3\delta\phi_3^3(-\tau) + O(4) \\ 0 \end{pmatrix}. \tag{38}$$

Then, system (30) is equivalent to the following operator equation:

$$\dot{u}_t = A(\mu)u_t + R(\mu)u_t, \tag{39}$$

where $u(t) = (x_1(t), y_1(t), x_2(t), y_2(t))^T$, $u_t = u(t + \theta)$, for $\theta \in [-\tau, 0]$.

For $\psi \in \mathcal{C}^1([0, \tau], \mathbb{R}^4)$, define

$$A^* \psi(s) = \begin{cases} -\frac{d\psi(s)}{ds}, & s \in (0, \tau], \\ \int_{-\tau}^0 \psi(-\xi) d\eta(\xi, 0), & s = 0. \end{cases} \tag{40}$$

For $\phi \in \mathcal{C}[-\tau, 0]$ and $\psi \in \mathcal{C}[0, \tau]$, define the bilinear form

$$\langle \psi(s), \phi(\theta) \rangle = \bar{\psi}(0)\phi(0) - \int_{-\tau}^0 \int_0^\theta \bar{\psi}(\xi - \theta) d\eta(\theta)\phi(\xi) d\xi, \tag{41}$$

where $\eta(\theta) = \eta(\theta, 0)$. Then, $A(0)$ and A^* are adjoint operators.

Let $q(\theta)$ and $q^*(s)$ be eigenvectors of $A(0)$ and A^* associated to $i\omega^*$ and $-i\omega^*$, respectively. It is not difficult with verify that

$$q(\theta) = \left(1, \frac{\gamma_1}{i\omega^*}, 1, \frac{\gamma_1}{i\omega^*} \right)^T e^{i\omega^*\theta}, \tag{42}$$

$$q^*(s) = \frac{1}{D} \left(1, \frac{\gamma_2}{i\omega^*}, 1, \frac{\gamma_2}{i\omega^*} \right) e^{i\omega^*s},$$

where

$$D = 2 + \frac{2\gamma_1\gamma_2}{\omega^{*2}} + 2\beta^*\delta\gamma_2\tau e^{-i\omega^*\tau}. \tag{43}$$

Then, $\langle q^*(s), q(\theta) \rangle = 1$, $\langle q^*(s), \bar{q}(\theta) \rangle = 0$.

Let u_t be the solution of (39) and define

$$z(t) = \langle q^*, u_t \rangle, \quad W(t, \theta) = u_t(\theta) - 2 \operatorname{Re} \{z(t)q(\theta)\}. \tag{44}$$

On the center manifold \mathcal{C}_0 , we have

$$W(t, \theta) = W(z(t), \bar{z}(t), \theta), \tag{45}$$

where

$$W(z, \bar{z}, \theta) = W_{20} \frac{z^2}{2} + W_{11}z\bar{z} + W_{02} \frac{\bar{z}^2}{2} + \dots, \tag{46}$$

z and \bar{z} are local coordinates for center manifold \mathcal{C}_0 in the direction of q^* and \bar{q}^* . Note that W is real if u_t is real. We only consider real solutions.

For solution u_t in \mathcal{C}_0 , since $\mu = 0$, we have

$$\begin{aligned} \dot{z}(t) &= i\omega^*z + \langle q^*(\theta), f(0, W + 2 \operatorname{Re} \{z(t)q(\theta)\}) \rangle \\ &= i\omega^*z + \bar{q}^*(0), f(0, W(z, \bar{z}, 0) + 2 \operatorname{Re} \{z(t)q(0)\}) \\ &= i\omega^*z + \bar{q}^*(0) f_0(z, \bar{z}). \end{aligned} \tag{47}$$

We rewrite this equation as

$$\dot{z}(t) = i\omega^*z + g(z, \bar{z}), \tag{48}$$

where

$$g(z, \bar{z}) = g_{20} \frac{z^2}{2} + g_{11}z\bar{z} + g_{02} \frac{\bar{z}^2}{2} + g_{21} \frac{z^2\bar{z}}{2} \dots \tag{49}$$

By (39) and (48), we have

$$\begin{aligned} \dot{W} &= \dot{u}_t - \dot{z}q - \dot{\bar{z}}\bar{q} \\ &= \begin{cases} AW - 2 \operatorname{Re} \{ \bar{q}^*(0) f_0 q(\theta) \}, & \theta \in [-\tau, 0), \\ AW - 2 \operatorname{Re} \{ \bar{q}^*(0) f_0 q(0) \} + f_0, & \theta = 0, \end{cases} \tag{50} \\ &= AW + H(z, \bar{z}, \theta), \end{aligned}$$

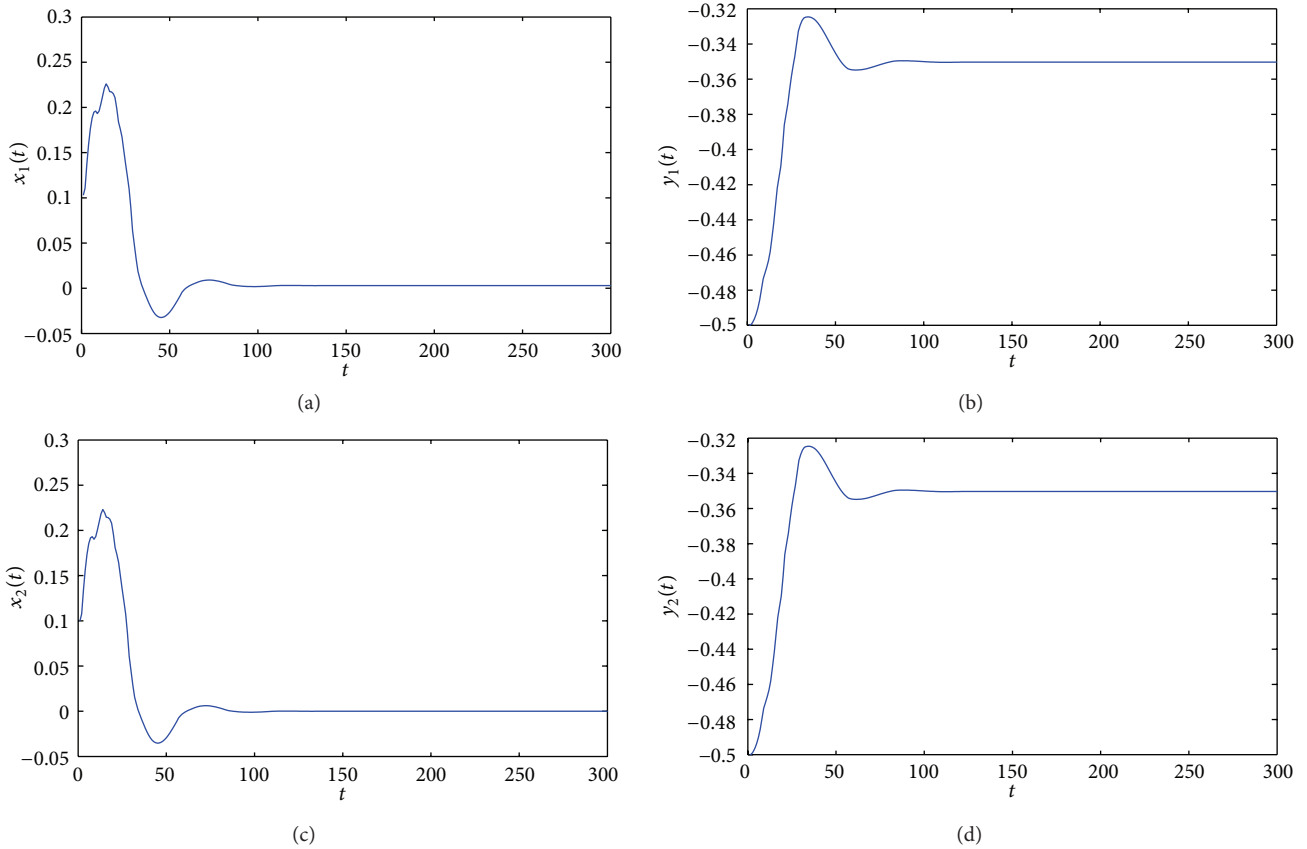


FIGURE 2: $\gamma_1 = 0.1, \gamma_2 = 2.5, \tau = 1.5, k = 1.9$, which means that condition (H_1) holds, and $\beta = 0.7 < \beta_0$. The initial value is $(0.1, -0.5, 0.1, -0.5)$.

where

$$H(z, \bar{z}, \theta) = H_{20}(\theta) \frac{z^2}{2} + H_{11}(\theta) z\bar{z} + H_{02}(\theta) \frac{\bar{z}^2}{2} + \dots \tag{51}$$

Expanding the above series and comparing the coefficients, we obtain

$$\begin{aligned} (A - 2i\omega^* I) W_{20}(\theta) &= -H_{20}(\theta), \\ A W_{11}(\theta) &= -H_{11}(\theta), \dots \end{aligned} \tag{52}$$

Notice that

$$\begin{aligned} q(\theta) &= \left(1, \frac{\gamma_1}{i\omega^*}, 1, \frac{\gamma_1}{i\omega^*} \right)^T e^{i\omega^* \theta}, \\ u_t(\theta) &= zq(\theta) + \bar{z}\bar{q}(\theta) + W(z, \bar{z}, \theta), \end{aligned} \tag{53}$$

where

$$\begin{aligned} W^{(i)}(z, \bar{z}, \theta) &= W_{20}^{(i)}(\theta) \frac{z^2}{2} + W_{11}^{(i)}(\theta) z\bar{z} \\ &+ W_{02}^{(i)}(\theta) \frac{\bar{z}^2}{2} + \dots, \quad i = 1, 2, 3, 4. \end{aligned} \tag{54}$$

Combing (38) and by straightforward computation, we can obtain the coefficients which will be used in determining the important quantities:

$$g_{20} = \frac{2\beta^* \gamma_2 \rho}{D} e^{-2i\omega^* \tau} (k^2 - k + 2),$$

$$g_{11} = \frac{2\beta^* \gamma_2 \rho}{D} (k^2 - k + 2),$$

$$g_{02} = \frac{2\beta^* \gamma_2 \rho}{D} e^{2i\omega^* \tau} (k^2 - k + 2),$$

$$\begin{aligned} g_{21} &= \frac{2\beta^* \gamma_2}{D} \left\{ \rho \left(e^{i\omega^* \tau} W_{20}^{(1)}(-\tau) + 2e^{-i\omega^* \tau} W_{11}^{(1)}(-\tau) \right) \right. \\ &+ k^2 \rho \left(e^{i\omega^* \tau} W_{20}^{(1)}(-\tau) + 2e^{-i\omega^* \tau} W_{11}^{(1)}(-\tau) \right) \\ &+ k(1-k)\rho \\ &\times \left(e^{-i\omega^* \tau} W_{11}^{(3)}(-\tau) + e^{i\omega^* \tau} \frac{W_{20}^{(3)}(-\tau)}{2} \right. \\ &\left. \left. + e^{i\omega^* \tau} \frac{W_{20}^{(1)}(-\tau)}{2} + e^{-i\omega^* \tau} W_{11}^{(1)}(-\tau) \right) \right\} \end{aligned}$$

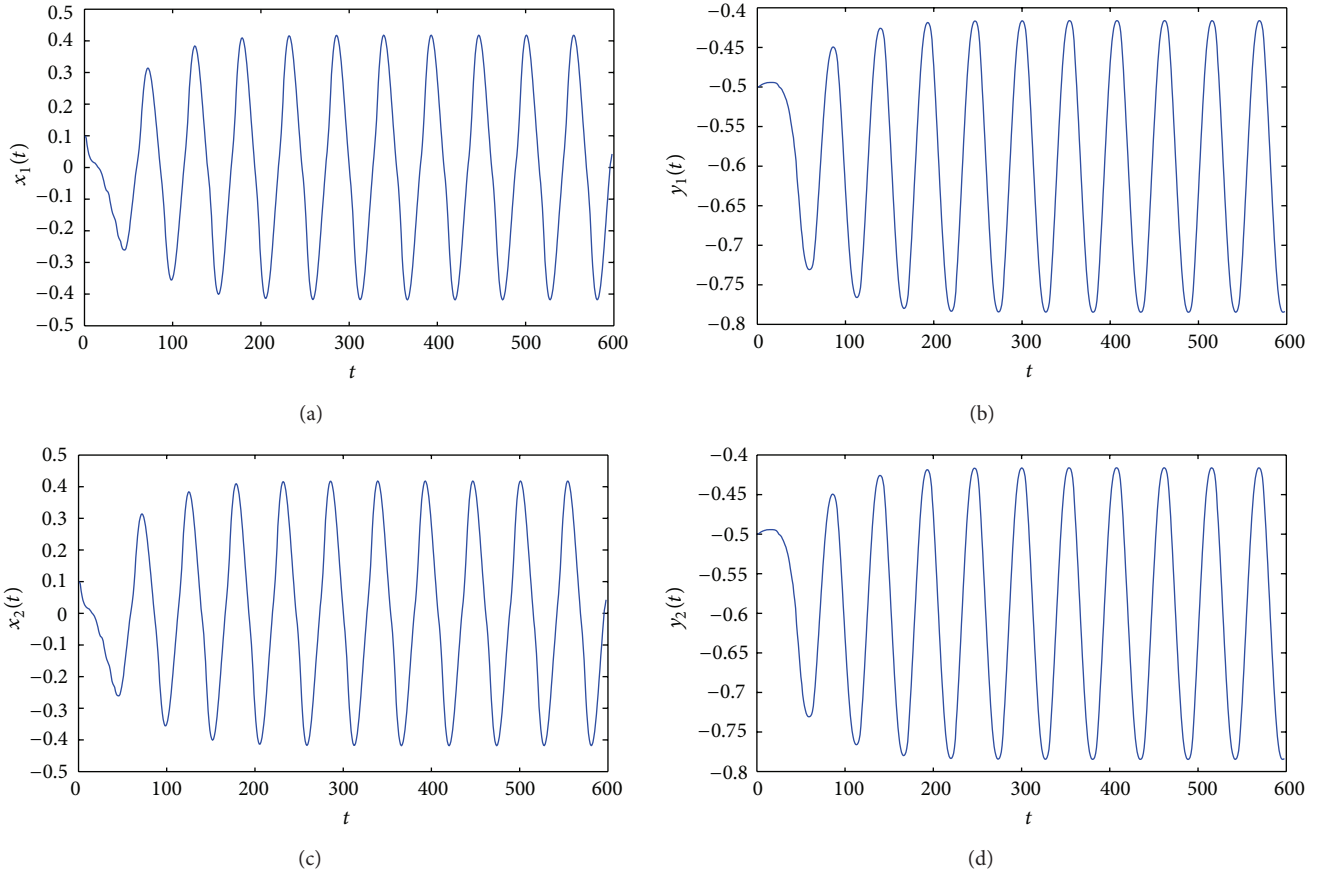


FIGURE 3: $\gamma_1 = 0.1, \gamma_2 = 2.5, \tau = 1.5, k = 1.9$, which means that condition (H_1) holds, and $\beta = 1.2 > \beta_0$. The initial value is $(0.1, -0.5, 0.1, -0.5)$.

$$\left. \begin{aligned}
 &+ (1 - k)^2 \rho \left(2e^{-i\omega^* \tau} W_{11}^{(3)}(-\tau) \right. \\
 &\left. + e^{i\omega^* \tau} W_{20}^{(3)}(-\tau) \right) - 4\delta e^{-i\omega^* \tau} \left. \right\}. \tag{55}
 \end{aligned}$$

We still need to compute $W_{20}(\theta)$ and $W_{11}(\theta)$, for $\theta \in [-\tau, 0)$. We have

$$\begin{aligned}
 H(z, \bar{z}, \theta) &= -\bar{q}^*(0) f_0 q(\theta) - q^*(0) \bar{f}_0 \bar{q}(\theta) \\
 &= -g(z, \bar{z}) q(\theta) - \bar{g}(z, \bar{z}) \bar{q}(\theta). \tag{56}
 \end{aligned}$$

Comparing the coefficients about $H(z, \bar{z}, \theta)$ gives that

$$\begin{aligned}
 H_{20}(\theta) &= -g_{20} q(\theta) - \bar{g}_{02} \bar{q}(\theta), \\
 H_{11} &= -g_{11} q(\theta) - \bar{g}_{11} \bar{q}(\theta). \tag{57}
 \end{aligned}$$

Then, from (52), we get

$$\begin{aligned}
 \dot{W}_{20}(\theta) &= 2i\omega^* W_{20}(\theta) + g_{20} q(\theta) + \bar{g}_{02} \bar{q}(\theta), \\
 \dot{W}_{11}(\theta) &= g_{11} q(\theta) + \bar{g}_{11} \bar{q}(\theta), \tag{58}
 \end{aligned}$$

which implies that

$$\begin{aligned}
 W_{20}(\theta) &= \frac{g_{20} q(0)}{-i\omega^*} e^{i\omega^* \theta} + \frac{\bar{g}_{02} \bar{q}(0)}{-3i\omega^*} e^{-i\omega^* \theta} + E e^{2i\omega^* \theta}, \\
 W_{11}(\theta) &= \frac{g_{11} q(0)}{i\omega^*} e^{i\omega^* \theta} + \frac{\bar{g}_{11} \bar{q}(0)}{-i\omega^*} e^{-i\omega^* \theta} + F. \tag{59}
 \end{aligned}$$

Here, E and F are both four-dimensional vectors and can be determined by setting $\theta = 0$ in $H(z, \bar{z}, \theta)$. In fact, from (38) and

$$H(z, \bar{z}, 0) = -2 \operatorname{Re} \{ \bar{q}^*(0) f_0 q(0) \} + f_0, \tag{60}$$

we have

$$\begin{aligned}
 H_{20}(0) &= -g_{20} q(0) - \bar{g}_{02} \bar{q}(0) \\
 &\quad + 2\beta^* \gamma_2 \rho e^{-2i\omega^* \tau} (1, 0, k^2 - k + 1, 0)^T, \\
 H_{11}(0) &= -g_{11} q(0) - \bar{g}_{11} \bar{q}(0) \\
 &\quad + 2\beta^* \gamma_2 \rho (1, 0, k^2 - k + 1, 0)^T. \tag{61}
 \end{aligned}$$

It follows from (52) and the definition of A that

$$\begin{aligned}
 \beta^* B W_{20}(0) + \beta^* C W_{20}(-\tau) &= 2i\omega^* W_{20}(0) - H_{20}(0), \\
 \beta^* B W_{11}(0) + \beta^* C W_{11}(-\tau) &= -H_{11}(0), \tag{62}
 \end{aligned}$$

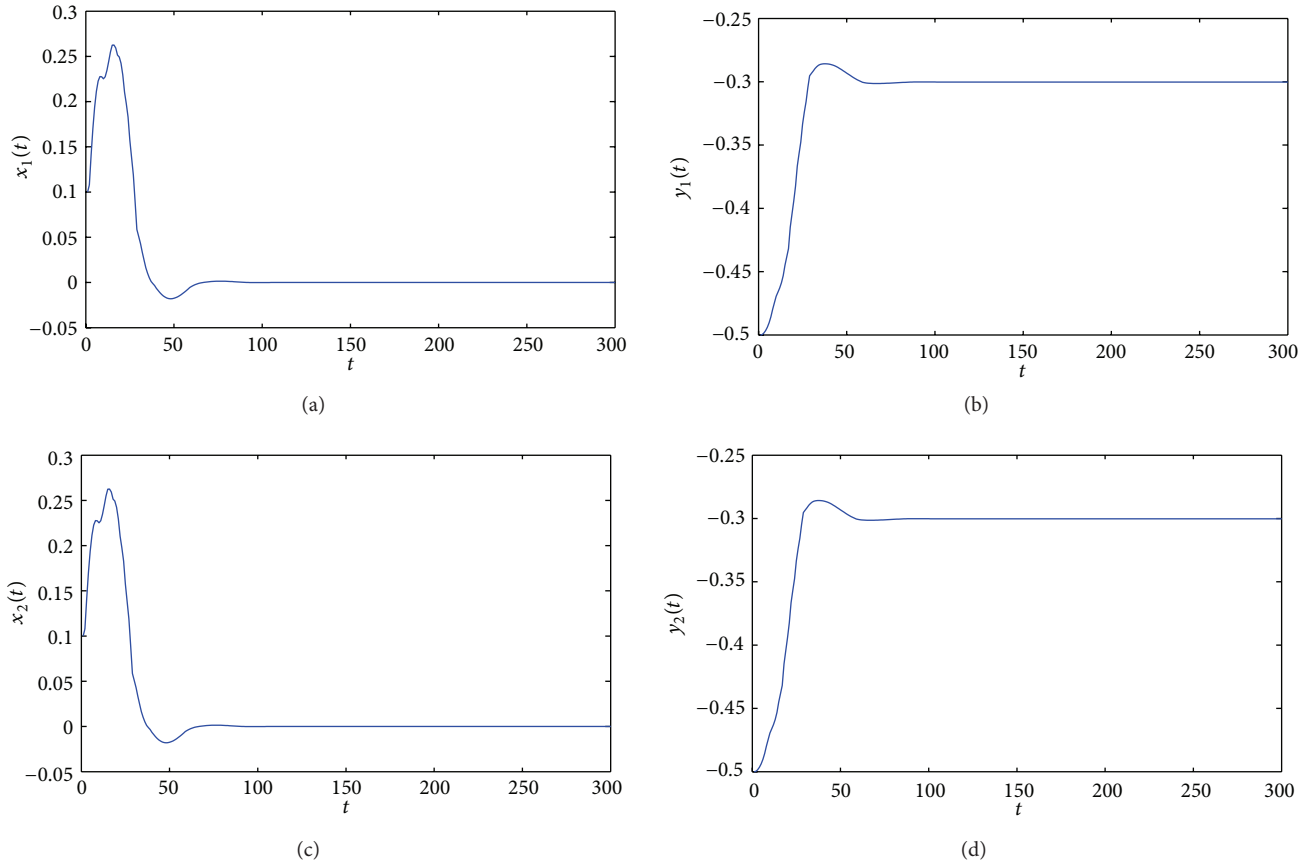


FIGURE 4: $\gamma_1 = 0.1$, $\gamma_2 = 2.5$, $\tau = 1.5$, $k = 3$, which means that condition (H_2) holds, and $\beta = 0.6 < \bar{\beta}_0$. The initial value is $(0.1, -0.5, 0.1, -0.5)$.

which implies that

$$\begin{aligned}
 E &= \left(B + e^{-2i\omega^* \tau} C - 2i\omega^* I \right)^{-1} \\
 &\times \left[B \left(\frac{g_{20}q(0)}{i\omega^*} + \frac{\bar{g}_{02}\bar{q}(0)}{3i\omega^*} \right) \right. \\
 &\quad + C \left(\frac{g_{20}q(0)}{i\omega^*} e^{-i\omega^* \tau} + \frac{\bar{g}_{02}\bar{q}(0)}{3i\omega^*} e^{i\omega^* \tau} \right) \\
 &\quad + \frac{1}{\beta^*} (g_{20}q(0) + \bar{g}_{02}\bar{q}(0)) \\
 &\quad \left. \times 2\beta^* \gamma_2 \rho e^{-2i\omega^* \tau} (1, 0, k^2 - k + 1, 0)^T \right], \\
 F &= (B + C)^{-1} \left[B \left(\frac{g_{11}q(0)}{-i\omega^*} + \frac{\bar{g}_{11}\bar{q}(0)}{i\omega^*} \right) \right.
 \end{aligned}$$

$$\begin{aligned}
 &\quad + C \left(\frac{g_{11}q(0)}{-i\omega^*} e^{-i\omega^* \tau} + \frac{\bar{g}_{11}\bar{q}(0)}{i\omega^*} e^{i\omega^* \tau} \right) \\
 &\quad + \frac{1}{\beta^*} (g_{11}q(0) + \bar{g}_{11}\bar{q}(0)) \\
 &\quad \left. - 2\beta^* \gamma_2 \rho (1, 0, k^2 - k + 1, 0)^T \right]. \tag{63}
 \end{aligned}$$

Consequently, the above g_{21} can be expressed by the parameters and delay in system (30). Thus, we can compute the following quantities:

$$\begin{aligned}
 c_1(0) &= \frac{i}{2\omega^*} \left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{20}|^2 \right) + \frac{g_{21}}{2}, \\
 \mu_2 &= -\frac{\operatorname{Re} c_1(0)}{\operatorname{Re} \lambda'(\beta^*)}, \\
 \beta_2 &= 2 \operatorname{Re} c_1(0), \\
 T_2 &= -\frac{\operatorname{Im} c_1(0) + \mu_2 \operatorname{Im} \lambda'(\beta^*)}{\omega^*},
 \end{aligned} \tag{64}$$

which determine the properties of bifurcating periodic solutions at the critical value τ_0 . The direction and stability of the

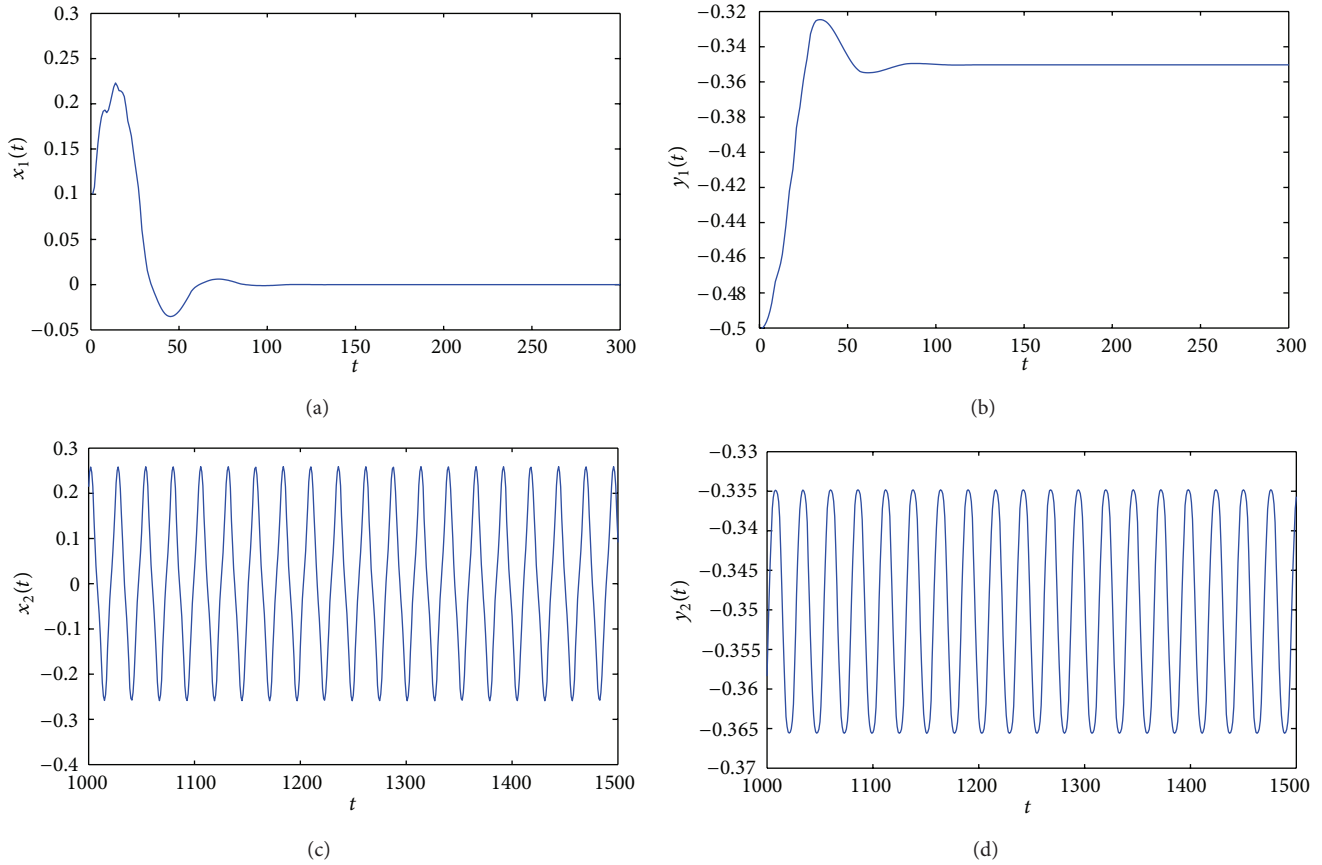


FIGURE 5: $\gamma_1 = 0.1, \gamma_2 = 2.5, \tau = 1.5, k = 3$, which means that condition (H_2) holds, and $\bar{\beta}_0 < \beta = 0.7 < \beta_0$. The initial value is $(0.1, -0.5, 0.1, -0.5)$.

Hopf bifurcation in the center manifold can be determined by μ_2 and β_2 , respectively. In fact, if $\mu_2 > 0$ ($\mu_2 < 0$), then the bifurcating periodic solutions are forward (backward); the bifurcating periodic solutions on the center manifold are stable (unstable) if $\beta_2 < 0$ ($\beta_2 > 0$); and T_2 determines the period of the bifurcating periodic solutions: the period increases (decreases) if $T_2 > 0$ ($T_2 < 0$).

From the discussion in Section 2, we have known that $\text{Re } \lambda'(\beta_j) > 0$; therefore; we have the following result.

Theorem 7. *The direction of the Hopf bifurcation for system (3) at the equilibrium $E(0, -\beta \cos^2 \varphi_0, 0, -\beta \cos^2 \varphi_0)$ when $\beta = \beta^*$ is forward (backward), and the bifurcating periodic solutions on the center manifold are stable (unstable) if $\text{Re}(c_1(0)) < 0$ (> 0). Particularly, the stability of the bifurcation periodic solutions of system (3) and the reduced equations on the center manifold are coincident at the first bifurcation value $\beta = \beta_0$.*

4. Numerical Simulations

In this section, we will carry out numerical simulations on system (3) at special values of β . We choose a set of data as follows:

$$\gamma_1 = 0.1, \quad \gamma_2 = 2.5, \quad \varphi_0 = \frac{\pi}{4}, \quad \tau = 1.5, \quad (65)$$

which are the same as those in [1]. Then, $\delta = 1, \rho = 0$.

Then, we can obtain

$$\begin{aligned} \omega_0 &\doteq 0.2225, & \omega_1 &\doteq 3.5677, \dots, \\ \bar{\omega}_0 &\doteq 1.7294, & \bar{\omega}_1 &\doteq 5.5531, \dots, \\ \beta_0 &\doteq 1.1008, & \beta_1 &\doteq 1.7434, \dots, \end{aligned} \quad (66)$$

$$\bar{\beta}_0 \doteq 1.3534, \quad \bar{\beta}_1 \doteq 2.5235, \dots, \quad k = 1.9,$$

$$\bar{\beta}_0 \doteq 0.6093, \quad \bar{\beta}_1 \doteq 1.1356, \dots, \quad k = 3.$$

From the analysis in Section 2, we know that $\beta(\bar{\beta})$ is increasing with respect to $\omega(\bar{\omega})$ when $\omega(\bar{\omega}) > \gamma_1 \gamma_2$, which means that

$$\beta_0 = \min \{\beta_j\}, \quad \bar{\beta}_0 = \min \{\bar{\beta}_j\}, \quad j = 0, 1, 2, \dots, \quad (67)$$

that is, $\beta_0(\bar{\beta}_0)$ is the first critical value at which system (3) undergoes a Hopf bifurcation.

When $k = 1.9$, by the previous results, it follows that

$$\begin{aligned} \lambda'(\beta_0) &\doteq 0.2440 - 0.0491i, & c_1(0) &\doteq -0.5373 + 0.1082i, \\ \mu_2 &\doteq 2.2020, & \beta_2 &\doteq -1.0746, & T_2 &\doteq -0.0018. \end{aligned} \quad (68)$$

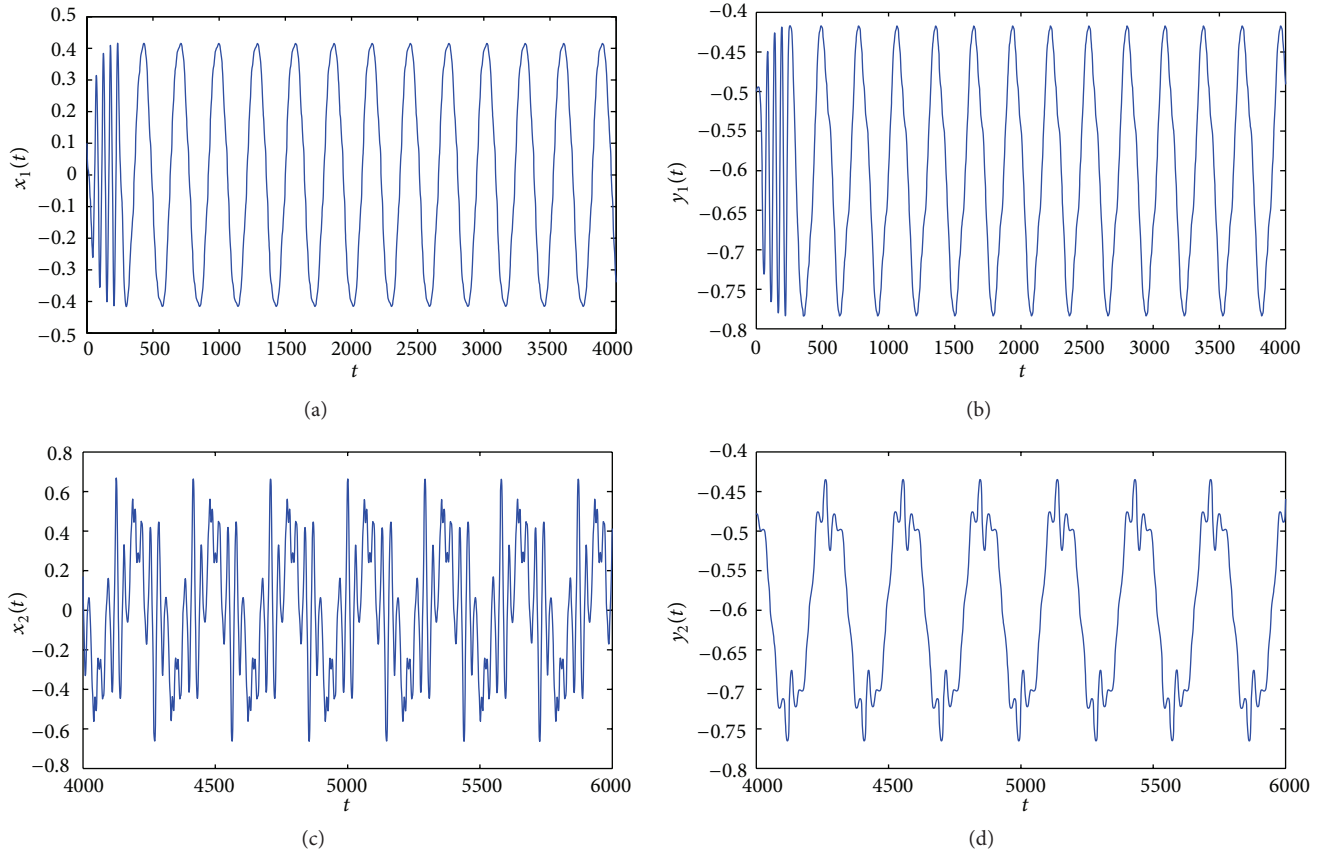


FIGURE 6: $\gamma_1 = 0.1, \gamma_2 = 2.5, \tau = 1.5, k = 3$, which means that condition (H_2) holds, and $\beta = 1.2 > \beta_0 > \bar{\beta}_0$. The initial value is $(0.1, -0.5, 0.1, -0.5)$.

Hence, we arrive at the following conclusion: the equilibrium E is asymptotically stable when $\beta \in [0, 1.1008)$ and unstable when $\beta \in (1.1008, +\infty)$, and, at the first critical value, the bifurcating periodic solutions are asymptotically stable, and the direction of the bifurcation is forward (see Figures 2 and 3).

When $k = 3$, we can get

$$\begin{aligned} \lambda'(\bar{\beta}_0) &= 0.9004 + 0.0924i, & c_1(0) &= -1.9116 + 0.7930i, \\ \mu_2 &= 2.1231, & \beta_2 &= -3.8232, & T_2 &= -0.5720. \end{aligned} \tag{69}$$

Then, we have the following: the equilibrium E is asymptotically stable when $\beta \in [0, 0.6093)$, and unstable when $\beta \in (0.6093, +\infty)$, and, at the first critical value, the bifurcating periodic solutions are asymptotically stable, and the direction of the bifurcation is forward (see Figures 4, 5, and 6).

5. Conclusion

Ravoori et al. [1] explored an experimental system of two nominally identical optoelectronic feedback loops coupled unidirectionally, which are described by system (3). In the experiment, they found that depending on the value of the

feedback strength β and delay τ , system (1) is capable of producing dynamics ranging from periodic oscillations to high-dimensional chaos [14, 15].

This paper investigates the stability and the existence of periodic solutions. We find that with the variety of the coupling strength k , even if all other parameters keep the same, the dynamical behavior can change greatly. In fact, it is clear that the first two equations, $x_1(t)$ and $y_1(t)$ are uncoupled with equations $x_2(t)$ and $y_2(t)$, so system (1) are independent of (2), which means that coupling strength k does not appear in (1). The characteristic equation of (1) has the same form as (6), so the first critical value β_0 is independent of k . The analysis of characteristic equation (7) shows that the value of k can affect the first critical value $\bar{\beta}_0$ definitely. And we draw a conclusion that when k is in an interval, in which $\beta_0 < \bar{\beta}_0$ holds, solutions of system (1) and (2) keep synchronous; when k belongs to the interval, in which $\bar{\beta}_0 < \beta_0$ holds, solutions of system (1) and (2) can also keep synchronous with $\beta < \bar{\beta}_0$, while they lose their synchronization when $\beta > \bar{\beta}_0$, no matter whether $\beta < \beta_0$ or not.

As a result, the modulation of the coupling strengths k together with the feedback strength β would be an efficient and an easily implementable method to control the behavior of the coupled chaotic oscillators.

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