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*Research Article*

# Bounds for the Combinations of Neuman-Sándor, Arithmetic, and Second Seiffert Means in terms of Contraharmonic Mean

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We give the greatest values $r_1, r_2$ and the least values $s_1, s_2$ in $(1/2, 1)$ such that the double inequalities $C(r_1a+(1-r_1)b, r_1b+(1-r_1)a) < \alpha A(a,b) + (1-\alpha)T(a,b) < C(s_1a + (1-s_1)b, s_1b + (1-s_1)a)$ and $C(r_2a + (1-r_2)b, r_2b + (1-r_2)a) < \alpha A(a,b) + (1-\alpha)M(a,b) < C(s_2a + (1-s_2)b, s_2b + (1-s_2)a)$ hold for any $\alpha \in (0,1)$ and all $a, b > 0$ with $a \neq b$, where $A(a,b)$, $M(a,b)$, $C(a,b)$, and $T(a,b)$ are the arithmetic, Neuman-Sándor, contraharmonic, and second Seiffert means of $a$ and $b$, respectively.

## 1. Introduction

For $a, b > 0$ with $a \neq b$, the Neuman-Sándor mean $M(a,b)$ [1], second Seiffert mean $T(a,b)$ [2] are defined by

$$M(a,b) = \frac{a-b}{2\sinh^{-1}((a-b)/(a+b))},$$
$$T(a,b) = \frac{a-b}{2\arctan((a-b)/(a+b))}, \tag{1}$$

respectively. Herein, $\sinh^{-1}(x) = \log(x + \sqrt{1+x^2})$ is the inverse hyperbolic sine function.

Let $H(a,b) = 2ab/(a+b)$, $G(a,b) = \sqrt{ab}$, $L(a,b) = (a-b)/(\log a - \log b)$, $P(a,b) = (a-b)/[4\arctan(\sqrt{a/b}) - \pi]$, $I(a,b) = 1/e(b^b/a^a)^{1/(b-a)}$, $A(a,b) = (a+b)/2$, $Q(a,b) = \sqrt{(a^2+b^2)/2}$, and $C(a,b) = (a^2+b^2)/(a+b)$ be the harmonic, geometric, logarithmic, first Seiffert, identric, arithmetic, quadratic, and contraharmonic means of two distinct positive real numbers $a$ and $b$, respectively. Then it is well known that the inequalities

$$\begin{aligned} H(a,b) &< G(a,b) < L(a,b) < P(a,b) \\ &< I(a,b) < A(a,b) < M(a,b) \\ &< T(a,b) < Q(a,b) < C(a,b) \end{aligned} \tag{2}$$

hold for all $a, b > 0$ with $a \neq b$.

Among means of two variables, the Neuman-Sándor, contraharmonic, and second Seiffert means have attracted the attention of several researchers. In particular, many remarkable inequalities and applications for these means can be found in the literature [3–15].

Neuman and Sándor [1, 16] proved that the inequalities

$$A(a,b) < M(a,b) < \frac{A(a,b)}{\log(1+\sqrt{2})},$$
$$\frac{\pi}{4}T(a,b) < M(a,b) < T(a,b),$$
$$M(a,b) < \frac{2A(a,b) + Q(a,b)}{3},$$
$$P(a,b)M(a,b) < A^2(a,b),$$

$$A(a,b)T(a,b) < M^2(a,b) < \frac{\left(A^2(a,b)+T^2(a,b)\right)}{2} \quad (3)$$

hold for all $a, b > 0$ with $a \neq b$.

Let $0 < a, b < 1/2$ with $a \neq b$, $a' = 1-a$ and $b' = 1-b$. Then the Ky Fan inequalities

$$\frac{G(a,b)}{G(a',b')} < \frac{L(a,b)}{L(a',b')} < \frac{P(a,b)}{P(a',b')} < \frac{A(a,b)}{A(a',b')} < \frac{M(a,b)}{M(a',b')} < \frac{T(a,b)}{T(a',b')} \quad (4)$$

can be found in [1].

Li et al. [17] proved that the double inequality $L_{p_0}(a,b) < M(a,b) < L_2(a,b)$ holds for all $a,b > 0$ with $a \neq b$, where $L_p(a,b) = [(b^{p+1}-a^{p+1})/((p+1)(b-a))]^{1/p}$ $(p \neq -1, 0)$, $L_0(a,b) = I(a,b)$ and $L_{-1}(a,b) = L(a,b)$ is the $p$th generalized logarithmic mean of $a$ and $b$, and $p_0 = 1.843\cdots$ is the unique solution of the equation $(p+1)^{1/p} = 2\log(1+\sqrt{2})$.

In [18], Neuman proved that the inequalities

$$\begin{aligned} &\alpha Q(a,b) + (1-\alpha)A(a,b) < M(a,b) \\ &\quad < \beta Q(a,b) + (1-\beta)A(a,b), \\ &\lambda C(a,b) + (1-\lambda)A(a,b) < M(a,b) \\ &\quad < \mu C(a,b) + (1-\mu)A(a,b) \end{aligned} \quad (5)$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $\alpha \leq [1-\log(1+\sqrt{2})]/[(\sqrt{2}-1)\log(1+\sqrt{2})] = 0.3249\cdots$, $\lambda \leq [1-\log(1+\sqrt{2})]/\log(1+\sqrt{2}) = 0.1345\cdots$, $\beta \geq 1/3$ and $\mu \geq 1/6$.

Zhao et al. [19] found the least values $\alpha_1, \alpha_2, \alpha_3$ and the greatest values $\beta_1, \beta_2, \beta_3$ such that the double inequalities

$$\begin{aligned} &\alpha_1 H(a,b) + (1-\alpha_1)Q(a,b) < M(a,b) \\ &\quad < \beta_1 H(a,b) + (1-\beta_1)Q(a,b), \\ &\alpha_2 G(a,b) + (1-\alpha_2)Q(a,b) < M(a,b) \\ &\quad < \beta_2 G(a,b) + (1-\beta_2)Q(a,b), \\ &\alpha_3 H(a,b) + (1-\alpha_3)C(a,b) < M(a,b) \\ &\quad < \beta_3 H(a,b) + (1-\beta_3)C(a,b) \end{aligned} \quad (6)$$

hold for all $a, b > 0$ with $a \neq b$.

In [20, 21], the authors proved that the double inequalities

$$\begin{aligned} &\alpha_1 T(a,b) + (1-\alpha_1)G(a,b) < A(a,b) \\ &\quad < \beta_1 T(a,b) + (1-\beta_1)G(a,b), \\ &\alpha_2 Q(a,b) + (1-\alpha_2)A(a,b) < T(a,b) \\ &\quad < \beta_2 Q(a,b) + (1-\beta_2)A(a,b), \\ &Q^{\alpha_3}(a,b)A^{1-\alpha_3}(a,b) < T(a,b) \\ &\quad < Q^{\beta_3}(a,b)A^{1-\beta_3}(a,b) \end{aligned} \quad (7)$$

hold for all $a, b > 0$ with $a \neq b$ if and only of $\alpha_1 \leq 3/5$, $\beta_1 \geq 4/\pi$, $\alpha_2 \leq (4-\pi)/[(\sqrt{2}-1)\pi]$, $\beta_2 \geq 2/3$, $\alpha_3 \leq 2/3$, and $\beta_3 \geq 4 - 2\log\pi/\log 2$.

For $\alpha, \beta, \lambda, \mu \in (1/2, 1)$, Chu et al. [22, 23] proved that the inequalities

$$\begin{aligned} &C(\alpha a + (1-\alpha)b, \alpha b + (1-\alpha)a) < T(a,b) \\ &\quad < C(\beta a + (1-\beta)b, \beta b + (1-\beta)a), \\ &Q(\lambda a + (1-\lambda)b, \lambda b + (1-\lambda)a) < T(a,b) \\ &\quad < Q(\mu a + (1-\mu)b, \mu b + (1-\mu)a) \end{aligned} \quad (8)$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $\alpha \leq (1+\sqrt{4/\pi-1})/2$, $\beta \geq (3+\sqrt{3})/6$, $\lambda \leq (1+\sqrt{16/\pi^2-1})/2$ and $\mu \geq (3+\sqrt{6})/6$.

The aim of this paper is to find the greatest values $r_1, r_2$ and the least values $s_1, s_2$ such that the double inequalities

$$\begin{aligned} &C(r_1 a + (1-r_1)b, r_1 b + (1-r_1)a) \\ &\quad < \alpha A(a,b) + (1-\alpha)T(a,b) \\ &\quad < C(s_1 a + (1-s_1)b, s_1 b + (1-s_1)a), \end{aligned} \quad (9)$$

$$\begin{aligned} &C(r_2 a + (1-r_2)b, r_2 b + (1-r_2)a) \\ &\quad < \alpha A(a,b) + (1-\alpha)M(a,b) \\ &\quad < C(s_2 a + (1-s_2)b, s_2 b + (1-s_2)a) \end{aligned} \quad (10)$$

hold for any $\alpha \in (0,1)$ and all $a, b > 0$ with $a \neq b$.

## 2. Lemmas

In order to prove our main results, we need three lemmas, which we present in this section.

**Lemma 1** (see [24, Theorem 1.25]). *For $-\infty < a < b < +\infty$, let $f, g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on $(a,b)$, let $g'(x) \neq 0$ on $(a,b)$. If $f'(x)/g'(x)$ is increasing (decreasing) on $(a,b)$, then so are*

$$\frac{f(x)-f(a)}{g(x)-g(a)}, \qquad \frac{f(x)-f(b)}{g(x)-g(b)}. \quad (11)$$

*If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.*

**Lemma 2.** *Let $u, \alpha \in (0,1)$ and*

$$f_{u,\alpha}(x) = ux^2 - (1-\alpha)\left(\frac{x}{\arctan x} - 1\right). \tag{12}$$

*Then $f_{u,\alpha}(x) > 0$ for all $x \in (0,1)$ if and only if $u \geq (1-\alpha)/3$ and $f_{u,\alpha}(x) < 0$ for all $x \in (0,1)$ if and only if $u \leq (1-\alpha)(4/\pi - 1)$.*

*Proof.* From (12), one has

$$f_{u,\alpha}(0^+) = 0, \tag{13}$$

$$f_{u,\alpha}(1^-) = u - (1-\alpha)\left(\frac{4}{\pi} - 1\right), \tag{14}$$

$$f'_{u,\alpha}(x) = 2x\left[u - \frac{1-\alpha}{2}g(x)\right], \tag{15}$$

where

$$g(x) = \frac{(1+x^2)\arctan x - x}{x(1+x^2)(\arctan x)^2}. \tag{16}$$

Let $g_1(x) = \arctan x - x/(1+x^2)$ and $g_2(x) = x(\arctan x)^2$, then

$$g(x) = \frac{g_1(x)}{g_2(x)}, \qquad g_1(0) = g_2(0) = 0, \tag{17}$$

$$\begin{aligned}\frac{g_1{}'(x)}{g_2{}'(x)} &= \frac{2x^2}{2x(1+x^2)\arctan x + (1+x^2)^2(\arctan x)^2} \\ &= \frac{1}{((1+x^2)\arctan x/x) + (1/2)\left[(1+x^2)\arctan x/x\right]^2}.\end{aligned} \tag{18}$$

It is not difficult to verify that the function $(1+x^2)\arctan x/x$ is strictly increasing on $(0,1)$. Then (17) and (18) together with Lemma 1 lead to the conclusion that $g(x)$ is strictly decreasing on $(0,1)$. Moreover, making use of L'Hôpital's rule, we get

$$g(0^+) = \frac{2}{3}, \tag{19}$$

$$g(1^-) = \frac{4(\pi-2)}{\pi^2}. \tag{20}$$

We divide the proof into four cases.

*Case 1.* $u \geq (1-\alpha)/3$. Then from (15) and (19) together with the monotonicity of $g(x)$, we clearly see that $f_{u,\alpha}(x)$ is strictly increasing on $(0,1)$. Therefore, $f_{u,\alpha}(x) > 0$ for all $x \in (0,1)$ follows from (13) and the monotonicity of $f_{u,\alpha}(x)$.

*Case 2.* $u \leq 2(1-\alpha)(\pi-2)/\pi^2$. Then from (15) and (20) together with the monotonicity of $g(x)$, we clearly see that $f_{u,\alpha}(x)$ is strictly decreasing on $(0,1)$. Therefore, $f_{u,\alpha}(x) < 0$ for all $x \in (0,1)$ follows from (13) and the monotonicity of $f_{u,\alpha}(x)$.

*Case 3.* $2(1-\alpha)(\pi-2)/\pi^2 < u \leq (1-\alpha)(4/\pi - 1)$. Then (14) leads to

$$f_{u,\alpha}(1^-) \leq 0. \tag{21}$$

From (15), (19), and (20) together with the monotonicity of $g(x)$, we clearly see that there exists unique $x_0 \in (0,1)$ such that $f_{u,\alpha}(x)$ is strictly decreasing on $(0, x_0]$ and strictly increasing on $[x_0, 1)$. Therefore, $f_{u,\alpha}(x) < 0$ for all $x \in (0,1)$ follows from (13) and (21) together with the piecewise monotonicity of $f_{u,\alpha}(x)$.

*Case 4.* $(1-\alpha)(4/\pi - 1) < u \leq (1-\alpha)/3$. Then (14) leads to

$$f_{u,\alpha}(1^-) > 0. \tag{22}$$

It follows from (15), (19), and (20) together with the monotonicity of $g(x)$, there exists unique $x_1 \in (0,1)$ such that $f_{u,\alpha}(x)$ is strictly decreasing on $(0, x_1]$ and strictly increasing on $[x_1, 1)$. Equation (13) and inequality (22) together with the piecewise monotonicity of $f_{u,\alpha}(x)$ lead to the conclusion that there exists $x_2 \in (x_1, 1)$ such that $f_{u,\alpha}(x) < 0$ for $x \in (0, x_2)$ and $f_{u,\alpha}(x) > 0$ for $x \in (x_2, 1)$. □

**Lemma 3.** *Let $\lambda, \alpha \in (0,1)$ and*

$$\varphi_{\lambda,\alpha}(x) = \lambda x^2 - (1-\alpha)\left(\frac{x}{\sinh^{-1}(x)} - 1\right). \tag{23}$$

*Then $\varphi_{\lambda,\alpha}(x) > 0$ for all $x \in (0,1)$ if and only if $\lambda \geq (1-\alpha)/6$ and $\varphi_{\lambda,\alpha}(x) < 0$ for all $x \in (0,1)$ if and only if $\lambda \leq (1-\alpha)(1 - \log(1+\sqrt{2}))/\log(1+\sqrt{2})$.*

*Proof.* From (23) we get

$$\varphi_{\lambda,\alpha}(0^+) = 0, \tag{24}$$

$$\varphi_{\lambda,\alpha}(1^-) = \lambda - \frac{(1-\alpha)\left[1 - \log\left(1+\sqrt{2}\right)\right]}{\log\left(1+\sqrt{2}\right)}, \tag{25}$$

$$\varphi'_{\lambda,\alpha}(x) = 2x\left[\lambda - \frac{1-\alpha}{2}\psi(x)\right], \tag{26}$$

where

$$\psi(x) = \frac{\sinh^{-1}(x) - x/\sqrt{1+x^2}}{x\left(\sinh^{-1}(x)\right)^2}. \tag{27}$$

Let $\psi_1(x) = \sinh^{-1}(x) - x/\sqrt{1+x^2}$ and $\psi_2(x) = x(\sinh^{-1}(x))^2$, then

$$\psi(x) = \frac{\psi_1(x)}{\psi_2(x)}, \qquad \psi_1(0) = \psi_2(0) = 0,$$

$$
\begin{aligned}
\frac{\psi_1'(x)}{\psi_2'(x)}
&= x^2 \times \left(\left(1+x^2\right)^{3/2}\left(\sinh^{-1}(x)\right)^2 \right. \\
&\qquad \left. + 2x\left(1+x^2\right)\sinh^{-1}(x)\right)^{-1} \\
&= \left(\left(\left(1+x^2\right)^{3/4}\sinh^{-1}(x)/x\right)^2 \right. \\
&\qquad \left. + 2\left(1+x^2\right)^{1/4}\left(\left(1+x^2\right)^{3/4}\sinh^{-1}(x)/x\right)\right)^{-1}.
\end{aligned} \tag{28}
$$

It is not difficult to verify that the function $(1+x^2)^{3/4}\sinh^{-1}(x)/x$ is strictly increasing on $(0,1)$. Then (28) together with Lemma 1 leads to the conclusion that $\psi(x)$ is strictly decreasing on $(0,1)$. Moreover, making use of L'Hôpital's rule, we have

$$\psi(0^+) = \frac{1}{3}, \tag{29}$$

$$\psi(1^-) = \frac{\sqrt{2}\log\left(1+\sqrt{2}\right)-1}{\sqrt{2}\log^2\left(1+\sqrt{2}\right)}. \tag{30}$$

We divide the proof into four cases.

*Case 1.* $\lambda \geq (1-\alpha)/6$. Then from (26) and (29) together with the monotonicity of $\psi(x)$, we clearly see that $\varphi_{\lambda,\alpha}(x)$ is strictly increasing on $(0,1)$. Therefore, $\varphi_{\lambda,\alpha}(x) > 0$ for all $x \in (0,1)$ follows from (24) and the monotonicity of $\varphi_{\lambda,\alpha}(x)$.

*Case 2.* $\lambda \leq (1-\alpha)[\sqrt{2}\log(1+\sqrt{2})-1]/[2\sqrt{2}\log^2(1+\sqrt{2})]$. Then from (26) and (30) together with the monotonicity of $\psi(x)$, we clearly see that $\varphi_{\lambda,\alpha}(x)$ is strictly decreasing on $(0,1)$. Therefore, $\varphi_{\lambda,\alpha}(x) < 0$ for all $x \in (0,1)$ follows from (24) and the monotonicity of $\varphi_{\lambda,\alpha}(x)$.

*Case 3.* $((1-\alpha)[\sqrt{2}\log(1+\sqrt{2})-1]/2\sqrt{2}\log^2(1+\sqrt{2})) < \lambda \leq ((1-\alpha)[1-\log(1+\sqrt{2})]/\log(1+\sqrt{2}))$. Then (25) leads to

$$\varphi_{\lambda,\alpha}(1^-) \leq 0. \tag{31}$$

From (26), (29), and (30) together with the monotonicity of $\psi(x)$, we clearly see that there exists $x_3 \in (0,1)$ such that $\varphi_{\lambda,\alpha}(x)$ is strictly decreasing on $(0, x_3]$ and strictly increasing on $[x_3, 1)$. Therefore, $\varphi_{\lambda,\alpha}(x) < 0$ for all $x \in (0,1)$ follows from (24) and (31) together with the piecewise monotonicity of $\varphi_{\lambda,\alpha}(x)$.

*Case 4.* $((1-\alpha)[1-\log(1+\sqrt{2})]/\log(1+\sqrt{2})) < \lambda < ((1-\alpha)/6)$. Then (25) leads to

$$\varphi_{\lambda,\alpha}(1^-) > 0. \tag{32}$$

It follows from (26), (29), and (30) together with the monotonicity of $\psi(x)$, there exists $x_4 \in (0,1)$ such that $\varphi_{\lambda,\alpha}(x)$ is strictly decreasing on $(0, x_4]$ and strictly increasing on $[x_4, 1)$. Equation (24) and inequality (32) together with the piecewise monotonicity of $\varphi_{\lambda,\alpha}(x)$ lead to the conclusion that there exists $x_5 \in (x_4, 1)$ such that $\varphi_{\lambda,\alpha}(x) < 0$ for $x \in (0, x_5)$ and $\varphi_{\lambda,\alpha}(x) > 0$ for $x \in (x_5, 1)$. □

## 3. Main Results

**Theorem 4.** *If* $\alpha \in (0,1)$ *and* $r_1, s_1 \in (1/2, 1)$, *then the double inequality*

$$
\begin{aligned}
&C\left(r_1 a + (1-r_1) b, r_1 b + (1-r_1) a\right) \\
&\quad < \alpha A(a,b) + (1-\alpha) T(a,b) \\
&\quad < C\left(s_1 a + (1-s_1) b, s_1 b + (1-s_1) a\right)
\end{aligned} \tag{33}
$$

*holds for all* $a, b > 0$ *with* $a \neq b$ *if and only if* $r_1 \leq [1 + \sqrt{(1-\alpha)(4-\pi)/\pi}]/2$ *and* $s_1 \geq [1 + \sqrt{(1-\alpha)/3}]/2$.

*Proof.* Since $A(a,b)$, $T(a,b)$, and $C(a,b)$ are symmetric and homogeneous of degree one, without loss of generality, we assume that $a > b$. Let $p \in (1/2, 1)$ and $x = (a-b)/(a+b)$, then $x \in (0,1)$ and

$$
\begin{aligned}
&C\left(pa + (1-p) b, pb + (1-p) a\right) \\
&\quad - [\alpha A(a,b) + (1-\alpha) T(a,b)] \\
&\quad = A(a,b)\left[(2p-1)^2 x^2 - (1-\alpha)\left(\frac{x}{\arctan x} - 1\right)\right].
\end{aligned} \tag{34}
$$

Therefore, Theorem 4 follows easily from Lemma 2 and (34). □

**Theorem 5.** *If* $\alpha \in (0,1)$ *and* $r_2, s_2 \in (1/2, 1)$, *then the double inequality*

$$
\begin{aligned}
&C\left(r_2 a + (1-r_2) b, r_2 b + (1-r_2) a\right) \\
&\quad < \alpha A(a,b) + (1-\alpha) M(a,b) \\
&\quad < C\left(s_2 a + (1-s_2) b, s_2 b + (1-s_2) a\right)
\end{aligned} \tag{35}
$$

*holds for all* $a, b > 0$ *with* $a \neq b$ *if and only if* $s_2 \geq [1 + \sqrt{(1-\alpha)/6}]/2$ *and* $r_2 \leq [1 + \sqrt{(1-\alpha)(1-\log(1+\sqrt{2}))/\log(1+\sqrt{2})}]/2$.

*Proof.* Since $A(a,b)$, $M(a,b)$, and $C(a,b)$ are symmetric and homogeneous of degree one, without loss of generality, we assume that $a > b$. Let $q \in (1/2, 1)$ and $x = (a-b)/(a+b)$, then $x \in (0,1)$ and

$$
\begin{aligned}
&C\left(qa + (1-q) b, qb + (1-q) a\right) \\
&\quad - [\alpha A(a,b) + (1-\alpha) M(a,b)] \\
&\quad = A(a,b)\left[(2q-1)^2 x^2 - (1-\alpha)\left(\frac{x}{\sinh^{-1}(x)} - 1\right)\right].
\end{aligned} \tag{36}
$$

Therefore, Theorem 5 follows easily from Lemma 3 and (36). □

*Remark 6.* If $\alpha = 0$, then Theorem 4 reduces to the first double inequality in (8).

**Corollary 7.** *If $\lambda, \mu \in (1/2, 1)$, then the double inequality*

$$C(\lambda a + (1-\lambda) b, \lambda b + (1-\lambda) a) < M(a,b) < C(\mu a + (1-\mu) b, \mu b + (1-\mu) a) \quad (37)$$

*holds for all $a, b > 0$ with $a \neq b$ if and only if $\lambda \leq [1 + \sqrt{1/\log(1+\sqrt{2}) - 1}]/2$ and $\mu \geq (6+\sqrt{6})/12$.*

*Proof.* Corollary 7 follows easily from Theorem 5 with $\alpha = 0$. □

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## References

[1] E. Neuman and J. Sándor, "On the Schwab-Borchardt mean," *Mathematica Pannonica*, vol. 14, no. 2, pp. 253–266, 2003.

[2] H.-J. Seiffert, "Aufgabe $\beta$ 16," *Die Wurzel*, vol. 29, pp. 221–222, 1995.

[3] Y.-M. Chu and B.-Y. Long, "Bounds of the Neuman-Sándor mean using power and identric means," *Abstract and Applied Analysis*, vol. 2012, Article ID 832591, 6 pages, 2012.

[4] Y.-M. Chu, B.-Y. Long, W.-M. Gong, and Y.-Q. Song, "Sharp bounds for Seiffert and Neuman-Sándor means in terms of generalized logarithmic means," *Journal of Inequalities and Applications*, vol. 2013, article 10, 2013.

[5] F. R. Villatoro, "Local error analysis of Evans-Sanugi, nonlinear one-step methods based on $\theta$-means," *International Journal of Computer Mathematics*, vol. 87, no. 5, pp. 1009–1022, 2010.

[6] F. R. Villatoro, "Stability by order stars for non-linear theta-methods based on means," *International Journal of Computer Mathematics*, vol. 87, no. 1–3, pp. 226–242, 2010.

[7] J. Pahikkala, "On contraharmonic mean and Phythagorean triples," *Elemente der Mathematik*, vol. 65, no. 2, pp. 62–67, 2010.

[8] O. Y. Ababneh and R. Rozita, "New third order Runge Kutta based on contraharmonic mean for stiff problems," *Applied Mathematical Sciences*, vol. 3, no. 5–8, pp. 365–376, 2009.

[9] S. Toader and G. Toader, "Complementaries of Greek means with respect to Gini means," *International Journal of Applied Mathematics & Statistics*, vol. 11, no. N07, pp. 187–192, 2007.

[10] Y. Lim, "The inverse mean problem of geometric and contraharmonic means," *Linear Algebra and Its Applications*, vol. 408, pp. 221–229, 2005.

[11] S. Toader and G. Toader, "Generalized complementaries of Greek means," *Pure Mathematics and Applications*, vol. 15, no. 2-3, pp. 335–342, 2004.

[12] Y.-M. Chu, M.-K. Wang, and Y.-F. Qiu, "Optimal Lehmer mean bounds for the geometric and arithmetic combinations of arithmetic and Seiffert means," *Azerbaijan Journal of Mathematics*, vol. 2, no. 1, pp. 3–9, 2012.

[13] P. A. Hästö, "A monotonicity property of ratios of symmetric homogeneous means," *JIPAM*, vol. 3, no. 5, article 71, 2002.

[14] Y.-M. Chu, M.-K. Wang, and Y.-F. Qiu, "An optimal double inequality between power-type Heron and Seiffert means," *Journal of Inequalities and Applications*, vol. 2010, Article ID 146945, 11 pages, 2010.

[15] M.-K. Wang, Y.-F. Qiu, and Y.-M. Chu, "Sharp bounds for Seiffert means in terms of Lehmer means," *Journal of Mathematical Inequalities*, vol. 4, no. 4, pp. 581–586, 2010.

[16] E. Neuman and J. Sándor, "On the Schwab-Borchardt mean. II," *Mathematica Pannonica*, vol. 17, no. 1, pp. 49–59, 2006.

[17] Y.-M. Li, B.-Y. Long, and Y.-M. Chu, "Sharp bounds for the Neuman-Sándor mean in terms of generalized logarithmic mean," *Journal of Mathematical Inequalities*, vol. 6, no. 4, pp. 567–577, 2012.

[18] E. Neuman, "A note on a certain bivariate mean," *Journal of Mathematical Inequalities*, vol. 6, no. 4, pp. 637–643, 2012.

[19] T.-H. Zhao, Y.-M. Chu, and B.-Y. Liu, "Optimal bounds for Neuman-Sándor mean in terms of the convex combinations of harmonic, geometric, quadratic, and contraharmonic means," *Abstract and Applied Analysis*, vol. 2012, Article ID 302635, 9 pages, 2012.

[20] Y.-M. Chu, C. Zong, and G.-D. Wang, "Optimal convex combination bounds of Seiffert and geometric means for the arithmetic mean," *Journal of Mathematical Inequalities*, vol. 5, no. 3, pp. 429–434, 2011.

[21] Y.-M. Chu, M.-K. Wang, and W.-M. Gong, "Two sharp double inequalities for Seiffert mean," *Journal of Inequalities and Applications*, vol. 2011, article 44, 2011.

[22] Y.-M. Chu and S.-W. Hou, "Sharp bounds for Seiffert mean in terms of contraharmonic mean," *Abstract and Applied Analysis*, vol. 2012, Article ID 425175, 6 pages, 2012.

[23] Y.-M. Chu, S.-W. Hou, and Z.-H. Shen, "Sharp bounds for Seiffert mean in terms of root mean square," *Journal of Inequalities and Applications*, vol. 2012, article 11, 2012.

[24] G. D. Anderson, M. K. Vamanamurthy, and M. K. Vuorinen, *Conformal Invariants, Inequalities, and Quasiconformal Maps*, Canadian Mathematical Society Series of Monographs and Advanced Texts, John Wiley & Sons, New York, NY, USA, 1997.