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Research Article

f-Orthomorphisms and f-Linear Operators on the Order Dual of an f-Algebra Revisited

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We give a necessary and sufficient condition on an f-algebra A for which orthomorphisms, f-linear operators, and f-orthomorphisms on the order dual $A^\sim$ are the same class of operators.

1. Introduction and Preliminaries

First of all, we point out that the standard book [1] is adopted in this paper as a unique source of unexplained terminology and notation.

Let L and M be Riesz spaces. The operator $T : L \to M$ is called order bounded if the image under T of an order bounded set in L is again an order bounded set in M. An order bounded operator $\pi : L \to L$ is said to be an orthomorphism if $|\pi x| \wedge |y| = 0$ whenever $|x| \wedge |y| = 0$ for all $x, y \in L$. The set of all orthomorphisms on L is denoted by $\mathrm{Orth}(L)$.

If L is a Riesz space $L^\sim$, the first order dual of L will be the Riesz space of all order bounded linear functionals on L. The second order dual (or order bidual) is denoted by $L^{\sim\sim}$. It is well known that $L^\sim$ is a Dedekind complete Riesz space. Define for each $x \in L$ the element $x^{\sim\sim} \in L^{\sim\sim}$ by $x^{\sim\sim}(f) = f(x)$ for all $f \in L^\sim$ and then define the mapping $\kappa : L \to L^{\sim\sim}$ by $\kappa(x) = x^{\sim\sim}$ for all $x \in L$. If $L^\sim$ separates the points of L then κ is injective and we can identify L with the Riesz subspace $\kappa(L)$. Throughout the paper, we only consider Archimedean Riesz spaces L with point separating order dual $L^\sim$.

From now on, A denotes an f-algebra, that is, a lattice ordered algebra in which $a \wedge b = 0$ implies that $a \cdot c \wedge b = 0$ for all $0 \le c \in A$. Following constructions in [2], a multiplication can be introduced in the order bidual $A^{\sim\sim}$ of A. This is accomplished in three steps as explained next. For every $f \in A^\sim$ and $x \in A$, we define $f \cdot x \in A^\sim$ by the following:

$$(f \cdot x)(y) = f(xy) \quad (y \in A). \tag{1}$$

Then, for $G \in A^{\sim\sim}$ and $f \in A^\sim$, we introduce $G \cdot f \in A^\sim$ by putting the following:

$$(G \cdot f)(x) = G(f \cdot x) \quad (x \in A). \tag{2}$$

Finally, let $F, G \in A^{\sim\sim}$ and define $FG \in A^{\sim\sim}$ by the following

$$(FG)(f) = F(G \cdot f) \quad (f \in A^\sim). \tag{3}$$

The latter equality defines the *Arens multiplication* in $A^{\sim\sim}$. Bernau and Huijsmans in [3] that $A^{\sim\sim}$ is an f-algebra with respect to the Arens multiplication. The band of all order continuous linear functionals on $A^\sim$ is denoted by $(A^\sim)^\sim_n$ and its disjoint complement in $A^{\sim\sim}$ by $(A^\sim)^\sim_s$. Observe that

$$A^{\sim\sim} = (A^\sim)^\sim_n \oplus (A^\sim)^\sim_s, \tag{4}$$

as $A^{\sim\sim}$ is Dedekind complete.

For each $F \in (A^\sim)^\sim_n$, we define the mapping $V_F : A^\sim \to A^\sim$ by the following:

$$V_F(f) = F \cdot f \quad \forall f \in A^\sim. \tag{5}$$

It is shown in [4, Theorem 5.2] that $V_F \in \mathrm{Orth}(A^\sim)$ and the mapping $V : F \to V(F) = V_F$ are an algebra and lattice homomorphism between $(A^\sim)^\sim_n$ and $\mathrm{Orth}(A^\sim)$.

The final paragraph of this introduction is devoted to the definition of the so-called f-orthomorphism and f-linear operator on the order dual.

Let A be an f-algebra. For all $f \in A^\sim$ define the set $R(f)$ by the following:

$$R(f) = \{F \cdot f : F \in (A^\sim)^\sim_n\}. \tag{6}$$

An order bounded operator $T : A^{\sim} \to A^{\sim}$ is said to be as follows:

(1) an f-orthomorphism if $R(Tf) \subseteq R(f)$ for each $f \in A^{\sim}$, the collection of all f-orthomorphisms on $A^{\sim}$ will be denoted by $\mathrm{Orth}(A^{\sim}, A^{\sim}, (A^{\sim})_n^{\sim})$;

(2) an f-linear with respect to $(A^{\sim})_n^{\sim}$ if $T(F \cdot f) = F \cdot Tf$ for all $f \in A^{\sim}$ and $F \in (A^{\sim})_n^{\sim}$, the set of all f-linear operators on $A^{\sim}$ will be denoted by $\mathscr{L}_b(A^{\sim}, A^{\sim}, (A^{\sim})_n^{\sim})$.

Turan showed [5, Proposition 2.6] that $\mathrm{Orth}(A^{\sim}, A^{\sim}, (A^{\sim})_n^{\sim}) = \mathscr{L}_b(A^{\sim}, A^{\sim}, (A^{\sim})_n^{\sim})$ whenever A is unital f-algebra (since A is topologically full with respect to itself). Recently, Feng et al. [6, Theorem 3.4] proved that for any f-algebra A we have the following:

$$\mathrm{Orth}(A^{\sim}) \subseteq \mathscr{L}_b\left(A^{\sim}, A^{\sim}, (A^{\sim})_n^{\sim}\right) \subseteq \mathrm{Orth}\left(A^{\sim}, A^{\sim}, (A^{\sim})_n^{\sim}\right). \tag{7}$$

This leads to a very natural question, namely, when does any f-orthomorphisms is an orthomorphism. A partial answer was already obtained by Feng et al. in [6, Theorem 3.4]. In this regards they proved that if A has the factorization property, then $\mathrm{Orth}(A^{\sim}) = \mathscr{L}_b(A^{\sim}, A^{\sim}, (A^{\sim})_n^{\sim}) = \mathrm{Orth}(A^{\sim}, A^{\sim}, (A^{\sim})_n^{\sim})$. In this paper, we give a complete answer to this question. In fact, we give a necessary and sufficient condition on an f-algebra A for which orthomorphisms, f-orthomorphisms, and f-linear operators on the order dual are precisely the same class of operators.

2. When f-Orthomorphisms Are Orthomorphisms

Let A be an f-algebra with separating order dual. Let J be the order ideal generated by all products of A; that is,

$$J = \{x \in A : |x| \leq y \cdot z \text{ for some } y, z \in A\}. \tag{8}$$

It is easy to see that J is an ℓ-ideal (ring and order ideal) in A. The annihilator

$$J^0 = \{f \in A^{\sim} : f(x) = 0 \ \forall x \in J\} \tag{9}$$

is always a band in $A^{\sim}$. Since $A^{\sim}$ is Dedekind complete Riesz space, then $A^{\sim} = J^0 \oplus (J^0)^d$. We denote by $P_0 : A^{\sim} \to J^0$ the band projection.

As we will see later, the set J^0 plays a key role in the proof of the main result of this paper. We start our discussion by a study of the connection between f-orthomorphism and orthomorphism in a special case.

Example 1. Suppose that the set $J = \{0\}$; hence $J^0 = A^{\sim}$. This implies that the product in A is given by $x \cdot y = 0$ for all $x, y \in A$. It is easy to see that

$$\mathscr{L}_b\left(A^{\sim}, A^{\sim}, (A^{\sim})_n^{\sim}\right) = \mathrm{Orth}\left(A^{\sim}, A^{\sim}, (A^{\sim})_n^{\sim}\right) = \mathscr{L}_b\left(A^{\sim}\right). \tag{10}$$

Thus, $\mathrm{Orth}(A^{\sim}) = \mathrm{Orth}(A^{\sim}, A^{\sim}, (A^{\sim})_n^{\sim})$ if and only if every order bounded operator is an orthomorphism. Hence, $\dim A^{\sim} = \dim A = 1$.

From now on, we may assume that the ideal $J \neq \{0\}$.

It is obvious that if A has the factorization property, then $J^0 = \{0\}$. In the next example we show that the condition "$J^0 = \{0\}$" is strictly weaker than the condition "has the factorization property." Recall that the condition "$J^0 = \{0\}$" is equivalent to the condition "$(A^{\sim})_n^{\sim}$ is a semi-prime f-algebra" (see [4, Corollary 6.3]).

Example 2. Let $C([0,1))$ be the Archimedean unital f-algebra of all real-valued continuous functions on $[0,1)$. A function $f \in C([0,1))$ is an eventual-polynomial if there is $r_f \in [0,1)$ and a (unique) polynomial P_f such that $f(r) = P_f(r)$ for all $r \in [r_f, 1)$. Let Y be the set of all eventual-polynomials in $C([0,1))$. Now, put the following:

$$A = \left\{f \in Y : P_f(0) = 0\right\}, \tag{11}$$

and observe that A is a semiprime f-subalgebra of Y. Clearly, A does not satisfy the factorization property. It is shown in [7, Example 7] that $(A^{\sim})_n^{\sim}$ is a unital f-algebra, hence semiprime, f-algebra. So $J^0 = \{0\}$.

We list some simple properties of the band J^0. The proof of the next lemma is straightforward and therefore omitted.

Lemma 3. *Let A be an f-algebra and $F \in A^{\sim\sim}$. Then*

(1) *$f \in J^0$ if and only if $f \cdot x$ for all $f \in A^{\sim}$ and $x \in A$.*

(2) *$F \cdot f = 0$ for all $f \in J^0$.*

(3) *$F \cdot f \in (J^0)^d$ for all $f \in A^{\sim}$.*

The next result is important in the context of this work; it is already proved in [6] but for the sake of completeness we partially reproduce the proof.

Proposition 4. *Let A be an f-algebra. Then* $\mathrm{Orth}(A^{\sim}) \subseteq \mathscr{L}_b(A^{\sim}, A^{\sim}, (A^{\sim})_n^{\sim}) \subseteq \mathrm{Orth}(A^{\sim}, A^{\sim}, (A^{\sim})_n^{\sim})$.

Proof. Let $\pi \in \mathrm{Orth}(A^{\sim})$; we have to show that $\pi(F \cdot f) = F \cdot \pi(f)$ for all $F \in (A^{\sim})_n^{\sim}$ and $f \in A^{\sim}$. Since $V_F \in \mathrm{Orth}(A^{\sim})$ which is a commutative algebra we get the following:

$$\pi(F \cdot f) = \pi \circ V_F(f) = V_F \circ \pi(f) = F \cdot \pi(f). \tag{12}$$

Thus, $\pi \in \mathscr{L}_b(A^{\sim}, A^{\sim}, (A^{\sim})_n^{\sim})$.

Let $T \in \mathscr{L}_b(A^{\sim}, A^{\sim}, (A^{\sim})_n^{\sim})$ and $f \in A^{\sim}$. Pick $x \in A$ and observe that

$$T(f \cdot x) = T\left(x^{\sim\sim} \cdot f\right) = x^{\sim\sim} \cdot Tf = Tf \cdot x. \tag{13}$$

Then for all $F \in (A^{\sim})_n^{\sim}$ we get the following:

$$\begin{aligned} F \cdot Tf(x) &= F(Tf \cdot x) = F(T(f \cdot x)) \\ &= F \circ T(f \cdot x) = F \circ T \cdot f(x) \quad \forall x \in A. \end{aligned} \tag{14}$$

Hence, $F \cdot Tf = T^{\sim}(F) \cdot f$ where $T^{\sim}$ is the order adjoint mapping of T defined by $T^{\sim}(F) = F \circ T$. Consequently, $T \in \mathrm{Orth}(A^{\sim}, A^{\sim}, (A^{\sim})_n^{\sim})$. □

Another lemma turns out to be useful for later purposes.

Lemma 5. *Let A be an f-algebra and $T \in \mathrm{Orth}(A^{\sim}, A^{\sim}, (A^{\sim})_n^{\sim})$. Then $|Tf| \wedge g \in J^0$ whenever $f \wedge g = 0$ in $A^{\sim}$.*

Proof. It suffices to show that $(|Tf| \wedge g) \cdot x = 0$ for all $0 \le x \in A$. Let $0 \le x \in A$. Since $V_{x^{\sim\sim}} \in \mathrm{Orth}(A^{\sim})$ we get the following:

$$\begin{aligned}(|Tf| \wedge g) \cdot x &= x^{\sim\sim} \cdot (|Tf| \wedge g) \\ &= x^{\sim\sim} \cdot |Tf| \wedge x^{\sim\sim} \cdot g \\ &= |x^{\sim\sim} \cdot Tf| \wedge x^{\sim\sim} \cdot g.\end{aligned} \tag{15}$$

There exists $G \in (A^{\sim})_n^{\sim}$ such that $x^{\sim\sim} \cdot Tf = G \cdot f$. Consequently,

$$\begin{aligned}(|Tf| \wedge g) \cdot x &= |G \cdot f| \wedge x^{\sim\sim} \cdot g \\ &\le G \cdot |f| \wedge x^{\sim\sim} \cdot g \\ &\le (G + x^{\sim\sim}) \cdot (|f| \wedge g) = 0.\end{aligned} \tag{16}$$

□

We have now gathered all of the ingredients for the proof of the principal theorem of this paper.

Theorem 6. *Let A be an f-algebra. Then the following are equivalents:*

(i) $\mathrm{Orth}(A^{\sim}) = \mathscr{L}_b(A^{\sim}, A^{\sim}, (A^{\sim})_n^{\sim}) = \mathrm{Orth}(A^{\sim}, A^{\sim}, (A^{\sim})_n^{\sim})$;

(ii) $J^0 = \{0\}$.

Proof. (i)⇒(ii) Arguing by contradiction, suppose that $J^0 \neq \{0\}$. There exists a nonzero positive element $h \in A^{\sim}$ such that $h(a) = 0$ for all $a \in J$. Let $0 \le g \in (J^0)^d$ and $0 \le x \in A$ such that $g(x) \neq 0$. Define the mapping $T : A^{\sim} \to A^{\sim}$ by $T(f) = f(x)h$. It is not hard to see that T is positive and for all $F \in (A^{\sim})_n^{\sim}$, $f \in A^{\sim}$ we have the following:

$$F \cdot Tf = f(x) F \cdot h = 0 \in R(f). \tag{17}$$

Thus, $T \in \mathrm{Orth}(A^{\sim}, A^{\sim}, A^{\sim})$. But T is not an orthomorphism since $g \wedge h = 0$ and $Tg \wedge h = g(x)h \wedge h \neq 0$.

(ii)⇒(i) In view of Proposition 4, it suffices to show that $\mathrm{Orth}(A^{\sim}, A^{\sim}, (A^{\sim})_n^{\sim}) \subseteq \mathrm{Orth}(A^{\sim})$. To do this, pick $T \in \mathrm{Orth}(A^{\sim}, A^{\sim}, (A^{\sim})_n^{\sim})$ and $f, g \in A^{\sim}$ such that $f \wedge g = 0$. According to Lemma 5,we obtain $|Tf| \wedge g \in J^0 = \{0\}$; that is, $|Tf| \wedge g = 0$, as required. □

It follows from Theorem 6 that if $\mathrm{Orth}(A^{\sim}) = \mathrm{Orth}(A^{\sim}, A^{\sim}, (A^{\sim})_n^{\sim})$ then $(A^{\sim})_n^{\sim}$; hence A, is a semiprime f-algebra.

We end this section with a consequence of Theorem 6. The notion of weak approximate unit plays a key role in the context of this study. Recall that an upward directed net $\{a_\tau : \tau \in T\}$ of positive elements in an f-algebra A is said to be an approximate unit if $\sup_\tau a_\tau \cdot x = x$ for all $0 \le x \in A$. An approximate unit $\{a_\tau\}$ is called weak approximate unit if $\sup_\tau f(a_\tau x) = f(x)$ for all $0 \le f \in A^{\sim}$ and $0 \le x \in A$ (see [4, Definitions 2.2 and 7.1]). It is well known that if A is a semi-prime f-algebra then A can be embedded as a ring and f-subalgebra in the unital f-algebra $\mathrm{Orth}(A)$.

In Theorem 6, we gave two necessary and sufficient conditions for which f-orthomorphisms are orthomorphisms. In the following proposition, we shall present a second one in terms of approximate unit. We have to impose, however, on A an extra condition, namely, the Stone condition (i.e., $x \wedge I \in A$ whenever x is positive in A, where I is the identity of $\mathrm{Orth}(A)$). It should be pointed out here that every uniformly complete semi-prime f-algebra A satisfies the Stone condition.

Proposition 7. *Consider in a semi-prime f-algebra A the following conditions.*

Theorem 8. (i) *A has a weak approximate unit.*

(ii) $\mathrm{Orth}(A^{\sim}) = \mathscr{L}_b(A^{\sim}, A^{\sim}, (A^{\sim})_n^{\sim}) = \mathrm{Orth}(A^{\sim}, A^{\sim}, (A^{\sim})_n^{\sim})$.

(i)⇒(ii) *Moreover, if A satisfies the Stone condition then* (i)⇔(ii).

Proof. (i)⇒(ii) Assume that $(a_\tau)_\tau$ is a weak approximate unit. Let $0 \le f \in J^0$, it follows from $f(a_\tau x) = 0$ for all $0 \le x \in A$ that $f(x) = 0$.

(ii)⇒(i) It follows from Theorem 6, that $(A^{\sim})_n^{\sim}$ is semi-prime. The result follows from [4, Theorem 7.2]. □

3. When f-Linear Operators Are Orthomorphisms

In this section, we study the connection between f-linear operators and orthomorphisms. From Proposition 4 and Theorem 6 if f-linear operators on the dual $A^{\sim}$ of an f-algebra A are orthomorphisms and if $J^0 = \{0\}$ then the two classes coincide. Next, we give an example of an f-algebra such that $J^0 \neq \{0\}$ in which f-linear operators are orthomorphisms.

Example 9. Let $A = \mathbb{R}^2$ equipped with the coordinatewise operations and ordering. Consider the multiplication defined by $(a, b) \cdot (c, d) = (ac, 0)$. It is easy to see that for all $f = (f_1, f_2) \in A^{\sim}$ and $F = (F_1, F_2)$ we have $f \cdot (a, b) = (f_1 a_1, 0)$ and $F \cdot f = (F_1 \cdot f_1, 0)$. Note that $J^0 = \{f = (0, f_2) : f_2 = 0\}$. Let $T := \begin{pmatrix} \alpha & \beta \\ \gamma & \sigma \end{pmatrix}$ an f-linear operator. An easy computation shows that $\beta = \gamma = 0$. Thus, T is an orthomorphism.

It seems natural therefore to ask under what condition we have $\mathrm{Orth}(A^{\sim}) = \mathscr{L}_b(A^{\sim}, A^{\sim}, (A^{\sim})_n^{\sim})$. The answer is given in the next theorem. First, let us discuss the ideal of nilpotents elements in the bidual of an f-algebra A. Let $F \in A^{\sim\sim}$ the absolute kernel or null ideal N_F of F is defined by the following:

$$N_F = \{f \in A^{\sim} : |F| (|f|) = 0\}. \tag{18}$$

It is evident that N_F is an order ideal. The disjoint complement of $C_F = N_F^d$ is called the carrier of F and is always a band in $A^\sim$. Given $0 \leq F \in A^{\sim\sim}$ such that $F \cdot f = 0$ for all $f \in A^\sim$ (F is nilpotent in $A^{\sim\sim}$), then $(J^0)^d \subseteq N_F$. Indeed, let $f \in C_F$, since C_F is a band and $V_{x^{\sim\sim}}$ is an orthomorphism in $A^\sim$, then $f \cdot x \in C_F$ for all $x \in A$. But, by hypothesis $f \cdot x \in N_F$, $f \cdot x \in N_F$, so $f \cdot x = 0$ for all $x \in A$. This implies that $f \in J^0$. Consequently, $C_F \subseteq J^0$ and $(J^0)^d \subseteq N_F$. For more information about the nilpotent elements in the bidual of f-algebra, the reader is referred to [4].

We are now in position to prove the main theorem in this section.

Theorem 10. *Let A be an f-algebra. The followings are equivalents.*

(i) $\mathrm{Orth}(A^\sim) = \mathscr{L}_b(A^\sim, A^\sim, (A^\sim)_n^\sim)$.

(ii) $\dim J^0 = 1$.

Proof. (i)⇒(ii) Suppose that $\dim J^0 \geq 2$. We can find therefore positive elements $h, k \in J^0$ such that $h \wedge k = 0$. Let $0 \leq a \in A$ such that $k(a) \neq 0$. Define the mapping $T : A^\sim \to A^\sim$ by the following:

$$T(f) = (P_0 \circ f)(a) \cdot h. \tag{19}$$

Since $F \cdot f \in (J^0)^d$ for all $F \in (A^\sim)_n^\sim$ and $f \in A^\sim$, we get $T(F \cdot f) = F \cdot Tf = 0$. Thus T is an f-linear operator. Clearly, T is not an orthomorphism since $h \wedge k = 0$ and $T(k) \wedge h = k(a)h \wedge h \neq 0$.

(ii)⇒(i) Let T be an f-linear mapping on $A^\sim$ and $0 \leq h \in J^0$. First, we show that $T((J^0)^d) \subset (J^0)^d$ and $T(J^0) \subset J^0$. There exists $\varphi \in A^{\sim\sim}$ such that $P_0 \circ T(f) = \varphi(f) \cdot h$. Since $P_0 \circ T$ is an f-linear operator, we derive that $\varphi(F \cdot f) = 0$ for all $F \in (A^\sim)_n^\sim$ and $f \in A^\sim$. That is $\varphi \cdot f = 0$ for all $f \in A^\sim$. By the remark above, we get $\varphi(f) = 0$ for all $f \in (J^0)^d$. This implies that $T(f) \in (J^0)^d$ whenever $f \in (J^0)^d$. Now, If $f \in J^0$ then

$$Tf \cdot x = x^{\sim\sim} \cdot Tf = T(x^{\sim\sim} \cdot f) = T(f \cdot x) = 0 \quad \forall x \in A. \tag{20}$$

This shows that $T(f) \in J^0$, in particular $T(h) = t \cdot h$ where $t \in \mathbb{R}$. Now, let $f \wedge g = 0$ in $A^\sim$. We have to show that $|Tf| \wedge g = 0$. Decompose f and g as $f = \alpha h + f_1$ and $g = \beta h + g_1$ with $\alpha, \beta \in \mathbb{R}$ and $f_1, g_1 \in (J^0)^d$. The decomposition of Tf is given by the following:

$$Tf = \alpha t \cdot h + T(f_2). \tag{21}$$

Since $f \wedge g = 0$, then $\alpha \cdot \beta = 0$ and $f_1 \wedge g_1 = 0$. On the other hand, according to Lemma 5, we have $|Tf| \wedge g \in J^0$. So

$$|Tf| \wedge g = (\alpha t \wedge \beta) \cdot h = 0 \tag{22}$$

as $\alpha \cdot \beta = 0$. This completes the proof of the theorem. □

References

[1] C. D. Aliprantis and O. Burkinshaw, *Positive Operators*, vol. 119 of *Pure and Applied Mathematics*, Academic Press, Orlando, Fla, USA, 1985.

[2] R. Arens, "The adjoint of a bilinear operation," *Proceedings of the American Mathematical Society*, vol. 2, pp. 839–848, 1951.

[3] S. J. Bernau and C. B. Huijsmans, "The order bidual of almost f-algebras and d-algebras," *Transactions of the American Mathematical Society*, vol. 347, no. 11, pp. 4259–4275, 1995.

[4] C. B. Huijsmans and B. de Pagter, "The order bidual of lattice ordered algebras," *Journal of Functional Analysis*, vol. 59, no. 1, pp. 41–64, 1984.

[5] B. Turan, "On f-linearity and f-orthomorphisms," *Positivity*, vol. 4, no. 3, pp. 293–301, 2000.

[6] Y. Feng, J. X. Chen, and Z. L. Chen, "f-Orthomorphisms and f- Linear Operators on the Order Dual of an -Algebra," *Abstract and Applied Analysis*, vol. 2012, Article ID 971560, 7 pages, 2012.

[7] K. Boulabiar and J. Jaber, "The order bidual of f-algebras revisited," *Positivity*, vol. 15, no. 2, pp. 271–279, 2011.