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Research Article

Four-Point Optimal Sixteenth-Order Iterative Method for Solving Nonlinear Equations

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We present an iterative method for solving nonlinear equations. The proposed iterative method has optimal order of convergence sixteen in the sense of Kung-Traub conjecture (Kung and Traub, 1974); it means that the iterative scheme uses five functional evaluations to achieve $16(=2^{5-1})$ order of convergence. The proposed iterative method utilizes one derivative and four function evaluations. Numerical experiments are made to demonstrate the convergence and validation of the iterative method.

1. Introduction

According to Kung and Traub conjecture, a multipoint iterative method without memory could achieve optimal convergence order 2^{n-1} by performing n evaluations of function or its derivatives [1]. In order to construct an optimal sixteenth-order convergent iterative method for solving nonlinear equations, we require four and eight optimal-order iterative schemes. Many authors have been developed the optimal eighth-order iterative methods, namely, Bi et al. [2], Bi et al. [3], Geum and Kim [4], Liu and Wang [5], Wang and Liu [6], and Soleymani et al. [7–9]. Some recent applications of nonlinear equation solvers in matrix inversion for regular or rectangular matrices have been introduced in [10–12].

For the proposed iterative method, we developed new optimal fourth- and eighth-orders iterative methods to construct optimal sixteenth-order iterative scheme. On the other hand, it is known that rational weight functions give a better convergence radius. By keeping this fact in mind, we introduced rational terms in weight functions to achieve optimal sixteenth order.

For the sake of completeness, we list some existing optimal sixteenth-order convergent methods. Babajee and Thukral [13] suggested 4-point sixteenth-order king family of iterative methods for solving nonlinear equations (BT):

$$\begin{aligned}
y_n &= x_n - \frac{f(x_n)}{f'(x_n)},\\
z_n &= y_n - \frac{1+\beta t_1}{1+(\beta-2)t_1}\frac{f(y_n)}{f'(x_n)},\\
w_n &= z_n - (\theta_0+\theta_1+\theta_2+\theta_3)\frac{f(y_n)}{f'(x_n)},\\
x_{n+1} &= w_n - (\theta_0+\theta_1+\theta_2+\theta_3+\theta_4+\theta_5+\theta_6+\theta_7)\frac{f(w_n)}{f'(x_n)},
\end{aligned} \tag{1}$$

where

$$t_1 = \frac{f(y_n)}{f(x_n)},\qquad t_2 = \frac{f(z_n)}{f(x_n)},\qquad t_3 = \frac{f(z_n)}{f(y_n)},$$

$$t_4 = \frac{f(w_n)}{f(x_n)},\qquad t_5 = \frac{f(w_n)}{f(z_n)},\qquad t_6 = \frac{f(w_n)}{f(y_n)},$$

$$\theta_0 = 1,\qquad \theta_1 = \frac{1+\beta t_1 + 3/2\beta t_1^2}{1+(\beta-2)t_1+(3/2\beta-1)t_1^2} - 1,$$

$$\theta_2 = t_3, \qquad \theta_3 = 4t_2, \qquad \theta_4 = t_5 + t_1 t_2,$$
$$\theta_5 = 2t_1 t_5 + 4(1-\beta) t_1^3 t_3 + 2t_2 t_3,$$
$$\theta_6 = 2t_6 + \left(7\beta^2 - \frac{47}{2}\beta + 14\right) t_3 t_1^4 + (2\beta - 3) t_2^2 + (5 - 2\beta) t_5 t_1^2 - t_3^3,$$
$$\theta_7 = 8t_4 + \left(-12\beta + 2\beta^2 + 12\right) t_5 t_1^3 - 4t_3^3 t_1 + \left(-2\beta^2 + 12\beta - 22\right) t_3^2 t_1^3 + \left(-10\beta^3 + \frac{127}{2}\beta^2 - 105\beta + 46\right) t_2 t_1^4. \tag{2}$$

In 2011, Geum and Kim [14] proposed a family of optimal sixteenth-order multipoint methods (GK2):

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= -K_f(u_n) \frac{f(y_n)}{f'(x_n)}, \\ s_n &= z_n - H_f(u_n, v_n, w_n) \frac{f(z_n)}{f'(x_n)}, \\ x_{n+1} &= s_n - W_f(u_n, v_n, w_n, t_n) \frac{f(s_n)}{f'(x_n)}, \end{aligned} \tag{3}$$

where

$$u_n = \frac{f(y_n)}{f(x_n)}, \qquad v_n = \frac{f(z_n)}{f(y_n)},$$
$$w_n = \frac{f(z_n)}{f(x_n)}, \qquad t_n = \frac{f(s_n)}{f(z_n)},$$
$$K_f(u_n) = \frac{1 + \beta u_n + (-9 + 5/2\beta) u_n^2}{1 + (\beta - 2) u_n + (-4 + \beta/2) u_n^2}, \tag{4}$$
$$H_f = \frac{1 + 2u_n + (2+\sigma) w_n}{1 - v_n + \sigma w_n},$$
$$W_f = \frac{1 + 2u_n}{1 - v_n - 2w_n - t_n} + G(u_n, v_n, w_n),$$

one of the choices for $G(u_n, v_n, w_n)$ along with $\beta = 24/11$ and $\sigma = -2$:

$$G(u_n, v_n, w_n) = -6u_n^3 v_n - \frac{244}{11} u_n^4 w_n + 6w_n^2 + u_n\left(2v_n^2 + 4v_n^3 + w_n - 2w_n^2\right). \tag{5}$$

In the same year, Geum and Kim [15] presented a biparametric family of optimally convergent sixteenth-order multipoint methods (GK1):

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= -K_f(u_n) \frac{f(y_n)}{f'(x_n)}, \\ s_n &= z_n - H_f(u_n, v_n, w_n) \frac{f(z_n)}{f'(x_n)}, \\ x_{n+1} &= s_n - W_f(u_n, v_n, w_n, t_n) \frac{f(s_n)}{f'(x_n)}, \end{aligned} \tag{6}$$

where

$$u_n = \frac{f(y_n)}{f(x_n)}, \qquad v_n = \frac{f(z_n)}{f(y_n)},$$
$$w_n = \frac{f(z_n)}{f(x_n)}, \qquad t_n = \frac{f(s_n)}{f(z_n)},$$
$$K_f(u_n) = \frac{1 + \beta u_n + (-9 + 5/2\beta) u_n^2}{1 + (\beta - 2) u_n + (-4 + \beta/2) u_n^2}, \tag{7}$$
$$H_f = \frac{1 + 2u_n + (2+\sigma) w_n}{1 - v_n + \sigma w_n},$$
$$W_f = \frac{1 + 2u_n + (2+\sigma) v_n w_n}{1 - v_n - 2w_n - t_n + 2(1+\sigma) v_n w_n} + G(u_n, w_n),$$

one of the choices for $G(u_n, w_n)$ along with $\beta = 2$ and $\sigma = -2$:

$$G(u_n, w_n) = -\frac{1}{2}\left[u_n w_n \left(6 + 12u_n + (24 - 11\beta) u_n^2 + u_n^3 \phi_1 + 4\sigma\right)\right] + \phi_2 w_n^2,$$
$$\phi_1 = \left(11\beta^2 - 66\beta + 136\right),$$
$$\phi_2 = \left(2u_n\left(\sigma^2 - 2\sigma - 9\right) - 4\sigma - 6\right). \tag{8}$$

2. A New Method and Convergence Analysis

The proposed sixteenth-order iterative method is described as follows (MA):

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$z_n = y_n - \frac{1 + 2t_1 - t_1^2}{1 - 6t_1^2} \frac{f(y_n)}{f'(x_n)},$$
$$w_n = z_n - \frac{1 - t_1 + t_3}{1 - 3t_1 + 2t_3 - t_2} \frac{f(z_n)}{f'(x_n)},$$

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fxn:= c1*(e+c2*e^2+c3*e^3+c4*e^4+c5*e^5+c6*e^6+c7*e^7+c8*e^8+c9*e^9+c10*e^10+
        c11*e^11+c13*e^12+c13*e^13+c14*e^14+c15*e^15+c16*e^16+c17*e^17);
dfxn:= diff (fxn, e);
ye:= simplify (taylor (e−fxn/dfxn, e = 0, 17));
fyn:= c1*(ye+c2*ye^2+c3*ye^3+c4*ye^4+c5*ye^5+c6*ye^6+c7*ye^7+c8*ye^8+c9*ye^9+
        c10*ye^10+c11*ye^11+c12*ye^12+c13*ye^13+c14*ye^14+c15*ye^15+c16*ye^16+c17*ye^17);
fyn:= simplify (taylor (fyn, e = 0, 17));
t1:= simplify (taylor (fyn/fxn, e = 0, 17));
fydfx:= simplify (taylor (fyn/dfxn, e = 0, 17));
ze:= factor (simplify (taylor (ye−(1+2*t1−t1^2)*fydfx/(1−6*t1^2), e = 0, 17)));
factor (simplify (taylor (ye−(1+2*t1−t1^2)*fydfx/(1−6*t1^2), e = 0, 7)));

 −c2*c3*e^4+(−2*c3^2+2*c3*c2^2+2*c2^4−2*c2*c4)*e^5+(−3*c2*c5−7*c3*c4+6*c2*c3^2+
  12*c3*c2^3+3*c4*c2^2−14*c2^5)*e^6+O(e^7)

fzn:= c1*(ze+c2*ze^2+c3*ze^3+c4*ze^4+c5*ze^5+c6*ze^6+c7*ze^7+c8*ze^8+c9*ze^9+
        c10*ze^10+c11*ze^11+c12*ze^12+c13*ze^13+c14*ze^14+c15*ze^15+c16*ze^16+c17*ze^17);
fzn:= simplify (taylor (fzn, e = 0, 17));
t2:= simplify (taylor (fzn/fyn, e = 0, 17));
t3:= simplify (taylor (fzn/fxn, e = 0, 17));
fzdfx:= simplify (taylor (fzn/dfxn, e = 0, 17));
we:= simplify (taylor (ze−(1−t1+t3)*fzdfx/(1−3*t1+2*t3−t2), e = 0, 17));
simplify (taylor (ze−(1−t1+t3)*fzdfx/(1−3*t1+2*t3−t2), e = 0, 10));

  −c4*c3*c2^2*e^8+(−2*c5*c3*c2^2−4*c2*c4*c3^2+4*c3*c4*c2^3−2*c4^2*c2^2+2*c4*c2^5
  −3*c3^3*c2^2+4*c3^2*c2^4+4*c3*c2^6)*e^9+O(e^10)

fwn:= c1*(we+c2*we^2+c3*we^3+c4*we^4+c5*we^5+c6*we^6+c7*we^7+c8*we^8+c9*we^9+c10*we^10+
        c12*we^12+c13*we^13+c14*we^14+c15*we^15+c16*we^16+c17*we^17);
fwn:= simplify (taylor (fwn, e = 0, 17));
fwdfx:= simplify (taylor (fwn/dfxn, e = 0, 17));
t4:= simplify (taylor (fwn/fxn, e = 0, 17));
t5:= simplify (taylor (fwn/fyn, e = 0, 17));
t6:= simplify (taylor (fwn/fzn, e = 0, 17));
q1:= simplify (taylor (1/(1−2*(t1+t1^2+t1^3+t1^4+t1^5+t1^6+t1^7)), e = 0, 17));
q2:= simplify (taylor (4*t3/(1−(31/4)*t3), e = 0, 17));
q3:= simplify (taylor (t2/(1−t2−20*t2^3), e = 0, 17));
q4:= simplify (taylor (8*t4/(1−t4)+2*t5/(1−t5)+t6/(1−t6), e = 0, 17));
q5:= simplify (taylor (15*t1*t3/(1−(131/15)*t3), e = 0, 17));
q6:= simplify (taylor (54*t1^2*t3/(1−t1^2*t3), e = 0, 17));
q7:= simplify (taylor (7*t2*t3+2*t1*t6+6*t6*t1^2+188*t3*t1^3+18*t6*t1^3+
                         9*t2^2*t3+648*t1^4*t3, e = 0, 17));
x[n+1]:= simplify (taylor (we−fwdfx*(q1+q2+q3+q4+q5+q6+q7), e = 0, 17));
for i to 16 do p:= factor (simplify (coeff (x[n+1], e, i))) end do;
                                    0
                                    0
                                    0
                                    0
                                    0
                                    0
                                    0
                                    0
                                    0
                                    0
                                    0
                                    0
                                    0
                                    0
                                    0
  −c4*c3*c2^2*(c5*c3*c2^2+2*c2*c4*c3^2−20*c3^4−51*c3^3*c2^2+522*c3^2*c2^4−2199*c3*c2^6
  +2*c2^8−30*c3*c4*c2^3+54*c4*c2^5)
```

Algorithm 1: The Maple code for finding the error equation.

Table 1: Set of six test nonlinear functions.

Functions	Roots
$f_1(x) = e^x \sin(x) + \log(1+x^2)$	$\alpha = 0$
$f_2(x) = (x-2)(x^{10}+x+1)e^{-x-1}$	$\alpha = 2$
$f_3(x) = \sin(x)^2 - x^2 + 1$	$\alpha = 1.40449\cdots$
$f_4(x) = e^{-x} - \cos(x)$	$\alpha = 0$
$f_5(x) = x^3 + \log(x)$	$\alpha = 0.70470949\cdots$

Table 2: Numerical comparison of absolute error $|x_n - \alpha|$, number of iterations = 3.

$(f_n(x), x_0)$	Iter/COC	MA	BT	GK1	GK2
f_1, 1.0	1	0.00268	0.00183	0.0111	0.00230
	2	$2.03e-37$	$1.71e-37$	$6.35e-24$	$5.61e-34$
	3	**2.47e − 583**	$3.53e-582$	$1.37e-363$	$1.03e-523$
	COC	16	16	16	16
f_2, 2.5	1	0.04086	0.0639	0.0296	0.00866
	2	$6.16e-9$	650.0	$5.35e-14$	$2.53e-21$
	3	$1.50e-121$	Divergent	$4.79e-201$	**1.89e − 317**
	COC	16.5	—	15.9	16.0
f_3, 2.5	1	0.0000326	0.0000303	0.000497	0.0000677
	2	$4.87e-73$	$1.70e-72$	$1.56e-51$	$1.14e-64$
	3	**3.11e − 1158**	$1.63e-1148$	$1.42e-811$	$4.52e-1021$
	COC	16	16	16	16
f_4, 1/6	1	$2.79e-7$	0.0000864	$1.28e-7$	0.000167
	2	$1.00e-109$	$1.18e-63$	$2.28e-107$	$9.28e-57$
	3	**2.80e − 1851**	$1.72e-1005$	$2.24e-1703$	$7.82e-893$
	COC	17	16	16	16
f_5, 3.0	1	0.0486	0.135	0.0949	0.0133
	2	$1.95e-22$	$1.81e-17$	$1.78e-19$	$1.11e-35$
	3	$8.46e-349$	$1.79e-271$	$6.86e-302$	**2.61e − 563**
	COC	16.0	16.0	15.9	16.0

$$x_{n+1} = w_n - \left(q_1 + q_2 + q_3 + q_4 + q_5 + q_6 + q_7\right) \frac{f(w_n)}{f'(x_n)}, \tag{9}$$

where

$$t_1 = \frac{f(y_n)}{f(x_n)}, \qquad t_2 = \frac{f(z_n)}{f(y_n)},$$

$$t_3 = \frac{f(z_n)}{f(x_n)}, \qquad t_4 = \frac{f(w_n)}{f(x_n)},$$

$$t_5 = \frac{f(w_n)}{f(y_n)}, \qquad t_6 = \frac{f(w_n)}{f(z_n)},$$

$$q_1 = \frac{1}{1 - 2\left(t_1 + t_1^2 + t_1^3 + t_1^4 + t_1^5 + t_1^6 + t_1^7\right)},$$

$$q_2 = \frac{4t_3}{1 - 31/4t_3}, \qquad q_3 = \frac{t_2}{1 - t_2 - 20t_2^3},$$

$$q_4 = \frac{8t_4}{1 - t_4} + \frac{2t_5}{1 - t_5} + \frac{t_6}{1 - t_6},$$

$$q_5 = \frac{15t_1t_3}{1 - 131/15t_3}, \qquad q_6 = \frac{54t_1^2t_3}{1 - t_1^2t_3},$$

$$q_7 = 7t_2t_3 + 2t_1t_6 + 6t_6t_1^2 + 188t_3t_1^3 + 18t_6t_1^3 + 9t_2^2t_3 + 648t_1^4t_3. \tag{10}$$

Theorem 1. *Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a sufficiently differentiable function, and $\alpha \in D$ is a simple root of $f(x) = 0$, for an open interval D. If x_0 is chosen sufficiently close to α, then the iterative scheme (9) converges to α and shows an order of convergence at least equal to sixteen.*

Proof. Let error at step n be denoted by $e_n = x_n - \alpha$ and $c_1 = f'(\alpha)$ and $c_k = (1/k!)(f^{(k)}(\alpha)/f'(\alpha))$, $k = 2, 3, \ldots$. We provided Maple based computer assisted proof in Algorithm 1 and got the following error equation:

$$e_{n+1} = -c_4c_3c_2^2\left(c_5c_3c_2^2 + 2c_4c_2c_3^2 - 20c_3^4 - 51c_3^3c_2^2 + 522c_3^2c_2^4\right.$$

$$- 2199c_3c_2^6 + 2c_2^8 - 30c_4c_3c_2^3$$
$$+54c_4c_2^5\Big) e_n^{16} + O\left(e_n^{17}\right). \tag{11}$$

□

3. Numerical Results

If the convergence order η is defined as

$$\lim_{x\to\infty} \frac{|e_{n+1}|}{|e_n|^{\eta}} = c \neq 0, \tag{12}$$

then the following expression approximates the computational order of convergence (COC) [16] as follows:

$$\rho \approx \frac{\ln |(x_{n+1} - \alpha) / (x_n - \alpha)|}{\ln |(x_n - \alpha) / (x_{n-1} - \alpha)|}, \tag{13}$$

where α is the root of nonlinear equation. A set of five nonlinear equations are used for numerical computations in Table 1. Three iterations are performed to calculate the absolute error ($|x_n - \alpha|$) and computational order of convergence. Table 2 shows absolute error and computational order of convergence, respectively.

4. Conclusion

An optimal sixteenth-order iterative scheme has been developed for solving nonlinear equations. A Maple program is provided to calculate error equation, which actually shows the optimal order of convergence in the sense of Kung-Traub conjecture. The computational order of convergence also verifies our claimed order of convergence. The proposed scheme uses four functions and one derivative evaluation per full cycle, which gives 1.741 as the efficiency index. We also have shown the validity of our proposed iterative scheme by comparing it with other existing optimal sixteenth-order iterative methods. The numerical results show that the performance of iterative scheme is competitive as compared to other methods.

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