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Research Article

Extragradient Method for Solutions of Variational Inequality Problems in Banach Spaces

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We introduce an iterative process which converges strongly to solutions of a certain variational inequity problem for η-inverse strongly accretive mappings in the set of common fixed points of finite family of strictly pseudocontractive mappings in Banach spaces. Our theorems improve and unify most of the results that have been proved for this important class of nonlinear operators.

1. Introduction

Let E be a real normed linear space with dual E^*. For $1 < q < \infty$, we denote by J_q the generalized duality mapping from E to 2^{E^*} defined by

$$J_q(x) := \left\{x^* \in E^* : \langle x, x^*\rangle = \|x\|^q,\ \|x^*\| = \|x\|^{q-1}\right\}, \quad (1)$$

where $\langle \cdot,\cdot\rangle$ denotes the duality pairing. In particular, $J = J_2$ is called the *normalized duality map*. It is well known (see e.g., [1]) that J_q is single valued if E is smooth and that

$$J_q(x) = \|x\|^{q-2}J(x), \quad x \neq 0. \quad (2)$$

In the sequel, we will denote the single-valued generalized map by j_q.

A mapping A with domain $D(A) \subseteq E$ and range $R(A)$ in E is called *α-strongly accretive* if there exist $\alpha \in (0,1)$ and $j_q(x-y) \in J_q(x-y)$ such that

$$\left\langle Ax - Ay, j_q(v-u)\right\rangle \geq \alpha\|x-y\|^q. \quad (3)$$

A is called *η-inverse strongly accretive* if there exist $\eta \in (0,1)$ and $j_q(x-y) \in J_q(x-y)$ such that

$$\begin{gathered}\left\langle Ax - Ay, j_q(x-y)\right\rangle \geq \eta\|Ax-Ay\|^q, \\ \text{for every } x, y \in D(A).\end{gathered} \quad (4)$$

Let C be a nonempty, closed, and convex subset of E and, let $A : C \to E$ be a nonlinear mapping. The variational inequality problem is to

$$\text{find } u \in C \text{ such that } \left\langle Au, j(v-u)\right\rangle \geq 0, \quad \forall v \in C, \quad (5)$$

for some $j(v-u) \in J(v-u)$. The set of solutions of variational inequality problem is denoted by $VI(C, A)$. If $E := H$, a real Hilbert space, the variational inequality problem reduces to

$$\text{find } u \in C \text{ such that } \langle Au, v-u\rangle \geq 0, \quad \forall v \in C, \quad (6)$$

which was introduced and studied by Stampacchia [2].

Variational inequality theory has emerged as an important tool in studying a wide class of related problems arising in mathematical, physical, regional, engineering, and nonlinear optimization sciences (see, for instance, [3–12]).

In 1976, Korpelevič [4] introduced the following well-known extragradient method:

$$\begin{aligned} y_n &= P_C(x_n - \gamma A x_n), \\ x_{n+1} &= P_C(x_n - \gamma A y_n), \quad n \geq 0, \end{aligned} \quad (7)$$

where P_C is the metric projection from $\mathbb{R}^n$ onto its subset C, for some $\gamma > 0$, and $A : C \to \mathbb{R}^n$ is an accretive operator. He proved that the sequence $\{x_n\}$ converges to a solution of the variational inequality (6).

Furthermore, Noor [6] proved that the iterative scheme, given by

$$y_n = P_C(x_n - \gamma A x_n),$$
$$x_{n+1} = P_C(y_n - \gamma A y_n), \quad n \geq 0, \tag{8}$$

where $A : C \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is an accretive operator, converges to a solution of the variational inequality (6).

We note that the above algorithms give strong convergence to a solution of the variational inequality (6). However, both algorithms fail, in general, to converge strongly in the setting of infinite-dimensional Hilbert spaces.

In 2006, Aoyama et al. [13] introduced and studied the following iterative algorithm in a uniformly convex and 2-uniformly smooth Banach spaces possessing weakly sequentially continuous duality mapping:

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) Q_C[x_n - \lambda_n A x_n], \quad n \geq 0, \tag{9}$$

where Q_C is a sunny nonexpansive retraction from E onto a closed and convex C, $A : C \to E$ is an η-inverse strongly accretive mapping and $\{\alpha_n\}$ and $\{\lambda_n\}$ subsets of real numbers, satisfy certain conditions. They proved that the sequence in (9) *converges weakly* to a point $z \in V(C, A)$.

Recently, Yao et al. [8] introduced and considered the following iterative method for η-strongly accretive mappings in a uniformly convex and 2-uniformly smooth Banach space possessing weakly sequentially continuous duality mapping:

$$y_n = Q_C(x_n - \lambda_n A x_n),$$
$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n Q_C(y_n - \lambda_n A y_n), \quad n \geq 0, \tag{10}$$

where Q_C is a sunny nonexpansive retraction from E onto C. They proved that the sequence $\{x_n\}$ defined by (10) converges strongly to $Q_{VI(C,A)}u$ provided that real sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\lambda_n\}$ satisfy certain conditions.

Let C be a nonempty subset of a real Banach space E. A mapping $T : C \to E$ is called λ-*strictly pseudocontractive* of Browder-Petryshyn type [14] if for all $x, y \in D(T)$ there exist $\lambda > 0$ and $j_q(x-y) \in J_q(x-y)$ such that

$$\langle Tx - Ty, j_q(x-y)\rangle \leq \|x-y\|^q - \lambda \|x - y - (Tx - Ty)\|^q. \tag{11}$$

T is called *Lipschitz* if there exists $L \geq 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\| \quad \forall x, y \in D(T). \tag{12}$$

If $L < 1$ in (12), then T is called *contraction*, while T is said to be *nonexpansive* if $L = 1$.

If $E = H$, a real Hilbert space, then (11) is equivalent to the inequality

$$\|Tx - Ty\|^2 \leq \|x-y\|^2 + k\|x - y - (Tx - Ty)\|^2, \quad k = (1 - 2\lambda), \tag{13}$$

and we can assume also that $k \geq 0$, so that $k \in [0, 1)$. A point $x \in C$ is a *fixed point of* T if $Tx = x$, and we denote by $F(T)$ the set of fixed points of T; that is, $F(T) = \{x \in C : Tx = x\}$.

In 2001, Yamada [7] introduced a hybrid steepest descent method which relates solutions of variational inequality problems with fixed point of mappings in Hilbert spaces. He proved that if T is nonexpansive self-map on C and A is an η-strongly accretive mapping from C into E satisfying certain conditions, then the sequence defined by

$$x_{n+1} = Tx_n - \mu\lambda_n A(Tx_n), \quad n \geq 0, \tag{14}$$

converges strongly to the unique solution of the variational inequality

$$\text{Find } x^* \in F(T) \text{ such that } \langle Ax^*, x - x^*\rangle \geq 0, \quad \forall x \in F(T). \tag{15}$$

The above results naturally bring us to the following question.

Question. Could we produce an iterative scheme which approximates a solution of variational inequality (5) for η-inverse strongly accretive mappings in Banach spaces?

In this paper, motivated by Yao et al. [8] and Yamada [7], it is our purpose to introduce an iterative scheme which converges strongly to a solution of the variational inequality (5) for η-inverse strongly accretive mapping in the set of common fixed points of finite family of strictly pseudocontractive mappings in a uniformly convex and q-uniformly smooth Banach space E possessing weakly sequentially continuous duality mapping. Our results complement or improve the results of Yao et al. [8], Aoyama et al. [13], and some authors.

2. Preliminaries

Let E be a real Banach space. The modulus of smoothness of E is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by $\rho_E(\tau) := \sup\{(1/2)(\|x+y\| + \|x-y\|) - 1 : \|x\| = 1, \|y\| = \tau\}$. If $\rho_E(\tau) > 0$ for all $\tau > 0$, then E is said to be *smooth*. If there exists a constant $c > 0$ and a real number $1 < q < \infty$, such that $\rho_E(\tau) \leq c\tau^q$, then E is said to be *q-uniformly smooth*.

If E is a real q-uniformly smooth Banach space, then by [15], the following geometric inequality holds:

$$\|x + y\|^q \leq \|x\|^q + q\langle y, j_q(x)\rangle + c_q\|y\|^q, \tag{16}$$

for all $x, y \in E$ and some real constant $c_q > 0$.

It is well known (see e.g., [16]) that

$$L_p(l_p) \text{ or } W_m^p \text{ is } \begin{cases} p\text{-uniformly smooth} & \text{if } 1 < p < 2, \\ 2\text{-uniformly smooth} & \text{if } p \geq 2. \end{cases} \tag{17}$$

The Banach space E is said to be *uniformly convex* if, given $\varepsilon > 0$, there exists $\delta > 0$, such that, for all $x, y \in E$ with $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x - y\| \geq \varepsilon$, $\|(1/2)(x+y)\| \leq 1 - \delta$. It is well known that L_p, ℓ_p, and Sobolev spaces W_m^p, $(1 < p < \infty)$ are uniformly convex.

Let $C \subseteq E$ be closed convex and Q a mapping of E onto C. Then, Q is said to be *sunny* if $Q(Q(x) + t(x - Q(x))) = Q(x)$

for all $x \in E$ and $t \geq 0$. A mapping Q of E into C is said to be a *retraction* if $Q^2 = Q$. If a mapping Q is a retraction, then $Q(z) = (z)$ for every $z \in R(Q)$, *range of* Q. A subset C of E is said to be a *sunny nonexpansive retract* of E if there exists a sunny nonexpansive retraction of E onto C, and it is said to be a *nonexpansive retract* of E if there exists a nonexpansive retraction of E onto C. *If $E = H$, the metric projection P_C is a sunny nonexpansive retraction from H to any closed convex subset of H.* Moreover, if C is a nonempty closed convex subset of a uniformly convex and uniformly smooth real Banach space E and T is a nonexpansive mapping of C into itself with $F(T) \neq \emptyset$, then the set $F(T)$ is a sunny nonexpansive retract of C.

In what follows, we will make use of the following lemmas.

Lemma 1 (see, e.g., [17]). *Let E be a smooth Banach space, and let K be a nonempty subset of E. Let $Q : E \to K$ be a retraction, and let J be the normalized duality map on E. Then, the following are equivalent:*

(i) *Q is sunny nonexpansive,*

(ii) *$\langle x - Q(x), J(y - Q(x))\rangle \leq 0$ for all $x \in E$ and $y \in K$.*

Lemma 2 (see [18]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1 - \alpha_n) a_n + \alpha_n\delta_n, \quad n \geq n_0, \tag{18}$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{\delta_n\} \subset \mathbb{R}$ satisfying the following conditions: $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\limsup_{n\to\infty}\delta_n \leq 0$. Then, $\lim_{n\to\infty}a_n = 0$.

Lemma 3 (see [13]). *Let C be a nonempty closed convex subset of a smooth Banach space E. Let Q_C be a sunny nonexpansive retraction from E onto C, and let A be an accretive operator of C into E. Then, for all $\lambda > 0$,*

$$VI(C, A) = F(Q_C(I - \lambda A)), \tag{19}$$

where $VI(C, A) = \{x^* \in C : \langle Ax^*, J(x - x^*)\rangle \geq 0, \ \forall x \in C\}$.

Lemma 4 (see [17]). *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space E, and let T be nonexpansive mapping of C into itself. If $\{x_n\}$ is a sequence of C such that $x_n \to x$ weakly and $x_n - Tx_n \to 0$ strongly, then x is a fixed point of T.*

Lemma 5 (see [19]). *Let E be a real Banach space. Then, for any given $x, y \in E$, the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle, \quad \forall j(x + y) \in J(x + y). \tag{20}$$

Lemma 6 (see [20]). *Let K be a nonempty closed convex subset of a strictly convex Banach space E. Let $T_i : K \to E$, $i = 1, 2, \ldots, r$, be a family of nonexpansive mappings such that $\bigcap_{i=1}^{r} F(T_i) \neq \emptyset$. Let $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_r$ be real numbers in $(0, 1)$ such that $\sum_{i=0}^{r} \alpha_i = 1$, and let $T := \alpha_0 I + \alpha_1 T_1 + \cdots + \alpha_r T_r$. Then, T is nonexpansive, and $F(T) = \bigcap_{i=1}^{r} F(T_i)$.*

Lemma 7 (see [21]). *Let C be a nonempty, closed and convex subset of a real uniformly convex and smooth Banach space E. Let $T_i : C \to E$, $i = 1, \ldots, N$, be λ_i-strictly pseudocontractive mappings such that $\bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $T := \theta_1 T_1 + \theta_2 T_2 + \cdots + \theta_N T_N$ with $\theta_1 + \theta_2 + \cdots + \theta_N = 1$. Then T is λ-strictly pseudocontractive with $\lambda := \min\{\lambda_i : i = 1, \ldots, N\}$ and $F(T) = \bigcap_{i=1}^{N} F(T_i)$.*

Lemma 8 (see [22]). *Let C be a nonempty closed and convex subset of a real q-uniformly smooth Banach space E for $1 < q < \infty$. Let $T : C \to E$ be a λ-strictly pseudocontractive mapping. Then, for $0 < \mu < \mu_0 = \min\{1, (q\lambda/c_q)^{1/(q-1)}\}$, where L is the Lipchitz constant of T and c_q is a constant in (16), the mapping $T_\mu x := (1 - \mu)x + \mu Tx$ is nonexpansive, and $F(T_\mu) = F(T)$.*

Lemma 9. *Let C be a nonempty closed and convex subset of a a real q-uniformly smooth Banach space E for $1 < q < \infty$. Let $A : C \to E$ be an η-inverse strongly accretive mapping. Then, for $0 < \gamma < (q\eta/c_q)^{1/(q-1)}$, the mapping $A_\mu x := (x - \gamma Ax)$ is nonexpansive.*

Proof. Now, using inequality (16), we get that

$$\begin{aligned}
&\|A_\gamma x - A_\gamma y\|^q \\
&\quad = \|(x - y) - \gamma(Ax - Ay)\|^q \\
&\quad \leq \|x - y\|^q - q\gamma\left\langle Ax - Ay, j_q(x - y)\right\rangle \\
&\qquad + \gamma^q c_q\|Ax - Ay\|^q \\
&\quad \leq \|x - y\|^q - q\gamma\eta\|Ax - Ay\|^2 + \gamma^q c_q\|Ax - Ay\|^2 \\
&\quad \leq \|x - y\|^q - \gamma\left(q\eta - \gamma^{q-1}c_q\right)\|Ax - Ay\|^q, \\
&\quad \leq \|x - y\|^q.
\end{aligned} \tag{21}$$

The proof is complete. □

Lemma 10 (see [10]). *Let E be a uniformly convex Banach space, and let $B_R(0)$ be a closed ball of E. Then, there exists a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that*

$$\begin{aligned}
&\|\alpha_0 x_0 + \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k\|^2 \\
&\qquad \leq \sum_{i=0}^{k}\alpha_i\|x_i\|^2 - \alpha_s\alpha_t g(\|x_s - x_t\|),
\end{aligned} \tag{22}$$

for $x_i \in B_R(0) := \{x \in E : \|x\| \leq R\}$, $i = 0, 1, 2, \ldots, k$ with $\sum_{i=0}^{k} \alpha_i = 1$.

Lemma 11 (see [5]). *Let $\{a_n\}$ be sequences of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$, for all $i \in \mathbb{N}$. Then, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$, and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1}, \qquad a_k \leq a_{m_k+1}. \tag{23}$$

In fact, $m_k = \max\{j \leq k : a_j < a_{j+1}\}$.

3. Main Results

Let C be a nonempty closed convex subset of a real uniformly convex and q-uniformly smooth Banach space E $(1 < q < \infty)$. Let $T_i : C \to C$, for $i = 1, \ldots, N$, be a λ_i-strictly pseudocontractive mappings, and let $A : C \to E$ be an η-inverse strongly accretive mapping. Then, in what follows, we will study the variational inequality

$$\text{Find } x^* \in \cap_{i=1}^{N} F(T_i) \text{ such that } \langle Ax^*, J(x - x^*)\rangle \geq 0, \quad \forall x \in \cap_{i=1}^{N} F(T_i), \tag{24}$$

and the following iteration process:

$$\begin{aligned} &x_0 \in C, \\ &z_n = c_n x_n + (1 - c_n) S x_n, \\ &x_{n+1} = Q_C\left[(1 - \alpha_n)\left(\beta_n x_n + (1 - \beta_n) Q_C[I - \gamma A] z_n\right)\right], \end{aligned} \tag{25}$$

where $S := [(1 - \mu)I + \mu T]$, for $T := \theta_1 T_1 + \theta_2 T_2 + \cdots + \theta_n T_N$, such that $\theta_1 + \theta_2 + \cdots + \theta_N = 1$, $0 < \mu < \mu_0 = \min\{1, (q\lambda/c_q)^{1/(q-1)}\}$, for $\lambda := \min\{\lambda_i : i = 1, 2, \ldots, N\}$, and $0 < \gamma < (q\eta/c_q)^{1/(q-1)}$, and c_q is the real number in (16). In addition, we assume $\{\alpha_n\} \subset (0, c) \subset (0, 1)$ and $\{\beta_n\}, \{c_n\} \subset [a, b] \subset (0, 1)$ as real sequences satisfying the following control conditions: (i) $\lim_{n\to\infty}\alpha_n = 0$, (ii) $\sum \alpha_n = \infty$, $\lim_{n\to\infty}(|\beta_n - \beta_{n-1}|/\alpha_{n-1}) = 0$, $\lim_{n\to\infty}(|\alpha_n - \alpha_{n-1}|/\alpha_{n-1}) = 0$, and $\lim_{n\to\infty}(|c_n - c_{n-1}|/\alpha_{n-1}) = 0$.

We now prove our main theorem.

Theorem 12. *Let C be a nonempty closed convex subset of a real uniformly convex and q-uniformly smooth Banach space E possessing weakly sequentially continuous duality mapping. Let $T_i : C \to C$, for $i = 1, \ldots, N$, be λ_i-strictly pseudocontractive mappings, and let $A : C \to E$ be η-inverse strongly accretive mapping. Let $\{x_n\}$ be a sequence defined by (25). Assume that $\mathcal{F} := F \cap VI(C, A) \neq \emptyset$, where $F = \cap_{i=1}^{N} F(T_i) = F(S)$. Then, $\{x_n\}$ converges strongly to $Q_{\mathcal{F}}(0)$, where $Q_{\mathcal{F}}$ is a sunny nonexpansive retraction of E onto $\mathcal{F}$, which is a solution of the variational inequality (24).*

Proof. By Lemmas 7 and 8 we have that S is nonexpansive. In addition, by Lemma 9 we get that $Q_C[I - \gamma A]$ is nonexpansive. Let $p \in \mathcal{F}$ and, let $y_n := \beta_n x_n + (1 - \beta_n) Q_C[I - \gamma A] z_n$. Then from (25), Lemmas 8 and 9 we have that

$$\begin{aligned} \|z_n - p\| &\leq c_n \|x_n - p\| + (1 - c_n)\|Sx_n - p\| \\ &\leq \beta_n \|x_n - p\| + (1 - \beta_n)\|x_n - p\| \\ &\leq \|x_n - p\|, \end{aligned} \tag{26}$$

$$\begin{aligned} \|y_n - p\| &\leq \beta_n \|x_n - p\| \\ &\quad + (1 - \beta_n)\left\|Q_C[I - \gamma A] z_n - Q_C[p - \gamma A p]\right\| \\ &\leq \beta_n \|x_n - p\| + (1 - \beta_n)\|z_n - p\| \\ &\leq \beta_n \|x_n - p\| + (1 - \beta_n)\|x_n - p\| \\ &\leq \|x_n - p\|. \end{aligned} \tag{27}$$

Thus, from (25) and (27), we get that

$$\begin{aligned} \|x_{n+1} - p\| &= \left\|Q_C\left[(1 - \alpha_n) y_n\right] - Q_C p\right\| \\ &\leq (1 - \alpha_n)\|y_n - p\| + \alpha_n \|p\| \\ &\leq (1 - \alpha_n)\|x_n - p\| + \alpha_n \|p\|. \end{aligned} \tag{28}$$

Therefore, by induction,

$$\|x_{n+1} - p\| \leq \max\{\|x_0 - p\|, \|p\|\}, \quad \forall n \geq 0, \tag{29}$$

which implies that $\{x_n\}$ and hence $\{y_n\}$, $\{z_n\}$, and $\{Ax_n\}$ are bounded. Furthermore, from (25), we obtain that

$$\begin{aligned} &\|z_{n+1} - z_n\| \\ &\quad = \|c_{n+1} x_{n+1} + (1 - c_{n+1}) S x_{n+1} \\ &\qquad - (c_n x_n + (1 - c_n) S x_n)\| \\ &\quad = \|(c_{n+1} x_{n+1} - c_n x_n) + (1 - c_{n+1}) S x_{n+1} - (1 - c_n) S x_n\| \\ &\quad \leq |c_{n+1} - c_n| \|x_{n+1}\| + c_n \|x_{n+1} - x_n\| \\ &\qquad + |c_{n+1} - c_n| \|Sx_{n+1}\| + (1 - c_n)\|x_{n+1} - x_n\| \\ &\quad \leq \|x_{n+1} - x_n\| + |c_{n+1} - c_n| \left[\|x_{n+1}\| + \|Sx_{n+1}\|\right], \end{aligned} \tag{30}$$

$$\begin{aligned} &\|y_{n+1} - y_n\| \\ &\quad = \|\beta_{n+1} x_{n+1} + (1 - \beta_{n+1}) Q_C[z_{n+1} - \gamma A z_{n+1}] \\ &\qquad - (\beta_n x_n + (1 - \beta_n) Q_C[z_n - \gamma A z_n])\| \\ &\quad = \|(\beta_{n+1} x_{n+1} - \beta_n x_n) + (1 - \beta_{n+1}) \\ &\qquad \times Q_C[z_{n+1} - \gamma A z_{n+1}] - (1 - \beta_n) Q_C[z_n - \gamma A z_n]\| \\ &\quad \leq |\beta_{n+1} - \beta_n| \|x_{n+1}\| + \beta_n \|x_{n+1} - x_n\| \\ &\qquad + |\beta_{n+1} - \beta_n| \|Q_C[z_{n+1} - \gamma A z_{n+1}]\| + (1 - \beta_n) \\ &\qquad \times \|z_{n+1} - z_n\| \\ &\quad \leq \beta_n \|x_{n+1} - x_n\| \\ &\qquad + |\beta_{n+1} - \beta_n| \left[\|x_{n+1}\| + \|Q_C[z_{n+1} - \gamma A z_{n+1}]\|\right] \\ &\qquad + (1 - \beta_n)\|z_{n+1} - z_n\| \\ &\quad \leq \beta_n \|x_{n+1} - x_n\| \\ &\qquad + |\beta_{n+1} - \beta_n| \left[\|x_{n+1}\| + \|Q_C[z_{n+1} - \gamma A z_{n+1}]\|\right] \\ &\qquad + (1 - \beta_n)\|x_{n+1} - x_n\| \\ &\qquad + |c_{n+1} - c_n| \left[\|x_{n+1}\| + \|Sx_{n+1}\|\right] \\ &\quad \leq \|x_{n+1} - x_n\| + |\beta_{n+1} - \beta_n| \\ &\qquad \times \left[\|x_{n+1}\| + \|Q_C[z_{n+1} - \gamma A z_{n+1}]\|\right] \\ &\qquad + |c_{n+1} - c_n| \left[\|x_{n+1}\| + \|Sx_{n+1}\|\right]. \end{aligned} \tag{31}$$

And, hence, from (25) and (31), we have that

$$\begin{aligned}
&\|x_{n+1}-x_n\| \\
&\quad = \|Q_C[(1-\alpha_n)y_n] - Q_C[(1-\alpha_{n-1})y_{n-1}]\| \\
&\quad \le \|(1-\alpha_n)y_n - (1-\alpha_{n-1})y_{n-1}\| \\
&\quad \le \|(1-\alpha_n)y_n - (1-\alpha_{n-1})y_n \\
&\qquad + (1-\alpha_{n-1})y_n - (1-\alpha_{n-1})y_{n-1}\| \\
&\quad \le |\alpha_n-\alpha_{n-1}|\,\|y_n\| + (1-\alpha_{n-1})\|y_{n-1}-y_n\| \\
&\quad \le (1-\alpha_{n-1})\|x_n-x_{n-1}\| + |\alpha_n-\alpha_{n-1}|M \\
&\qquad + |\beta_n-\beta_{n-1}|M + |c_n-c_{n-1}|M,
\end{aligned} \tag{32}$$

for some $M > 0$. Thus, using the properties of $\{\alpha_n\}$, $\{\beta_n\}$, $\{c_n\}$, (32), and Lemma 2, we obtain that $\|x_{n+1}-x_n\| \to 0$, as $n \to \infty$, which implies from (31) that $\|y_{n+1}-y_n\| \to 0$, as $n \to \infty$. Again from (25), we have that $\|x_{n+1}-y_n\| = \|Q_C[(1-\alpha_n)y_n] - Q_C y_n\| = \alpha_n\|y_n\| \to 0$, as $n \to \infty$. Consequently,

$$\|x_n - y_n\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{33}$$

Now, we prove that $\{x_n\}$ converges strongly to the point $x^* = Q_{\mathcal{F}}(0)$. Let $t_n = Q_C[I-\gamma A]z_n$, and let $d_n = (1-\alpha_n)y_n$. Then, since $\alpha_n \to 0$, we have that

$$\|d_n - y_n\| = \alpha_n\|y_n\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{34}$$

Furthermore, from (25), Lemma 5, and Lemma 10, we get that

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2 \\
&\quad = \|Q_C[(1-\alpha_n)y_n] - Q_C x^*\|^2 \\
&\quad \le \|\alpha_n(-x^*) + (1-\alpha_n)(y_n-x^*)\|^2 \\
&\quad \le (1-\alpha_n)\|y_n-x^*\|^2 - 2\alpha_n\langle x^*, j(d_n-x^*)\rangle \\
&\quad \le (1-\alpha_n)\big[\beta_n\|x_n-x^*\|^2 + (1-\beta_n)\|t_n-x^*\|^2 \\
&\qquad\qquad -\beta_n(1-\beta_n)g(\|t_n-x_n\|)\big] \\
&\qquad - 2\alpha_n\langle x^*, j(d_n-x^*)\rangle \\
&\quad \le (1-\alpha_n)\beta_n\|x_n-x^*\|^2 + (1-\alpha_n)(1-\beta_n)\|z_n-x^*\|^2 \\
&\qquad - \beta_n(1-\beta_n)(1-\alpha_n)g(\|t_n-x_n\|) \\
&\qquad - 2\alpha_n\langle x^*, j(d_n-x^*)\rangle \\
&\quad \le (1-\alpha_n)\beta_n\|x_n-x^*\|^2 + (1-\alpha_n)(1-\beta_n) \\
&\qquad \times \big[c_n\|x_n-p\|^2 + (1-c_n)\|Sx_n-p\|^2 \\
&\qquad\qquad - c_n(1-c_n)g(\|Sx_n-x_n\|)\big] \\
&\qquad - \beta_n(1-\beta_n)(1-\alpha_n)g(\|t_n-x_n\|) \\
&\qquad - 2\alpha_n\langle x^*, j(d_n-x^*)\rangle \\
&\quad \le (1-\alpha_n)\beta_n\|x_n-x^*\|^2 + (1-\alpha_n)(1-\beta_n) \\
&\qquad \times \big[c_n\|x_n-p\|^2 + (1-c_n)\|Sx_n-p\|^2 \\
&\qquad\qquad - c_n(1-c_n)g(\|Sx_n-x_n\|)\big] \\
&\qquad - \beta_n(1-\beta_n)(1-\alpha_n)g(\|t_n-x_n\|) \\
&\qquad - 2\alpha_n\langle x^*, j(d_n-x^*)\rangle,
\end{aligned} \tag{35}$$

which implies that

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2 \\
&\quad \le (1-\alpha_n)\|x_n-x^*\|^2 - c_n(1-c_n)(1-\alpha_n)(1-\beta_n) \\
&\qquad \times g(\|Sx_n-x_n\|) \\
&\qquad - \beta_n(1-\beta_n)(1-\alpha_n)g(\|t_n-x_n\|) \\
&\qquad - 2\alpha_n\langle x^*, j(d_n-x^*)\rangle
\end{aligned} \tag{36}$$

$$\le (1-\alpha_n)\|x_n-x^*\|^2 - 2\alpha_n\langle x^*, j(d_n-x^*)\rangle. \tag{37}$$

Now, following the method of proof of Lemma 3.2 of Maingé [5], we consider two cases.

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\|x_n-x^*\|\}$ is decreasing for all $n \ge n_0$. Then, we get that $\{\|x_n-x^*\|\}$ is convergent. Thus, from (36) and the fact that $\alpha_n \to 0$, as $n \to \infty$, we have that

$$g(\|Sx_n-x_n\|) \longrightarrow 0, \qquad g(\|t_n-x_n\|) \longrightarrow 0, \qquad \text{as } n \longrightarrow \infty, \tag{38}$$

which implies that

$$\begin{gathered}
\|Sx_n-x_n\| \longrightarrow 0, \\
\|t_n-x_n\| = \|Q_C[I-\gamma A]x_n - x_n\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.
\end{gathered} \tag{39}$$

In addition, since $\{d_n\}$ is bounded subset of a reflexive space E, we can choose a subsequence $\{d_{n_i}\}$ of $\{d_n\}$ such that $d_{n_i} \rightharpoonup z$ and $\limsup_{n\to\infty}\langle x^*, j(d_n-x^*)\rangle = \lim_{i\to\infty}\langle x^*, j(d_{n_i}-x^*)\rangle$. This implies from (34) and (33) that $x_{n_i} \rightharpoonup z$. Then, from (39) and Lemma 4, we have that $z \in F(S) = \cap_{i=1}^{N} F(T_i)$. Moreover, from (39) and Lemma 4, we have that $z \in F(Q_C[I-\gamma A])$, and by Lemma 3, we get

$z \in VI(C, A)$, and hence $z \in \mathcal{F}$. Therefore, using the fact that E has a weakly sequentially continuous duality mapping and Lemma 1, we immediately obtain that

$$\limsup_{n\to\infty} \langle x^*, j(d_n - x^*)\rangle = \lim_{i\to\infty} \left\langle x^*, j\left(d_{n_i} - x^*\right)\right\rangle = \langle x^*, j(z - x^*)\rangle \geq 0. \tag{40}$$

Then, it follows from (37), (40), and Lemma 2 that $\|x_n - x^*\| \to 0$, as $n \to \infty$. Consequently, $x_n \to x^* = Q_{\mathcal{F}}0$.

Case 2. Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\left\|x_{n_i} - x^*\right\| < \left\|x_{n_i+1} - x^*\right\|, \tag{41}$$

for all $i \in \mathbb{N}$. Then, by Lemma 11, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$, and

$$\begin{aligned} \left\|x_{m_k} - x^*\right\| &\leq \left\|x_{m_k+1} - x^*\right\|, \\ \left\|x_k - x^*\right\| &\leq \left\|x_{m_k+1} - x^*\right\|, \end{aligned} \tag{42}$$

for all $k \in \mathbb{N}$. Now, from (36) and the fact that $\alpha_n \to 0$, we get that $x_{m_k} - Sx_{m_k} \to 0$ and $\|Q_C[I - \gamma A]x_{m_k} - x_{m_k}\| \to 0$, as $k \to \infty$. Thus, like in Case 1, we obtain that

$$\limsup_{k\to\infty} \left\langle x^*, j\left(d_{m_k} - x^*\right)\right\rangle \geq 0. \tag{43}$$

Moreover, from (37), we have that

$$\left\|x_{m_k+1} - x^*\right\|^2 \leq \left(1 - \alpha_{m_k}\right)\left\|x_{m_k} - x^*\right\|^2 - 2\alpha_{m_k}\left\langle x^*, j\left(d_{m_k} - x^*\right)\right\rangle, \tag{44}$$

which implies from (42) and (44) that

$$\begin{aligned} \alpha_{m_k}\left\|x_{m_k} - x^*\right\|^2 &\leq \left\|x_{m_k} - x^*\right\|^2 - \left\|x_{m_k+1} - x^*\right\|^2 \\ &\quad - 2\alpha_{m_k}\left\langle x^*, j\left(d_{m_k} - x^*\right)\right\rangle \\ &\leq -2\alpha_{m_k}\left\langle x^*, j\left(d_{m_k} - x^*\right)\right\rangle. \end{aligned} \tag{45}$$

Now, since $\alpha_{m_k} > 0$, we obtain that

$$\left\|x_{m_k} - x^*\right\|^2 \leq -2\left\langle x^*, j\left(d_{m_k} - x^*\right)\right\rangle, \tag{46}$$

and using (43), we get that $\|x_{m_k} - x^*\| \to 0$. This together with (44) implies that $\|x_{m_k+1} - x^*\| \to 0$, as $k \to \infty$. But $\|x_k - x^*\| \leq \|x_{m_k+1} - x^*\|$, for all $k \in \mathbb{N}$; thus, we obtain that $x_k \to x^*$. Therefore, from both cases, we can conclude that $\{x_n\}$ converges strongly to $x^* = P_{\mathcal{F}}(0)$, which is a solution of the variational inequality (24), and the proof is complete. □

If in Theorem 12, we consider that $N = 1$, we get the following corollary.

Corollary 13. *Let C be a nonempty closed convex subset of a real uniformly convex and q-uniformly smooth Banach space E possessing weakly sequentially continuous duality mapping. Let $T : C \to C$ be a λ-strictly pseudocontractive mapping, and let $A : C \to E$ be an η-inverse strongly accretive mapping. Let $\{x_n\}$ be a sequence defined by (25), where $S := [(1-\mu)I + \mu T]$. Assume that $F := F(T) \cap VI(C, A) \neq \emptyset$. Then, $\{x_n\}$ converges strongly to $Q_F(0)$ which is a solution of the variational inequality*

$$\textit{Find } x^* \in F(T) \textit{ such that } \langle Ax^*, J(x - x^*)\rangle \geq 0, \quad \forall x \in F(T). \tag{47}$$

If in Theorem 12, we assume that T_i, for $i = 1, 2, \ldots, N$, are nonexpansive, we get the following corollary.

Corollary 14. *Let C be a nonempty closed convex subset of a real uniformly convex and q-uniformly smooth Banach space E possessing weakly sequentially continuous duality mapping. Let $T_i : C \to C$, for $i = 1, 2, \ldots, N$, be nonexpansive mappings, and let $A : C \to E$ be an η-inverse strongly accretive mapping. For $x_0 \in C$, let the sequence $\{x_n\}$ be generated iteratively by*

$$\begin{aligned} z_n &= c_n x_n + (1 - c_n) T x_n, \\ x_{n+1} &= Q_C\left[(1 - \alpha_n)\left(\beta_n x_n + (1 - \beta_n) Q_C[I - \gamma A] z_n\right)\right], \end{aligned} \tag{48}$$

where $T := \theta_0 I + \theta_1 T_1 + \cdots + \theta_N T_N$ for $\{\theta_i\}_{i=1}^N$, $\{\alpha_n\}$, $\{\beta_n\}$, γ are as in (24). Assume that $\mathcal{F} := F \cap VI(C, A) \neq \emptyset$, where $F := \cap_{i=1}^N F(T_i) = F(T)$. Then, $\{x_n\}$ converges strongly to $Q_{\mathcal{F}}(0)$, which is a solution of the variational inequality problem (24).

Proof. Lemma 6 and the method of proof of Theorem 12 provide the required assertion. □

If in Corollary 14, we consider that $N = 1$, we get the following corollary.

Corollary 15. *Let C be a nonempty closed convex subset of a real uniformly convex and q-uniformly smooth Banach space E possessing weakly sequentially continuous duality mapping. Let $T : C \to C$ be a nonexpansive mapping, and let $A : C \to E$ be an η-inverse strongly accretive mapping. For $x_0 \in C$, let the sequence $\{x_n\}$ be generated iteratively by*

$$\begin{aligned} z_n &= c_n x_n + (1 - c_n) T x_n, \\ x_{n+1} &= Q_C\left[(1 - \alpha_n)\left(\beta_n x_n + (1 - \beta_n) Q_C[I - \gamma A] z_n\right)\right]. \end{aligned} \tag{49}$$

Assume that $F := F(T) \cap VI(C, A) \neq \emptyset$. Then, $\{x_n\}$ converges strongly to $Q_F(0)$, which is a solution of the variational inequality problem

$$\textit{Find } x^* \in F(T) \textit{ such that } \langle Ax^*, J(x - x^*)\rangle \geq 0, \quad \forall x \in F(T). \tag{50}$$

If in Corollary 14, we assume that $T = T_1 = T_2 = \cdots = T_N = I$, we obtain the following corollary.

Corollary 16. *Let C be a nonempty closed convex subset of a real uniformly convex and q-uniformly smooth Banach space E possessing weakly sequentially continuous duality mapping. Let $A : C \to E$ be an η-inverse strongly accretive mapping. For $x_0 \in C$, let the sequence $\{x_n\}$ be generated iteratively by*

$$x_{n+1} = Q_C\left[\left(1-\alpha_n\right)\left(\beta_n x_n + \left(1-\beta_n\right) Q_C\left[I-\gamma A\right] x_n\right)\right]. \tag{51}$$

Assume that $VI(C, A) \neq \emptyset$. Then, $\{x_n\}$ converges strongly to $Q_{VI(C,A)}(0)$, where $Q_{VI(C,A)}$ is a sunny nonexpansive retraction of E onto $VI(C, A)$.

If in Theorem 12, we assume that A is an α-strongly accretive and L-Lipschitzian continuous mapping, we obtain the following corollary.

Corollary 17. *Let C be a nonempty closed convex subset of a real uniformly convex and q-uniformly smooth Banach space E possessing weakly sequentially continuous duality mapping. Let $T_i : C \to C$, for $i = 1, \ldots, N$, be λ_i-strictly pseudocontractive mappings, and Let $A : C \to E$ be an α-strongly accretive and L-Lipschitzian continuous mapping. Let $\{x_n\}$ be a sequence defined by (25) for $\eta = \alpha/L^2$. Assume that $\mathscr{F} := F \cap VI(C, A) \neq \emptyset$, where $F := \cap_{i=1}^{N} F(T_i) = F(S)$. Then, $\{x_n\}$ converges strongly to $Q_{\mathscr{F}}(0)$, which is a solution of the variational inequality problem*

$$\text{Find } x^* \in \cap_{i=1}^{N} F\left(T_i\right) \text{ such that } \left\langle Ax^*, J\left(x - x^*\right)\right\rangle \geq 0, \quad \forall x \in \cap_{i=1}^{N} F\left(T_i\right). \tag{52}$$

Proof. We note that if A is an α-strongly accretive and L-Lipschitzian continuous mapping of C into E, then we have that

$$\langle Ax - Ay, j\left(x-y\right)\rangle \geq \alpha\|x-y\|^2 \geq \frac{\alpha}{L^2}\|Ax - Ay\|^2, \quad \forall x, y \in C, \tag{53}$$

and hence, A is an η-inverse strongly accretive mapping with $\eta = \alpha/L^2$. Thus, the conclusion follows from Theorem 12. □

If $E = H$, a real Hilbert space, then E is a uniformly convex and q-uniformly smooth Banach space E for $1 < q < \infty$ possessing weakly sequentially continuous duality mapping. In this case, we have that $Q_C = P_C$, projection mapping from H onto C. Thus, we have the following corollary.

Corollary 18. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T_i : C \to C$, for $i = 1, \ldots, N$, be λ_i-strictly pseudocontractive mappings, and let $A : C \to E$ be an η-inverse strongly accretive mapping. For $x_0 \in C$, let the sequence $\{x_n\}$ be generated iteratively by*

$$\begin{aligned} z_n &= c_n x_n + \left(1-c_n\right) S x_n, \\ x_{n+1} &= P_C\left[\left(1-\alpha_n\right)\left(\beta_n x_n + \left(1-\beta_n\right) P_C\left[I-\gamma A\right] z_n\right)\right], \end{aligned} \tag{54}$$

where $S := [(1-\mu)I + \mu T]$, for $T := \theta_1 T_1 + \theta_2 T_2 + \cdots + \theta_n T_N$, such that $\theta_1 + \theta_2 + \cdots + \theta_N = 1$, $0 < \mu < \min\{1, 2\lambda\}$, for $\lambda := \min\{\lambda_i : i = 1, 2, \ldots, N\}$, and $0 < \gamma < 2\eta$. Assume that $\mathscr{F} := F \cap VI(C, A) \neq \emptyset$, where $F = \cap_{i=1}^{N} F(T_i) = F(S)$. Then, $\{x_n\}$ converges strongly to $P_{\mathscr{F}}(0)$, which is a solution of the variational inequality

$$\text{Find } x^* \in \cap_{i=1}^{N} F\left(T_i\right) \text{ such that } \left\langle Ax^*, x - x^*\right\rangle \geq 0, \quad \forall x \in \cap_{i=1}^{N} F\left(T_i\right). \tag{55}$$

Remark 19. Theorem 12 complements Theorem 3.2 of Yao et al. [8] in more general Banach spaces for η-inverse strongly accretive mappings. Moreover, Theorem 12 improves Theorem 3.1 of Aoyama et al. [13] and Theorem 3.7 of Saejung et al. [23] in the sense that our convergence is strong in the set of common fixed points of finite family of strictly pseudocontractive mappings.

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