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Research Article

On Cyclic Generalized Weakly C-Contractions on Partial Metric Spaces

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We give new results of a cyclic generalized weakly C-contraction in partial metric space. The results of this paper extend, generalize, and improve some fixed point theorems in the literature.

1. Introduction and Preliminaries

The notion of partial metric space [1], represented by the abbreviation PMS, departs from the usual metric spaces due to removing the assumption of self-distance. In other words, in PMS self-distance needs not to be zero. This interesting distance function is defined by Matthews [1], as a generalization metric to study in computer science, in particular, to get a more efficient programs in computer science. In the remarkable publication of Matthews [1], a characterization of the Banach Contraction Principle was given in the context of PMS. Due to its wide application potential [2–6], PMS and its topological properties are considered by many authors [7–25]. Very recently, Haghi et al. [26] proved that some obtained results in the context of PMS can be deduced from earlier results in the setting of usual metric space.

In the sequel, $\mathbb{R}^+$, $\mathbb{N}^*$ will represent the set of all real nonnegative numbers and the set of all positive natural numbers, respectively. Moreover, we use the abbreviations MS, CMS, PMS, and CMPS for metric space, complete metric space, partial metric space, and complete partial metric space, respectively. Let Λ be the collection of function $\varphi : [0,1) \to [0,1)$ which is nondecreasing, continuous together with the property $\varphi(t) > 0$ for $t \in (0,1)$ and $\varphi(0) = 0$. The following definition introduced by Chatterjea [27] to generalize the Banach contraction principle.

Definition 1. Suppose that (X, d) is an MS. A mapping $T : X \to X$ is said to be a C-contraction if there exists $\alpha \in (0, 1/2)$ such that the following inequality holds:

$$d(Tx, Ty) \leq \alpha\left(d(x, Ty) + d(y, Tx)\right) \quad \forall x, y \in X. \tag{1}$$

Moreover, Chatterjea [27] reported that every C-contraction $T : X \to X$ has a unique fixed point, where (X, d) is a complete metric space. Recently, Choudhury [28] introduced a generalization of C-contraction inspired by the notion of weak ϕ-contraction (see, e.g., [29, 30]).

Definition 2. Suppose that (X, d) is an MS. A self-mapping T on X is called a weakly C-contractive if

$$\begin{aligned} d(Tx, Ty) \leq{} & \frac{1}{2}\left[d(x, Ty) + d(y, Tx)\right] \\ & - \varphi\left(d(x, Ty), d(y, Tx)\right), \end{aligned} \tag{2}$$

for all $x, y \in X$, where the mapping $\varphi : [0, +\infty)^2 \to [0, +\infty)$ is continuous and has the following property:

$$\varphi(x, y) = 0 \quad \text{iff } x = y = 0. \tag{3}$$

The notion of weakly C-contractive can be also called a weak C-contraction. In [28], the author proves that on the setting of CMS, every weak C-contraction possesses a unique fixed point.

On other hand, in 2003, Kirk et al. [31] introduce cyclic contraction and give a characterization of the celebrated fixed-point theorem of Banach (known also as the Banach contraction mapping principle) in the set-up cyclic contraction. The authors [31] introduced the notion of cyclic representation in the following way.

Definition 3 (see [31]). Suppose that (X, d) is an MS and T is a self-mapping on X. Let m be a natural number and let X_i, $i = 1, \ldots, m$ be nonempty sets. Then, $X = \bigcup_{i=1}^{m} X_i$ is called a cyclic representation of X with respect to T if

$$X_1 \subset X_2, \ldots, T(X_{m-1}) \subset X_m, \quad T(X_m) \subset X_1, \tag{4}$$

where $X_{m+1} = X_1$.

Kirk et al. [31] prove that a self-mapping T, on a cyclic representation of X, possesses a fixed point if

$$d(Tx, Ty) \leq \varphi(d(x, y)), \quad \forall x \in X_i, \; y \in X_{i+1}, \quad i \in \{1, 2, \ldots, m\}, \tag{5}$$

where (X, d) is a CMS and $\varphi : [0, 1) \rightarrow [0, 1)$ is a function, upper semicontinuous from the right and $0 \leq \varphi(t) < t$ for $t > 0$.

Recently, Păcurar and Rus [32] generalize the result of Kirk et al. [31] via the notion of cyclic ϕ-contraction. Following the paper of Păcurar and Rus [32], the notion of cyclic weak-ϕ-contraction was introduced by Karapınar [33]. Let Λ be the collection of function $\varphi : [0, 1) \rightarrow [0, 1)$ which is nondecreasing, continuous together with the property $\varphi(t) > 0$ for $t \in (0, 1)$ and $\varphi(0) = 0$.

Definition 4 (see [33]). Suppose that (X, d) is an MS and T is a self-mapping on X. Let m be a natural number and let X_i, $i = 1, \ldots, m$, be nonempty closed sets. Assume that $X = \bigcup_{i=1}^{m} X_i$ is a cyclic representation of X with respect to T. A mapping $T : X \rightarrow X$ is said to be a cyclic weaker φ-contraction if there exists $\varphi \in \Lambda$ such that

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)), \tag{6}$$

for any $x \in X_i$, $y \in X_{i+1}$, $i = 1, 2, \ldots, m$, where $X_{m+1} = X_1$.

The author [33] shows that a self-mapping T, on a cyclic representation of X, possesses a fixed point if T is a cyclic weaker φ-contraction on a CMS (X, d).

In the last decade, the existence and uniqueness of a fixed point of various cyclic contractions in the context of PMS have been investigated and improved by several authors, see, for example, [7, 11].

In this paper, we derive some fixed-point result on certain cyclic contractions in the setup of complete PMS. Presented results of the paper extend, improve, and generalize some recent results on the topic in the literature. Among them, we list a few of them as follows: [7, 11, 13, 17, 28, 34].

For the sake of completeness, we call up some basic definitions and essential results in PMS. For more details, see, for example, [1, 7, 8, 17, 22].

Definition 5. Let X be a nonempty set. A function $p : X \times X \rightarrow [0, \infty)$ is called partial metric if the following conditions hold:

(p_1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$,

(p_2) $p(x, x) \leq p(x, y)$,

(p_3) $p(x, y) = p(y, x)$,

(p_4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$,

for all $x, y, z \in X$. A pair (X, p) is called partial metric space.

It is evident that if $p(x, y) = 0$, then $x = y$ due to assumptions (p_1) and (p_2). However, if $x = y$, then $p(x, y)$ need not be 0. It is also known that a PMS generates a topology which is T_0. We say that a sequence $\{x_n\}$ is convergent to a point $x \in X$ in (X, p) if $\lim_{n \to \infty} p(x, x_n) = p(x, x)$, denoted as $x_n \rightarrow x$ $(n \rightarrow \infty)$ or $\lim_{n \to \infty} x_n = x$, with respect to the corresponding topology. We underline the simple fact that a limit of a sequence in a PMS need not be unique. Notice also that the function $p(\cdot, \cdot)$ need not be continuous; that is, $x_n \rightarrow x$ and $y_n \rightarrow y$ need not yield $p(x_n, y_n) \rightarrow p(x, y)$.

There is strong correlation between partial metric and metric. For example, a mapping $d_p : X \times X \rightarrow \mathbb{R}^+$ given by

$$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y) \tag{7}$$

forms a metric on X, where p is a partial metric. It is called the corresponding metric of partial metric.

Example 6. Let $X = [0, \infty)$. The pair (X, p) is an elementary example of a PMS, where $p(x, y) = \max\{x, y\}$ for all $x, y \in \mathbb{R}^+$. Notice that the corresponding metric is

$$d_p(x, y) = 2\max\{x, y\} - x - y = |x - y|. \tag{8}$$

Example 7. The mapping $p(x, y) = \rho(x, y) + c$ forms a partial metric on X. Note also that the corresponding metric is $d_p(x, y) = 2d(x, y)$, where (X, ρ) is a metric space and $c \geq 0$ is arbitrary.

For the further nontrivial examples of PMS, they can be found in [1–6].

Definition 8. Let (X, p) be a PMS. Then

(1) a sequence $\{x_n\}$ is called a Cauchy if the limit of $p(x_n, x_m)$ as $n, m \rightarrow \infty$ exists (and is finite). If every Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$ such that $p(x, x) = \lim_{n,m \to \infty} p(x_n, x_m)$, then the space (X, p) is called complete,

(2) let $\{x_n\}$ be a sequence in (X, p). If $\lim_{n,m \to \infty} p(x_n, x_m) = 0$, then the sequence $\{x_n\}$ is called 0-Cauchy. Analogously, if every 0-Cauchy sequence in X converges to a point $x \in X$ such that $p(x, x) = 0$, then the space (X, p) is called 0-complete and denoted by 0-CPMS [22].

This lemma can be found in some recent publication on the topic, see, for example, [2–6].

Lemma 9. *Let (X, p) be a PMS. Then*

(a) *a sequence $\{x_n\}$ is Cauchy in (X, p) if and only if it is a Cauchy sequence in the metric space (X, d_p),*

(b) *a PMS (X, p) is complete if and only if the metric space (X, d_p) is complete. Furthermore, $\lim_{n\to+\infty} d_p(x_n, x) = 0$ if and only if*

$$p(x,x) = \lim_{n\to+\infty} p(x_n, x) = \lim_{n,m\to+\infty} p(x_n, x_m). \tag{9}$$

Lemma 10. *Let (X, p) be a PMS.*

(a) *If $p(x_n, z) \to p(z,z) = 0$ as $n \to \infty$, then $p(x_n, y) \to p(z, y)$ as $n \to \infty$ for each $y \in X$ [8, 9, 18].*

(b) *If (X, p) is complete, then it is 0-complete [22].*

The converse assertion of (b) does not hold; for the counter examples, see [22]. Note that every closed subset of a 0-CPMS is 0-complete.

Let Φ be the class of functions $\varphi : [0,\infty)^3 \to [0,\infty)$ which is lower semicontinuous and satisfying $\varphi(x,y,z) = 0 \Leftrightarrow x = y = z = 0$.

In what follows we introduce the notion of a cyclic generalized weakly C-contraction in PMS.

Definition 11. Assume that (X, p) is a PMS and m is a natural number. Suppose that $X_1, X_2, \ldots, X_m$ are closed nonempty subsets of (X, d_p) and $Y = \bigcup_{i=1}^m X_i$ is a cyclic representation of Y with respect to T; a mapping $T : Y \to Y$ is said to be a cyclic generalized weakly C-contraction if

$$\begin{aligned} p(Tx, Ty) \le{}& \frac{1}{4}\left[p(x, Tx) + p(y, Ty) + p(x, Ty) + p(y, Tx)\right] \\ &- \varphi\Big(p(x, Tx), p(y, Ty) \\ &\quad \frac{1}{2}\left[p(x, Ty) + p(y, Tx)\right]\Big), \end{aligned} \tag{10}$$

for any $x \in X_i$, $y \in X_{i+1}$, $i = 1, 2, \ldots, m$, where $X_{m+1} = X_1$ and $\varphi \in \Phi$.

In this paper, we establish a fixed point theorem for cyclic generalized weakly C-contractions in the frame of CMPS.

2. Main Results

We present the fundamental result of this paper as follows.

Theorem 12. *Assume that (X, p) is a 0-CPMS and $T : Y \to Y$ is a cyclic generalized weakly C-contraction. Then, the mapping T has a unique fixed point $z \in \bigcap_{i=1}^n X_i$, and $p(z,z) = 0$.*

Proof. Take $x_0 \in Y$; that is, there is some i_0 with $x_0 \in X_{i_0}$. Since $T(X_{i_0}) \subseteq X_{i_0+1}$ implies that $Tx_0 \in X_{i_0+1}$, we find $x_1 \in X_{i_0+1}$ such that $Tx_0 = x_1$. By using the same argument, we construct the sequence $x_{n+1} = Tx_n$, where $x_n \in X_{i_n}$. Consequently, for $n \ge 0$, there exists $i_n \in \{1, 2, \ldots, m\}$ such that $x_n \in X_{i_n}$ and $x_{n+1} \in X_{i_n+1}$. We suppose that $x_n \ne x_{n+1}$ for all n. Indeed, if $x_{n_0} = x_{n_0+1}$ for some n_0, then we conclude that $Tx_{n_0} = x_{n_0}$; that is, x_{n_0} is the desired fixed point of T. Consequently, the proof is completed.

Due to (10), we derive that

$$p(x_n, x_{n+1}) = p(Tx_{n-1}, Tx_n) \tag{11}$$

$$\begin{aligned} \le{}& \frac{1}{4}\left[p(x_{n-1}, Tx_{n-1}) + p(x_n, Tx_n) + p(x_{n-1}, Tx_n) + p(x_n, Tx_{n-1})\right] \\ &- \varphi\Big(p(x_{n-1}, Tx_{n-1}), p(x_n, Tx_n), \\ &\quad \frac{1}{2}\left[p(x_{n-1}, Tx_n) + p(x_n, Tx_{n-1})\right]\Big) \end{aligned} \tag{12}$$

$$\begin{aligned} ={}& \frac{1}{4}\left[p(x_{n-1}, x_n) + p(x_n, x_{n+1}) + p(x_{n-1}, x_{n+1}) + p(x_n, x_n)\right] \\ &- \varphi\Big(p(x_{n-1}, x_n), p(x_n, x_{n+1}), \frac{1}{2}\left[p(x_{n-1}, x_{n+1}) + p(x_n, x_n)\right]\Big) \end{aligned} \tag{13}$$

$$\le \frac{1}{4}\left[p(x_{n-1}, x_n) + p(x_n, x_{n+1}) + p(x_{n-1}, x_{n+1}) + p(x_n, x_n)\right] \tag{14}$$

$$\le \frac{1}{2}\left[p(x_{n-1}, x_n) + p(x_n, x_{n+1})\right] \text{ by } (p_4), \tag{15}$$

for all $n \ge 1$. As a result, we find that

$$p(x_n, x_{n+1}) \le p(x_{n-1}, x_n), \tag{16}$$

for all $n \ge 1$. We set $t_n = p(x_n, x_{n-1})$. On the occasion of the facts above, $\{t_n\}$ is a nonincreasing sequence of nonnegative real numbers. Consequently, there exists $L \ge 0$ such that

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = L. \tag{17}$$

We will prove that $L = 0$. Suppose, to the contrary, that $L > 0$. From (14) and (15) we derive that

$$\begin{aligned} p(x_n, x_{n+1}) &\le \frac{1}{4}\left[p(x_{n-1}, x_n) + p(x_n, x_{n+1}) + p(x_{n-1}, x_{n+1}) + p(x_n, x_n)\right] \\ &\le \frac{1}{2}\left[p(x_{n-1}, x_n) + p(x_n, x_{n+1})\right], \end{aligned} \tag{18}$$

for any $n \in \mathbb{N}^*$. Letting $n \to \infty$ in (18), we have

$$\begin{aligned} L &= \lim_{n\to\infty} p(x_{n+1}, x_n) \\ &\le \lim_{n\to\infty} \frac{1}{4}\left[2L + p(x_{n-1}, x_{n+1}) + p(x_n, x_n)\right] \le L. \end{aligned} \tag{19}$$

This yields that

$$\lim_{n\to\infty} p(x_n, x_n) + p(x_{n-1}, x_{n+1}) = 2L. \tag{20}$$

On the other hand, by (13) we have

$$\begin{aligned} t_{n+1} &\leq \frac{1}{4}\left[t_n + t_{n+1} + p(x_{n-1}, x_{n+1}) + p(x_n, x_n)\right] \\ &\quad - \varphi\left(t_n, t_{n+1}, \frac{1}{2}\left[p(x_{n-1}, x_{n+1}) + p(x_n, x_n)\right]\right) \\ &\leq \frac{1}{2}\left[t_n + t_{n+1}\right]. \end{aligned} \tag{21}$$

Letting $n \to \infty$ in inequality (21), we get that

$$L \leq L - \varphi(L, L, L) \leq L. \tag{22}$$

Since $\varphi(x, y, z) = 0 \Leftrightarrow x = y = z = 0$, we get $L = 0$.

Due to (p_2), we have $0 \leq p(x_n, x_n) \leq p(x_n, x_{n+1})$. Hence, $\lim_{n\to\infty} p(x_n, x_n) = 0$. Then, by (20) we conclude that $\lim_{n\to\infty} p(x_{n-1}, x_{n+1}) = 0$.

Hence, we have

$$\begin{aligned} \lim_{n\to\infty} p(x_n, x_{n+1}) &= \lim_{n\to\infty} p(x_n, x_n) \\ &= \lim_{n\to\infty} p(x_{n-1}, x_{n+1}) = 0. \end{aligned} \tag{23}$$

We assert that the sequence $\{x_n\}$ is Cauchy. To reach this goal, the standard techniques in the literature will be used (see, e.g., [17]). For the sake of completeness, we explicitly prove that $\{x_n\}$ is Cauchy. First assert that

(K) for each $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that if $r, q \geq n$ with $r - q \equiv 1(m)$, then $p(x_r, x_q) < \varepsilon$.

Suppose, to the contrary, that there is $\varepsilon > 0$ such that for all $n \in \mathbb{N}$ if $r_n > q_n \geq n$ with $r_n - q_n \equiv 1(m)$, then

$$p(x_{q_n}, x_{r_n}) \geq \varepsilon. \tag{24}$$

We examine the case $n > 2m$. So, taking $q_n \geq n$ into account, we can choose r_n with $r_n > q_n$ in a way that it is the smallest integer satisfying $r_n - q_n \equiv 1(m)$ and $p(x_{q_n}, x_{r_n}) \geq \varepsilon$. Hence, $p(x_{q_n}, x_{r_n - m}) \leq \varepsilon$, by using the triangular inequality

$$\begin{aligned} \varepsilon &\leq p(x_{q_n}, x_{r_n}) \\ &\leq p(x_{q_n}, x_{r_n - m}) + \sum_{i=1}^{m} p(x_{r_n - i}, x_{r_{n-i+1}}) - \sum_{i=1}^{m} p(x_{r_n - i}, x_{r_n - i}) \\ &< \varepsilon + \sum_{i=1}^{m} p(x_{r_n - i}, x_{r_{n-i+1}}). \end{aligned} \tag{25}$$

Letting $n \to \infty$ in (25) and keeping in mind $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$, we obtain

$$\lim_{n\to\infty} p(x_{q_n}, x_{r_n}) = \varepsilon. \tag{26}$$

Again, by (p_4)

$$\begin{aligned} \varepsilon &\leq p(x_{q_n}, x_{r_n}) \\ &\leq p(x_{q_n}, x_{q_n+1}) + p(x_{q_n+1}, x_{r_n+1}) + p(x_{r_n+1}, x_{r_n}) \\ &\quad - p(x_{q_n+1}, x_{q_n+1}) - p(x_{r_n+1}, x_{r_n+1}) \\ &\leq p(x_{q_n}, x_{q_n+1}) + p(x_{q_n+1}, x_{q_n}) + p(x_{q_n}, x_{r_n}) \\ &\quad + p(x_{r_n}, x_{r_n+1}) + p(x_{r_n+1}, x_{r_n}) \\ &\quad - p(x_{q_n}, x_{q_n}) - p(x_{r_n}, x_{r_n}) \\ &\leq 2p(x_{q_n}, x_{q_n+1}) + p(x_{q_n}, x_{r_n}) + 2p(x_{r_n}, x_{r_n+1}). \end{aligned} \tag{27}$$

Taking (23) and (26) into account, we get

$$\lim_{n\to\infty} p(x_{q_n+1}, x_{r_n+1}) = \varepsilon, \tag{28}$$

as $n \to \infty$ in (26).

By (p_4) we have the following inequalities:

$$\begin{aligned} p(x_{q_n}, x_{r_n+1}) &\leq p(x_{q_n}, x_{r_n}) + p(x_{r_n}, x_{r_n+1}) \\ &\quad - p(x_{r_n}, x_{r_n}) \\ &\leq p(x_{q_n}, x_{r_n}) + p(x_{r_n}, x_{r_n+1}), \\ p(x_{q_n}, x_{r_n}) &\leq p(x_{q_n}, x_{r_n+1}) + p(x_{r_n}, x_{r_n+1}) \\ &\quad - p(x_{r_n+1}, x_{r_n+1}) \\ &\leq p(x_{q_n}, x_{r_n+1}) + p(x_{r_n}, x_{r_n+1}). \end{aligned} \tag{29}$$

Letting $n \to \infty$ in (29) we derived that

$$\lim_{n\to\infty} p(x_{q_n}, x_{r_n+1}) = \varepsilon. \tag{30}$$

Again by (p_4) we have

$$\begin{aligned} p(x_{r_n}, x_{q_n+1}) &\leq p(x_{r_n}, x_{r_n+1}) + p(x_{r_n+1}, x_{q_n+1}) \\ &\quad - p(x_{r_n+1}, x_{r_n+1}) \\ &\leq p(x_{r_n}, x_{r_n+1}) + p(x_{r_n+1}, x_{q_n+1}), \\ p(x_{r_n+1}, x_{q_n+1}) &\leq p(x_{r_n+1}, x_{r_n}) + p(x_{r_n}, x_{q_n+1}) - p(x_{r_n}, x_{r_n}) \\ &\leq p(x_{r_n+1}, x_{r_n}) + p(x_{r_n}, x_{q_n+1}). \end{aligned} \tag{31}$$

Letting $n \to \infty$ in (31) we derived that

$$p(x_{r_n}, x_{q_n+1}) = \varepsilon. \tag{32}$$

Since x_{q_n} and x_{r_n} lie in distinct adjacently labeled sets X_i and X_{i+1} for certain $1 \le i \le m$, keeping in mind that T is a cyclic generalized weakly C-contraction, we have

$$\begin{aligned}
p\left(x_{q_n+1}, x_{r_n+1}\right) &= p\left(Tx_{q_n}, Tx_{r_n}\right) \\
&\le \frac{1}{4}\left[p\left(x_{q_n}, Tx_{q_n}\right) + p\left(x_{r_n}, Tx_{r_n}\right)\right. \\
&\quad \left. + p\left(x_{q_n}, Tx_{r_n}\right) + p\left(x_{r_n}, Tx_{q_n}\right)\right] \\
&\quad - \varphi\Big(p\left(x_{q_n}, Tx_{q_n}\right), p\left(x_{r_n}, Tx_{r_n}\right), \\
&\quad \frac{1}{2}\left[p\left(x_{q_n}, Tx_{r_n}\right) + p\left(x_{r_n}, Tx_{q_n}\right)\right]\Big) \\
&\le \frac{1}{4}\left[p\left(x_{q_n}, x_{q_n+1}\right) + p\left(x_{r_n}, x_{r_n+1}\right)\right. \\
&\quad \left. + p\left(x_{q_n}, x_{r_n+1}\right) + p\left(x_{r_n}, x_{q_n+1}\right)\right] \\
&\quad - \varphi\Big(p\left(x_{q_n}, x_{q_n+1}\right), p\left(x_{r_n}, x_{r_n+1}\right), \\
&\quad \frac{1}{2}\left[p\left(x_{q_n}, x_{r_n+1}\right) + p\left(x_{r_n}, x_{q_n+1}\right)\right]\Big).
\end{aligned} \tag{33}$$

Taking into account (23), (26), (28), (30), (32), and the lower semicontinuity of φ, letting $n \to \infty$ in the inequality above, we find that

$$\varepsilon \le \frac{1}{4}[0 + 0 + \varepsilon + \varepsilon] - \varphi(0, 0, \varepsilon) \le \frac{1}{2}\varepsilon, \tag{34}$$

which is a contradiction. Hence, (K) holds.

We are ready to show that the sequence $\{x_n\}$ is Cauchy. Fix $\varepsilon > 0$. Due to the assumptions, one can find $n_0 \in \mathbb{N}$ such that if $r, q \ge n_0$ with $r - q \equiv 1(m)$,

$$p\left(x_r, x_q\right) \le \frac{\varepsilon}{2}. \tag{35}$$

Since $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$, we also find $n_1 \in \mathbb{N}$ such that

$$p\left(x_n, x_{n+1}\right) \le \frac{\varepsilon}{2m}, \tag{36}$$

for any $n \ge n_1$. Assume that $r, s \ge \max\{n_0, n_1\}$ and $s > r$. Consequently, there is a $k \in \{1, 2, \ldots, m\}$ with $s - r \equiv k(m)$. Therefore, $s - r + \alpha \equiv 1(m)$ for $\alpha = m - k + 1$. Thus, we obtain for $j \in \{1, \ldots, m\}$, $s + j - r \equiv 1(m)$

$$\begin{aligned}
p\left(x_r, x_s\right) &\le p\left(x_r, x_{s+j}\right) + p\left(x_{s+j}, x_{s+j-1}\right) + \cdots + p\left(x_{s+1}, x_s\right) \\
&\quad - \left[p\left(x_{s+j}, x_{s+j}\right) + \cdots + p\left(x_{s+1}, x_{s+1}\right)\right] \\
&\le p\left(x_r, x_{s+j}\right) + p\left(x_{s+j}, x_{s+j-1}\right) + \cdots \\
&\quad + p\left(x_{s+1}, x_s\right).
\end{aligned} \tag{37}$$

By (35) and (36) together with the last inequality, we find that

$$p\left(x_r, x_s\right) \le \frac{\varepsilon}{2} + j \times \frac{\varepsilon}{2m} \le \frac{\varepsilon}{2} + m \times \frac{\varepsilon}{2m} = \varepsilon, \tag{38}$$

which yields that the sequence $\{x_n\}$ is Cauchy. Regarding that ε is arbitrary, we conclude that $\{x_n\}$ is a 0-Cauchy sequence.

Taking into account that Y is closed in (X, p), we observe that (Y, p) is also 0-complete. Thus, there exists $x \in Y = \bigcup_{i=1}^{m} X_i$ such that $\lim_{n\to\infty} x_n = x$ in (Y, p); equivalently

$$p(x, x) = \lim_{n\to\infty} p\left(x, x_n\right) = \lim_{n,m\to\infty} p\left(x_n, x_m\right) = 0. \tag{39}$$

Now, we assert that x is a fixed point of T. First, we observed that the sequence (x_n) has infinite terms in each X_i for $i \in \{1, 2, \ldots, m\}$, since $\lim_{n\to\infty} x_n = x$ and as $Y = \bigcup_{i=1}^{m} X_i$ is a cyclic representation of Y with respect to T. Assume that $x \in X_i$ and $Tx \in X_{i+1}$. We consider a subsequence x_{n_k} of (x_n) with $x_{n_k} \in X_{i-1}$. Notice that such subsequence exists due to the above-mentioned comment. By applying the contractive condition, we find

$$\begin{aligned}
p\left(x_{n_{k+1}}, Tx\right) &= p\left(Tx_{n_k}, Tx\right) \\
&\le \frac{1}{4}\left[p\left(x_{n_k}, Tx_{n_k}\right) + p(x, Tx)\right. \\
&\quad \left. + p\left(x_{n_k}, Tx\right) + p\left(x, Tx_{n_k}\right)\right] \\
&\quad - \varphi\Big(p\left(x_{n_k}, Tx_{n_k}\right), p(x, Tx), \\
&\quad \frac{1}{2}\left[p\left(x_{n_k}, Tx\right) + p\left(x, Tx_{n_k}\right)\right]\Big) \\
&= \frac{1}{4}\left[p\left(x_{n_k}, x_{n_k+1}\right) + p(x, Tx)\right. \\
&\quad \left. + p\left(x_{n_k}, Tx\right) + p\left(x, x_{n_k+1}\right)\right] \\
&\quad - \varphi\Big(p\left(x_{n_k}, x_{n_k+1}\right), p(x, Tx), \\
&\quad \frac{1}{2}\left[p\left(x_{n_k}, Tx\right) + p\left(x, x_{n_k+1}\right)\right]\Big).
\end{aligned} \tag{40}$$

Letting $n \to \infty$ and by using $x_{n_k} \to x$, together with the lower semicontinuity of φ, we get

$$\begin{aligned}
p(x, Tx) &\le \frac{1}{2} p(x, Tx) - \varphi\left(0, p(x, Tx), \frac{1}{2} p(x, Tx)\right) \\
&\le \frac{1}{2} p(x, Tx).
\end{aligned} \tag{41}$$

So $p(x, Tx) = 0$ which yields that $Tx = x$. We will prove the uniqueness of x to complete the proof. Suppose, on the contrary, that $y, z \in X$ are distinct fixed points of T. We observe that $y, z \in \bigcap_{i=1}^{m} X_i$, since T is cyclic mapping and

$y, z \in X$ are fixed points of T. Due to mentioned contractive condition, we derive that

$$\begin{aligned} p(y,z) &= p(Ty,Tx) \\ &\leq \frac{1}{4}[p(y,Ty)+p(z,Tz)+p(y,Tz)+p(z,Ty)] \\ &\quad - \varphi\Big(p(y,Ty), p(z,Tz), \\ &\qquad \frac{1}{2}[p(y,Tz)+p(z,Ty)]\Big), \end{aligned} \tag{42}$$

that is,

$$\begin{aligned} p(y,z) &\leq \frac{1}{2}p(y,z) - \varphi\Big(0,0,\frac{1}{2}[p(y,z)+p(z,y)]\Big) \\ &\leq \frac{1}{2}p(y,z). \end{aligned} \tag{43}$$

This gives us $p(y,z)=0$; that is, $y=z$. □

Corollary 13. *Suppose that (X,p) is a 0-CPMS, $m \in \mathbb{N}$, $X_1, X_2, \ldots, X_m$ are nonempty closed subsets of X. Let $T : Y \to Y$ be and let $Y = \bigcup_{i=1}^m X_i$. $Y = \bigcup_{i=1}^m X_i$ be a cyclic representation of X with respect to T.*

If there exists $\beta \in [0,1/4)$ such that

$$\begin{aligned} p(Tx,Ty) \leq \beta[&p(x,Tx)+p(y,Ty) \\ &+p(x,Ty)+p(y,Tx)], \end{aligned} \tag{44}$$

for any $x \in X_i$, $y \in X_{i+1}$, $i = 1,2,\ldots,m$, where $X_{m+1} = X_1$, then, T has a fixed point $z \in \bigcap_{i=1}^n X_i$ and $p(z,z)=0$.

Proof. Let $\beta \in [0,1/4)$. Hence, it suffices to take the function $\varphi : [0,+\infty)^3 \to [0,+\infty)$ defined by $\varphi(a,b,c) = (1/4-\beta)(a+b+2c)$. It is evident that φ satisfies the following conditions:

(1) $\varphi(a,b,c)=0$ if and only if $a=b=c=0$, and

(2) φ is lower semi-continuous.

The results follow when we apply Theorem 12. □

Theorem 14. *Suppose that (X,d) is a 0-CPMS. If the mapping $T : X \to X$ satisfies*

$$\begin{aligned} p(Tx,Ty) &\leq \frac{1}{4}[p(x,Tx)+p(y,Ty) \\ &\qquad +p(x,Ty)+p(y,Tx)] \\ &\quad - \varphi\Big(p(x,Tx), p(x,Ty), \\ &\qquad \frac{1}{2}[p(x,Ty)+p(y,Tx)]\Big), \end{aligned} \tag{45}$$

for any $x, y \in X$, where $\phi \in \Phi$, then, it has a unique fixed point $z \in X$ with $p(z,z)=0$.

Proof. It is sufficient to take $X_i = X$ for $i = 1,\ldots,m$ in Theorem 12. □

Remark 15. Let us remark that if in Definition 11 we consider the following condition

$$\begin{aligned} p(Tx,Ty) \leq \max\Big\{&p(x,x), p(y,y), \\ &\frac{1}{4}[p(x,Tx)+p(y,Ty)+p(x,Ty) \\ &\quad +p(y,Tx)]\Big\} \\ &- \varphi\Big(p(x,Tx), p(x,Ty), \\ &\quad \frac{1}{2}[p(x,Ty)+p(y,Tx)]\Big), \end{aligned} \tag{46}$$

instead of (10), then by following the lines in the proof of Theorem 12, we obtain the same conclusions in our results.

Example 16. Let $X = [0,1]$ and $p(x,y) = \max\{x,y\}$. It is clear that (X,p) is a 0-complete partial metric space. Fix $m \in \mathbb{N}$ and define $Y = \cup_{i=1}^m X_i$, where $X_i = [0,(1/2^i)]$ for $i = 1,2,\ldots,m$. Let $T : X \to X$ and let $\varphi : [0,\infty)^3 \to [0,\infty)^3$ be defined as

$$\varphi(p,q,r) = \begin{cases} 0 & \text{if } p=q=r=0, \\ \dfrac{(p+q+r)}{36} & \text{otherwise}, \end{cases} \tag{47}$$

respectively. Then, all the conditions of Theorem 12 are satisfied. Hence, T has a unique fixed point, namely, 0.

Remark 17. Notice that we get the same results if we replace 0-CPMS with CPMS.

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