Research Article

A Note on Certain Modular Equations about Infinite Products of Ramanujan

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Ramanujan proposed additive formulae of theta functions that are related to modular equations about infinite products. Employing these formulaes, we derived some identities on infinite products. In the same spirit, we also could present elementary and simple proofs of certain Ramanujan's modular equations on infinite products.

1. Introduction

Q-theory is undoubtedly one of the most famous and useful mathematical theorems, such as Andrews-Askey type integral [1]

$$\int_c^d \frac{(qt/c, qt/d, ft, rst; q)_\infty}{(at, bt, et, st; q)_\infty} \, {}_3\phi_2 \left[\begin{matrix} r, bt, \dfrac{c}{g} & q, sg \\ rst, bc; & \end{matrix} \right] d_q t$$

$$= \frac{d(1-q)(q, c/d, qd/c, abcd, df, rsd; q)_\infty}{(ac, bc, ad, bd, sd, de; q)_\infty}$$

$$\times \sum_{k=0}^\infty \frac{(f/e, ad, sd, bd; q)_k (ce)^k}{(q, df, rsd, abcd; q)_k} \tag{1}$$

$$\times {}_3\phi_2 \left[\begin{matrix} r, bdq^k, \dfrac{acdq^k}{g} & q, sg \\ rsdq^k, abcdq^k; & \end{matrix} \right],$$

Askey-Roy type integral [2]

$$\int_{-\pi}^\pi \left(\left(P_n\left(e^{i\theta}, f\right) P_m\left(e^{i\theta}, g\right) \right. \right.$$

$$\left. \times \left(\frac{\rho e^{i\theta}}{d}, qde^{-i\theta}, \rho ce^{-i\theta}, \frac{qe^{i\theta}}{(c\rho)}; q \right)_\infty \right)$$

$$\times \left(\left(ae^{i\theta}, be^{i\theta}, ce^{-i\theta}, de^{-i\theta}; q\right)_\infty \right)^{-1} \right) d\theta$$

$$= \frac{2\pi(af; q)_n (bg; q)_m (abcd, \rho c/d, dq/(\rho c), \rho, q/\rho; q)_\infty}{a^n b^m (q, bc, bd, ac, ad; q)_\infty}$$

$$\times \sum_{k=0}^n \frac{(q^{-n}, ac, ad; q)_k q^k}{(q, af, abcd; q)_k} \, {}_3\phi_2 \left[\begin{matrix} q^{-m}, bc, bd \\ bg, abcdq^k; \end{matrix} \; q, q \right], \tag{2}$$

Moment integrals [3]

$$\int_{-\infty}^\infty \frac{P_n(w, c) P_m(w, d)}{(aw, bw; q)_\infty} \, d\alpha^{(s,t)}(w)$$

$$= \frac{(ac; q)_n (bd; q)_m (abst; q)_\infty}{a^n b^m (as, at, bs, bt; q)_\infty} \tag{3}$$

$$\times \sum_{k=0}^n \frac{(q^{-n}, as, at; q)_k q^k}{(q, ac, abst; q)_k} \, {}_3\phi_2 \left[\begin{matrix} q^{-m}, bs, bt \\ bd, abstq^k; \end{matrix} \; q, q \right],$$

(where $P_n(a, b) = (a - b) \cdots (a - bq^{n-1})$), q-Fractional Calculus Equations [4] and q-Calculus [5]. For more information, please refer to [1–5].

The theta functions are very useful tool in researching q-series, especially in dealing with the form of the equation similar to above formulas, whose left-hand side is summation and right-hand side is integral. The additive identities of

theta are one of the important of Ramanujan's contributions. Using it, we gave elementary and simple proofs of certain Ramanujan's modular equations on infinite products. For more information, please refer to [1–7].

In his notebook [8, pages 34–38], Ramanujan defines the following theta functions:

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2} \tag{4}$$

$$= (-a; ab)_\infty (-b; ab)_\infty (ab; ab)_\infty, \quad |ab| < 1,$$

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q; -q)_\infty}{(q; -q)_\infty}, \tag{5}$$

$$\psi(q) := f(q, q^3) = \sum_{n=1}^{\infty} q^{n(n-1)/2} = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty}, \tag{6}$$

where

$$(a; q)_\infty := \prod_{n=0}^{\infty} (1 - aq^n), \quad |q| < 1, \tag{7}$$

sometimes written as

$$(a, b, c, \ldots; q)_\infty = (a; q)_\infty (b; q)_\infty (c; q)_\infty \cdots. \tag{8}$$

The infinite products are from the Jacobi triple product identity [8, page 35].

In the course of deduction, we used the following simple fact [9, 10]:

$$\left(a^k; q^k\right)_\infty = \left(a, a\omega_k, \ldots, a\omega_k^{k-1}; q\right)_\infty,$$
$$(a, q)_\infty = \left(a, aq, aq^2, \ldots, aq^{k-1}; q^k\right)_\infty. \tag{9}$$

By definition of Ramanujan theta functions one can easily verify the following identities [8, page 45]:

$$f(a,b) + f(-a,-b) = 2f\left(a^3 b, ab^3\right),$$
$$f(a,b) - f(-a,-b) = 2af\left(\frac{b}{a}, \frac{a}{b} a^4 b^4\right). \tag{10}$$

From (4), if $ab = cd = p$, we have

$$f(a,b) f(c,d) = \sum_{m,n=-\infty}^{\infty} p^{(m^2+n^2)/2 - (m+n)/2} a^m c^n. \tag{11}$$

Thus setting $m - n = 2j$ and $m + n = 2k$, we find that

$$f(a,b) f(c,d) + f(-a,-b) f(-c,-d)$$

$$= \sum_{\substack{n,m=-\infty \\ m,n,\text{even}}}^{+\infty} p^{(m^2+n^2)/2 - (m+n)/2} a^m c^n$$

$$= 2 \sum_{j,k=-\infty}^{\infty} p^{j^2+k^2-k} a^{j+k} c^{k-j} \tag{12}$$

$$= 2 \sum_{j,k=-\infty}^{\infty} p^{k(k-1)} (ac)^k p^{j(j-1)} (bc)^{-j}$$

$$= 2 f(ac, bd) f(ad, bc).$$

Thus when $ab = cd$, we have

$$f(a,b) f(c,d) + f(-a,-b) f(-c,-d)$$
$$= 2f(ac, bd) f(ad, bc). \tag{13}$$

Similarly we have that

$$f(a,b) f(c,d) - f(-a,-b) f(-c,-d)$$
$$= 2af\left(\frac{b}{c}, \frac{abcd}{b/c}\right) f\left(\frac{b}{d}, \frac{abcd}{b/d}\right). \tag{14}$$

The special case of these identities can be written as the following form by using Jacobian theta function [6, 7]:

$$2\theta_1 (x + y \mid 2\tau) \theta_1 (x - y \mid 2\tau)$$
$$= \theta_3 (y \mid \tau) \theta_4 (x \mid \tau) - \theta_3 (x \mid \tau) \theta_4 (y \mid \tau). \tag{15}$$

The authors of [6, 7] give simple proofs and very important use of it.

In the above two identities, putting $c = a$ and $d = b$, we easily obtain

$$f^2(a,b) + f^2(-a,-b) = 2f\left(a^2, b^2\right) \varphi(ab), \tag{16}$$

$$f^2(a,b) - f^2(-a,-b) = 4af\left(\frac{b}{a}, \frac{a}{b} a^2 b^2\right) \psi\left(a^2 b^2\right). \tag{17}$$

2. Main Results

The sums and products of infinite are used in many domains of mathematics, such as Partition Functions [11–14], Fractal Geometry [9], Fractional Calculus [10], Fractal Time Series [4], and so on. Then the equations of it are concentrated by several mathematicians and engineers [15–18]. At the same time, it can be used in dynamic equations, differential equations [19], and partial differential equations [20].

This paper has two main purposes. The first is to derive the identities as follows: for $|q| < 1$,

$$\frac{(q; q)_\infty^2}{(q^3; q^3)_\infty^2} + 3 \frac{(-q; q)_\infty^2}{(-q^3; q^3)_\infty^2}$$

$$= 4\omega \frac{(q; q)_\infty^2}{(q^3; q^3)_\infty^2} \frac{f(q\omega, q^3\omega^2) f(q\omega^2, q^3\omega)}{f^2(\omega, q\omega^2)}, \tag{18}$$

$$\frac{(q; q)_\infty^2}{(q^3; q^3)_\infty^2} - 3 \frac{(-q; q)_\infty^2}{(-q^3; q^3)_\infty^2} = 2 \frac{(q; q)_\infty^2}{(q^3; q^3)_\infty^2} \frac{f(\omega^2, q\omega)}{f^2(\omega, q\omega^2)} \varphi(q), \tag{19}$$

$$\frac{(q;q)_\infty^2}{(q^5;q^5)_\infty^2} - 5\frac{(-q;q)_\infty^2}{(-q^5;q^5)_\infty^2}$$

$$= 4\zeta\frac{(q;q)_\infty^2}{(q^5;q^5)_\infty^2}$$

$$\times \frac{f\left(q\zeta^2,q\zeta^3\right)f\left(\zeta,q^2\zeta^4\right)f\left(\zeta^3,q^2\zeta^2\right)f\left(q\zeta,q\zeta^4\right)}{f^2\left(\zeta,q\zeta^4\right)f^2\left(\zeta^2,q\zeta^3\right)},$$

$$(20)$$

in which $\omega = \exp(2i\pi/3)$ and $\zeta = \exp(2i\pi/5)$. In the same way, we are able to give the simple and elementary proofs of the following identities of Ramanujan [8, 11, 12]:

$$\frac{(-q;q)_\infty^2}{(-q^3;q^3)_\infty^2} + \frac{(q;q)_\infty^2}{(q^3;q^3)_\infty^2} = 2\frac{\left(-q^3;q^6\right)_\infty^2}{(q^3;q^6)_\infty^2}\left(-q^2,-q^4;q^6\right)_\infty,$$

$$(21)$$

$$\frac{(-q;q)_\infty^2}{(-q^3;q^3)_\infty^2} - \frac{(q;q)_\infty^2}{(q^3;q^3)_\infty^2} = 4q\frac{\left(-q^6;q^6\right)_\infty^2}{(q^3;q^6)_\infty^2}\left(-q,-q^5;q^6\right)_\infty,$$

$$(22)$$

$$\frac{(-q;q)_\infty^2}{(-q^5;q^5)_\infty^2} - \frac{(q;q)_\infty^2}{(q^5;q^5)_\infty^2}$$

$$= 4q\frac{\left(-q,-q^2,-q^3,-q^4,-q^6,-q^7,-q^8,-q^9;q^{10}\right)_\infty}{(q^5;q^{10})_\infty^4}.$$

$$(23)$$

3. Modular Equations of Infinite Productions

In this section, we first give the two sets refinement about the identities (18) and (20).

Theorem 1. *For $|q| < 1$,*

$$\frac{(q;q)_\infty}{(q^3;q^3)_\infty} - i\sqrt{3}\frac{(-q;q)_\infty}{(-q^3;q^3)_\infty} = 2\omega\frac{(q;q)_\infty}{(q^3;q^3)_\infty}\frac{f\left(q\omega,q^3\omega^2\right)}{f\left(\omega,q\omega^2\right)},$$

$$\frac{(q;q)_\infty}{(q^3;q^3)_\infty} + i\sqrt{3}\frac{(-q;q)_\infty}{(-q^3;q^3)_\infty} = 2\frac{(q;q)_\infty}{(q^3;q^3)_\infty}\frac{f\left(q\omega^2,q^3\omega\right)}{f\left(\omega,q\omega^2\right)}.$$

$$(24)$$

Proof. Note that $1 + \omega + \omega^2 = 0$ and $\omega - \omega^2 = i\sqrt{3}$. By (9), we get that

$$i\sqrt{3}\frac{(-q;q)_\infty}{(-q^3;q^3)_\infty} \div \frac{(q;q)_\infty}{(q^3;q^3)_\infty}$$

$$= \frac{\left(\omega - \omega^2\right)(q\omega;q)_\infty\left(q\omega^2;q\right)_\infty}{(-q\omega;q)_\infty(-q\omega^2;q)_\infty}$$

$$= \frac{\omega\left(1 - \omega\right)\left(1 + \omega\right)(q\omega;q)_\infty\left(q\omega^2;q\right)_\infty}{(1 + \omega)\left(-q\omega;q\right)_\infty(-q\omega^2;q)_\infty}$$

$$= -\frac{(\omega;q)_\infty\left(q\omega^2;q\right)_\infty}{(-\omega;q)_\infty(-q\omega^2;q)_\infty}$$

$$= -\frac{f\left(-\omega,-q\omega^2\right)}{f\left(\omega,q\omega^2\right)}.$$

$$(25)$$

From (10), we have

$$f\left(\omega,q\omega\right) - f\left(-\omega,-q\omega\right) = 2f\left(q\omega^2,q^3\omega\right), \qquad (26)$$

$$f\left(\omega,q\omega\right) + f\left(-\omega,-q\omega\right) = 2\omega f\left(q\omega,q^3\omega^2\right). \qquad (27)$$

Dividing by $f(\omega,q\omega)$, respectively, and then applying (25), we derive

$$i\sqrt{3}\frac{(-q;q)_\infty}{(-q^3;q^3)_\infty} \div \frac{(q;q)_\infty}{(q^3;q^3)_\infty} - 1$$

$$= -1 - \frac{f\left(-\omega,-q\omega^2\right)}{f\left(\omega,q\omega^2\right)} = -2\omega\frac{f\left(q\omega,q^3\omega^2\right)}{f\left(\omega,q\omega^2\right)},$$

$$(28)$$

$$i\sqrt{3}\frac{(-q;q)_\infty}{(-q^3;q^3)_\infty} \div \frac{(q;q)_\infty}{(q^3;q^3)_\infty} + 1$$

$$= 1 - \frac{f\left(-\omega,-q\omega^2\right)}{f\left(\omega,q\omega^2\right)} = 2\frac{f\left(q\omega^2,q^3\omega\right)}{f\left(\omega,q\omega^2\right)}.$$

$$(29)$$

Multiplying by $(q;q)_\infty/(q^3;q^3)_\infty$, respectively, we complete the proofs of (24). □

Proof of (19). Let $\alpha = \exp(i\pi/6)$; then it is easy to know that $\omega = i\alpha$ and $\alpha + 1/\alpha = -\sqrt{3}$.

One has

$$\sqrt{3}\frac{(-q;q)_\infty}{(-q^3;q^3)_\infty} \div \frac{(q;q)_\infty}{(q^3;q^3)_\infty}$$

$$= -\left(\alpha + \frac{1}{\alpha}\right)\frac{(q\omega;q)_\infty\left(q\omega^2;q\right)_\infty}{(-q\omega;q)_\infty(-q\omega^2;q)_\infty}$$

$$= -\frac{1}{\alpha}\left(1 + \alpha^2\right)\frac{(q\omega;q)_\infty\left(q\omega^2;q\right)_\infty}{(-q\omega;q)_\infty(-q\omega^2;q)_\infty}$$

$$= -\frac{1}{\alpha}\left(1 - i\alpha\right)\left(1 + i\alpha\right)\frac{(q\omega;q)_\infty\left(q\omega^2;q\right)_\infty}{(-q\omega;q)_\infty(-q\omega^2;q)_\infty}$$

$$= -\frac{1}{\alpha}\left(1 - \omega\right)\left(1 + \omega\right)\frac{(q\omega;q)_\infty\left(q\omega^2;q\right)_\infty}{(-q\omega;q)_\infty(-q\omega^2;q)_\infty}$$

$$= -\frac{1}{\alpha}(1+\omega)^2 \frac{(\omega;q)_\infty (q\omega^2;q)_\infty}{(-\omega;q)_\infty (-q\omega^2;q)_\infty}$$

$$= -\frac{\omega}{\alpha}\frac{f\left(-\omega,-q\omega^2\right)}{f\left(\omega,q\omega^2\right)} = -i\frac{f\left(-\omega,-q\omega^2\right)}{f\left(\omega,q\omega^2\right)}. \tag{30}$$

Then we obtain that

$$1 + \sqrt{3}\frac{(-q;q)_\infty}{(-q^3;q^3)_\infty} \div \frac{(q;q)_\infty}{(q^3;q^3)_\infty} = 1 - i\frac{f\left(-\omega,-q\omega^2\right)}{f\left(\omega,q\omega^2\right)},$$

$$1 - \sqrt{3}\frac{(-q;q)_\infty}{(-q^3;q^3)_\infty} \div \frac{(q;q)_\infty}{(q^3;q^3)_\infty} = 1 + i\frac{\left(-\omega,-q\omega^2\right)}{f\left(\omega,q\omega^2\right)}. \tag{31}$$

In (16), let $a = \omega$ and $b = q\omega^2$ then we have that

$$f^2\left(\omega,q\omega^2\right) + f^2\left(-\omega,-q\omega^2\right) = 2f\left(\omega^2,q^2\omega\right)\varphi(q). \tag{32}$$

Dividing by $f^2(\omega,q\omega)$, respectively, we arrive at

$$1 + \frac{f^2\left(-\omega,-q\omega^2\right)}{f^2\left(\omega,q\omega^2\right)} = \frac{2f\left(\omega^2,q^2\omega\right)\varphi(q)}{f^2\left(\omega,q\omega^2\right)}. \tag{33}$$

Multiplying (31), combining with (33), and then multiplyed by $(q;q)_\infty^2/(q^3;q^3)_\infty^2$, we are able to obtain (19). □

Theorem 2. *For $|q| < 1$,*

$$\frac{(q;q)_\infty}{(q^5;q^5)_\infty} - \sqrt{5}\frac{(-q;q)_\infty}{(-q^5;q^5)_\infty}$$

$$= 2\zeta\frac{(q;q)_\infty}{(q^5;q^5)_\infty}\frac{f\left(q\zeta^2,q\zeta^3\right)f\left(\zeta,q^2\zeta^4\right)}{f\left(\zeta,q\zeta^4\right)f\left(\zeta^2,q\zeta^3\right)}, \tag{34}$$

$$\frac{(q;q)_\infty}{(q^5;q^5)_\infty} + \sqrt{5}\frac{(-q;q)_\infty}{(-q^5;q^5)_\infty}$$

$$= 2\frac{(q;q)_\infty}{(q^5;q^5)_\infty}\frac{f\left(\zeta^3,q^2\zeta^2\right)f\left(q\zeta,q\zeta^4\right)}{f\left(\zeta,q\zeta^4\right)f\left(\zeta^2,q\zeta^3\right)}. \tag{35}$$

Proof. First we recall that $\zeta^5 = 1$, $\zeta + \zeta^4 - \zeta^2 - \zeta^3 = \sqrt{5}$ and $1 + \zeta + \zeta^2 + \zeta^3 + \zeta^4 = 0$. Using (9), we have

$$\sqrt{5}\frac{(-q;q)_\infty}{(-q^5;q^5)_\infty} \div \frac{(q;q)_\infty}{(q^5;q^5)_\infty}$$

$$= \left(\zeta + \zeta^4 - \zeta^2 - \zeta^3\right)$$

$$\times \frac{(q\zeta;q)_\infty (q\zeta^2;q)_\infty (q\zeta^3;q)_\infty (q\zeta^4;q)_\infty}{(-q\zeta;q)_\infty (-q\zeta^2;q)_\infty (-q\zeta^3;q)_\infty (-q\zeta^4;q)_\infty}$$

$$= \left(\zeta(1-\zeta)\left(1-\zeta^2\right)(1+\zeta)\left(1+\zeta^2\right)\right.$$

$$\times (q\zeta;q)_\infty \left(q\zeta^2;q\right)_\infty \left(q\zeta^3;q\right)_\infty \left(q\zeta^4;q\right)_\infty \right)$$

$$\times \left((1+\zeta)\left(1+\zeta^2\right)(-q\zeta;q)_\infty \right.$$

$$\left. \times \left(-q\zeta^2;q\right)_\infty \left(-q\zeta^3;q\right)_\infty \left(-q\zeta^4;q\right)_\infty \right)^{-1}$$

$$= -\frac{f\left(-\zeta,-q\zeta^4\right)f\left(-\zeta^2,-q\zeta^3\right)}{f\left(\zeta,q\zeta^4\right)f\left(\zeta^2,q\zeta^3\right)}. \tag{36}$$

Then we know easily that

$$1 + \sqrt{5}\frac{(-q;q)_\infty}{(-q^5;q^5)_\infty} \div \frac{(q;q)_\infty}{(q^5;q^5)_\infty}$$

$$= 1 - \frac{f\left(-\zeta,-q\zeta^4\right)f\left(-\zeta^2,-q\zeta^3\right)}{f\left(\zeta,q\zeta^4\right)f\left(\zeta^2,q\zeta^3\right)},$$

$$1 - \sqrt{5}\frac{(-q;q)_\infty}{(-q^5;q^5)_\infty} \div \frac{(q;q)_\infty}{(q^5;q^5)_\infty}$$

$$= 1 + \frac{f\left(-\zeta,-q\zeta^4\right)f\left(-\zeta^2,-q\zeta^3\right)}{f\left(\zeta,q\zeta^4\right)f\left(\zeta^2,q\zeta^3\right)}. \tag{37}$$

In (13) and (14), setting $a = \zeta$, $b = q\zeta^4$, $c = \zeta^2$, and $d = q\zeta^3$, we get that

$$f\left(\zeta,q\zeta^4\right)f\left(\zeta^2,q\zeta^3\right) + f\left(-\zeta,-q\zeta^4\right)f\left(-\zeta^2,-q\zeta^3\right)$$

$$= 2f\left(\zeta^3,q^2\zeta^2\right)f\left(q\zeta^4,q\zeta\right),$$

$$f\left(\zeta,q\zeta^4\right)f\left(\zeta^2,q\zeta^3\right) - f\left(-\zeta,-q\zeta^4\right)f\left(-\zeta^2,-q\zeta^3\right)$$

$$= 2\zeta f\left(q\zeta^2,q\zeta^3\right)f\left(\zeta,q^2\zeta^4\right). \tag{38}$$

Dividing the above two equations by $f(\zeta,q\zeta^4)f(\zeta^2,q\zeta^3)$, respectively, and then combining with (37), we obtain that

$$1 + \sqrt{5}\frac{(-q;q)_\infty}{(-q^5;q^5)_\infty} \div \frac{(q;q)_\infty}{(q^5;q^5)_\infty} = 2\frac{f\left(\zeta^3,q^2\zeta^2\right)f\left(q\zeta,q\zeta^4\right)}{f\left(\zeta,q\zeta^4\right)f\left(\zeta^2,q\zeta^3\right)}, \tag{39}$$

$$1 - \sqrt{5}\frac{(-q;q)_\infty}{(-q^5;q^5)_\infty} \div \frac{(q;q)_\infty}{(q^5;q^5)_\infty} = 2\zeta\frac{f\left(q\zeta^2,q\zeta^3\right)f\left(\zeta,q^2\zeta^4\right)}{f\left(\zeta,q\zeta^4\right)f\left(\zeta^2,q\zeta^3\right)}. \tag{40}$$

Multiplied by $(q;q)_\infty/(q^5;q^5)_\infty$, the identities (39) and (40) become (34) and (35).

Multiplying the two refinements in Theorems 1 and 2, respectively, we obtain the identities (18) and (20). Using the same method, we can obtain refinement identities of (21) and (22) which are similar to Theorems 1 and 2; then we can deduce (21), (22), and (23) easily. The details of proofs are omitted. □

The following conclusion can be obtained easily.

Corollary 3. *For* $|q| < 1$,

$$
\frac{(q;q)_\infty^4}{(q^3;q^3)_\infty^4} - 9\frac{(-q;q)_\infty^4}{(-q^3;q^3)_\infty^4}
$$

$$
= 8\omega \frac{(q;q)_\infty^4}{(q^3;q^3)_\infty^4} \tag{41}
$$

$$
\times \frac{f\left(\omega^2, q\omega\right) f\left(q\omega, q^3\omega^2\right) f\left(q\omega^2, q^3\omega\right)}{f^4\left(\omega, q\omega^2\right)} \varphi\left(q\right).
$$

Acknowledgments

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