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Research Article

Strong Convergence Results for Equilibrium Problems and Fixed Point Problems for Multivalued Mappings

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Using viscosity approximation method, we study strong convergence to a common element of the set of solutions of an equilibrium problem and the set of common fixed points of a finite family of multivalued mappings satisfying the condition (E) in the setting of Hilbert space. Our results improve and extend some recent results in the literature.

1. Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let C be a nonempty closed convex subset H. A subset $C \subset H$ is called proximal if, for each $x \in H$, there exists an element $y \in C$ such that

$$\|x-y\| = \operatorname{dist}(x, C) = \inf\{\|x-z\| : z \in C\}. \tag{1}$$

A single-valued mapping $T : C \to C$ is said to be nonexpansive, if

$$\|Tx-Ty\| \le \|x-y\|, \quad \forall x, y \in C. \tag{2}$$

Let P_C be a nearest point projection of H into C; that is, for $x \in H$, $P_C x$ is a unique nearest point in C with the property

$$\|x-P_C x\| := \inf\{\|x-y\| : y \in C\}. \tag{3}$$

We denote by $CB(C)$, $K(C)$, and $P(C)$ the collection of all nonempty closed bounded subsets, nonempty compact subsets, and nonempty proximal bounded subsets of C respectively. The Hausdorff metric H on $CB(H)$ is defined by

$$H(A, B) := \max\left\{\sup_{x\in A} \operatorname{dist}(x, B), \sup_{y\in B} \operatorname{dist}(y, A)\right\}, \tag{4}$$

for all $A, B \in CB(H)$.

Let $T : H \to 2^H$ be a multivalued mapping. An element $x \in H$ is said to be a fixed point of T, if $x \in Tx$ and the set of fixed points of T is denoted by $F(T)$.

A multivalued mapping $T : H \to CB(H)$ is called

(i) nonexpansive if

$$H(Tx, Ty) \le \|x-y\|, \quad x, y \in H; \tag{5}$$

(ii) quasi-nonexpansive if $F(T) \neq \emptyset$ and $H(Tx, Tp) \le \|x-p\|$ for all $x \in H$ and all $p \in F(T)$.

Recently, García-Falset et al. [1] introduced a new condition on single-valued mappings, called condition (E), which is weaker than nonexpansiveness.

Definition 1. A mapping $T : H \to H$ is said to satisfy condition (E_μ) provided that

$$\|x-Ty\| \le \mu\ \|x-Tx\| + \|x-y\|, \quad x, y \in H. \tag{6}$$

We say that T satisfies condition (E) whenever T satisfies (E_μ) for some $\mu \ge 1$.

Recently, Abkar and Eslamian [2, 3] generalized this condition for multivalued mappings as follows.

Definition 2. A multivalued mapping $T : H \to CB(H)$ is said to satisfy condition (E) provided that

$$H(Tx, Ty) \leq \mu \ \operatorname{dist}(x, Tx) + \|x - y\|, \quad x, y \in H, \quad (7)$$

for some $\mu \geq 1$.

It is obvious that every nonexpansive multivalued mapping $T : H \to CB(H)$ satisfies the condition (E), and every mapping $T : H \to CB(H)$ which satisfies the condition (E) with nonempty fixed point set $F(T)$ is quasi-nonexpansive.

Example 3. Let us define a mapping T on $[0, 3]$ by

$$T(x) = \begin{cases} \left[0, \dfrac{x}{3}\right], & x \neq 3 \\ [1, 2] & x = 3. \end{cases} \quad (8)$$

It is easy to see that T satisfies the condition (E) but is not nonexpansive. Indeed, for $x, y \in [0, 3)$, $H(Tx, Ty) = |(x - y)/3| \leq |x - y|$. Let $x = 0$ and $y = 3$. Then $H(Tx, Ty) = 2 \leq 3 = |x - y|$. If $x \in (0, 3)$ and $y = 3$, then, we have $\operatorname{dist}(x, Tx) = 2x/3$ and $\operatorname{dist}(y, Ty) = 1$; hence

$$H(Tx, Ty) = 2 - \frac{x}{3} \leq 3 - x + \frac{4x}{3} = |x - y| + 2 \operatorname{dist}(x, Tx). \quad (9)$$

Thus, T satisfies the condition (E). However, T is not nonexpansive; indeed for $x = 3$ and $y = 7/3$, $H(Tx, Ty) = 11/9 > 2/3 = |x - y|$.

Let $\Psi : C \times C \to \mathbb{R}$ be a bifunction. The equilibrium problem associated with the bifunction Ψ and the set C is:

$$\text{find } x \in C \quad \text{such that } \Psi(x, y) \geq 0, \ \forall y \in C. \quad (10)$$

Such a point $x \in C$ is called the solution of the equilibrium problem. The set of solutions is denoted by $EP(\Psi)$.

A broad class of problems in optimization theory, such as variational inequality, convex minimization, and fixed point problems, can be formulated as an equilibrium problem; see [4, 5]. In the literature, many techniques and algorithms have been proposed to analyze the existence and approximation of a solution to equilibrium problem; see [6]. Many researchers have studied various iteration processes for finding a common element of the set of solutions of the equilibrium problems and the set of fixed points of a class of nonlinear mappings. For example, see [7–22].

Fixed points and fixed point iteration process for nonexpansive mappings have been studied extensively by many authors to solve nonlinear operator equations, as well as variational inequalities; see, for example, [23–28]. In the recent years, fixed point theory for multivalued mappings has been studied by many authors; see [29–40] and the references therein.

In this paper, using viscosity approximation method, we study strong convergence to a common element of the set of solutions of an equilibrium problem and the set of common fixed points of a finite family of multivalued mappings satisfying the condition (E) in the setting of Hilbert space. Our results improve and extend some recent results in the literature.

2. Preliminaries

For solving the equilibrium problem, we assume that the bifunction Ψ satisfies the following conditions:

(A1) $\Psi(x, x) = 0$ for any $x \in C$;

(A2) Ψ is monotone; that is, $\Psi(x, y) + \Psi(y, x) \leq 0$ for any $x, y \in C$;

(A3) Ψ is upper-hemicontinuous; that is, for each $x, y, z \in C$,

$$\limsup_{t \to 0^+} \Psi(tz + (1 - t)x, y) \leq \Psi(x, y); \quad (11)$$

(A4) $\Psi(x, .)$ is convex and lower semicontinuous for each $x \in C$.

Lemma 4 (see [4]). *Let C be a nonempty closed convex subset of H and let Ψ be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)–(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$\Psi(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \quad \forall y \in C. \quad (12)$$

Lemma 5 (see [6]). *Assume that $\Psi : C \times C \to \mathbb{R}$ satisfies (A1)–(A4). For $r > 0$ and $x \in H$, define a mapping $S_r : H \to C$ as follows:*

$$S_r x = \left\{z \in C : \Psi(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C\right\}. \quad (13)$$

Then, the following hold:

(i) *S_r is single valued;*

(ii) *S_r is firmly nonexpansive; that is, for any $x, y \in H$,*

$$\|S_r x - S_r y\|^2 \leq \langle S_r x - S_r y, x - y\rangle; \quad (14)$$

(iii) *$F(S_r) = EP(\Psi)$;*

(iv) *$EP(\Psi)$ is closed and convex.*

Lemma 6 (see [41]). *Let H be a real Hilbert space. Then, for all $x, y, z \in H$ and $\alpha, \beta, \gamma \in [0, 1]$ with $\alpha + \beta + \gamma = 1$ one has*

$$\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x - y\|^2 - \alpha\gamma\|x - z\|^2 - \beta\gamma\|z - y\|^2. \quad (15)$$

Lemma 7. *For every x and y in a Hilbert space H, the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle. \quad (16)$$

Lemma 8 (see [42]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ a sequence in $(0, 1)$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{\gamma_n\}$ a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} \gamma_n < \infty$, and $\{\beta_n\}$ a sequence of real numbers with* $\limsup_{n \to \infty} \beta_n \leq 0$. *Suppose that the following inequality holds:*

$$a_{n+1} \leq (1 - \alpha_n) a_n + \alpha_n \beta_n + \gamma_n, \quad n \geq 0. \quad (17)$$

Then, $\lim_{n \to \infty} a_n = 0$.

Lemma 9 (see [43]). *Let $\{u_n\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that $u_{n_i} < u_{n_i+1}$ for all $i \geq 0$. For every sufficiently large number $n \geq n_0$, define an integer sequence $\{\tau(n)\}$ as*

$$\tau(n) = \max\{k \leq n : u_k < u_{k+1}\}. \tag{18}$$

Then, $\tau(n) \to \infty$ as $n \to \infty$ and for all $n \geq n_0$,

$$\max\{u_{\tau(n)}, u_n\} \leq u_{\tau(n)+1}. \tag{19}$$

Lemma 10 (see [20]). *Let C be a closed convex subset of a real Hilbert space H. Let $T : C \to CB(C)$ be a quasi-nonexpansive multivalued mapping. If $F(T) \neq \emptyset$ and $T(p) = \{p\}$ for all $p \in F(T)$. Then $F(T)$ is closed and convex.*

Lemma 11 (see [20]). *Let C be a closed convex subset of a real Hilbert space H. Let $T : C \to P(C)$ be a multivalued mapping such that P_T is quasi-nonexpansive and $F(T) \neq \emptyset$, where $P_T(x) = \{y \in Tx : \|x - y\| = \mathrm{dist}(x, Tx)\}$. Then, $F(T)$ is closed and convex.*

Lemma 12 (see [16, 20]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to K(C)$ be a multivalued mapping satisfying the condition (E). If x_n converges weakly to v and $\lim_{n\to\infty} \mathrm{dist}(x_n, Tx_n) = 0$, then $v \in Tv$.*

3. A Strong Convergence Theorem

Theorem 13. *Let C be a nonempty closed convex subset of a real Hilbert space H and Ψ a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)–(A4). Let $T_i : C \to CB(C)$ $(i = 1, 2, \ldots, m)$ be a finite family of multivalued mappings, each satisfying condition (E). Assume further that $\mathcal{F} = \bigcap_{i=1}^{m} F(T_i) \bigcap EP(\Psi) \neq \emptyset$ and $T_i(p) = \{p\}, (i = 1, 2, \ldots, m)$ for each $p \in \mathcal{F}$. Let f be a k-contraction of C into itself. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated the following algorithm:*

$$\begin{gathered} x_0 \in C, \\ u_n \in C \text{ such that } \Psi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C \\ y_{n,1} = a_{n,1}u_n + b_{n,1}x_n + c_{n,1}z_{n,1}, \\ y_{n,2} = a_{n,2}u_n + b_{n,2}z_{n,1} + c_{n,2}z_{n,2}, \\ y_{n,3} = a_{n,3}u_n + b_{n,3}z_{n,2} + c_{n,3}z_{n,3} \\ \vdots \\ y_{n,m} = a_{n,m}u_n + b_{n,m}z_{n,m-1} + c_{n,m}z_{n,m}, \\ x_{n+1} = \vartheta_n f(x_n) + (1 - \vartheta_n) y_{n,m}, \quad \forall n \geq 0, \end{gathered} \tag{20}$$

where $z_{n,1} \in T_1(u_n)$, $z_{n,k} \in T_k(y_{n,k-1})$ for $k = 2, \ldots, m$, and $\{a_{n,i}\}$, $\{b_{n,i}\}$, $\{c_{n,i}\}$, $\{\vartheta_n\}$, and $\{r_n\}$ satisfy the following conditions:

(i) $\{a_{n,i}\}, \{b_{n,i}\}, \{c_{n,i}\} \subset [a, b] \subset (0, 1), a_{n,i} + b_{n,i} + c_{n,i} = 1, (i = 1, 2, \ldots, m)$,

(ii) $\{\vartheta_n\} \subset (0, 1)$, $\lim_{n\to\infty}\vartheta_n = 0$, $\sum_{n=1}^{\infty} \vartheta_n = \infty$,

(iii) $\{r_n\} \subset (0, \infty)$, *and* $\liminf_{n\to\infty} r_n > 0$.

Then, the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to $q \in \mathcal{F}$, where $q = P_{\mathcal{F}} f(q)$.

Proof. Let $Q = P_{\mathcal{F}}$. It is easy to see that Qf is a contraction. By Banach contraction principle, there exists a $q \in \mathcal{F}$ such that $q = P_{\mathcal{F}} f(q)$. From Lemma 5 for all $n \geq 0$, we have

$$\|u_n - q\| = \|S_{r_n}x_n - S_{r_n}q\| \leq \|x_n - q\|. \tag{21}$$

We show that $\{x_n\}$ is bounded. Since, for each $i = 1, 2, \ldots, m$, T_i satisfies the condition (E) and we have

$$\begin{aligned} \|y_{n,1} - q\| &= \|a_{n,1}u_n + b_{n,1}x_n + c_{n,1}z_{n,1} - q\| \\ &\leq a_{n,1}\|u_n - q\| + b_{n,1}\|x_n - q\| + c_{n,1}\|z_{n,1} - q\| \\ &= a_{n,1}\|u_n - q\| + b_{n,1}\|x_n - q\| + c_{n,1}\,\mathrm{dist}(z_{n,1}, T_1q) \\ &\leq a_{n,1}\|u_n - q\| + b_{n,1}\|x_n - q\| + c_{n,1}H(T_1u_n, T_1q) \\ &\leq a_{n,1}\|u_n - q\| + b_{n,1}\|x_n - q\| + c_{n,1}\|u_n - q\| \\ &\leq \|x_n - q\|, \end{aligned} \tag{22}$$

$$\begin{aligned} \|y_{n,2} - q\| &= \|a_{n,2}u_n + b_{n,2}z_{n,1} + c_{n,2}z_{n,2} - q\| \\ &\leq a_{n,2}\|u_n - q\| + b_{n,2}\|z_{n,1} - q\| + c_{n,2}\|z_{n,2} - q\| \\ &= a_{n,2}\|u_n - q\| + b_{n,2}\,\mathrm{dist}(z_{n,1}, T_1q) + c_{n,2}\,\mathrm{dist}(z_{n,2}, T_2q) \\ &\leq a_{n,2}\|u_n - q\| + b_{n,2}H(T_1u_n, T_1q) + c_{n,2}H(T_2y_{n,1}, T_2q) \\ &\leq a_{n,2}\|u_n - q\| + b_{n,2}\|u_n - q\| + c_{n,2}\|y_{n,1} - q\| \\ &\leq \| x_n - q \|. \end{aligned} \tag{23}$$

By continuing this process, we obtain

$$\|y_{n,m} - q\| \leq \|x_n - q\|. \tag{24}$$

This implies that

$$\begin{aligned}
&\|x_{n+1}-q\| \\
&= \|\vartheta_n f x_n + (1-\vartheta_n) y_n - q\| \\
&\le \vartheta_n \|f x_n - q\| + (1-\vartheta_n)\|y_n - q\| \\
&\le \vartheta_n (\|f x_n - f q\| + \|f q - q\|) + (1-\vartheta_n)\|x_n - q\| \\
&\le \vartheta_n k \|x_n - q\| + \vartheta_n \|f q - q\| + (1-\vartheta_n)\|x_n - q\| \\
&= (1-\vartheta_n(1-k))\|x_n - q\| + \vartheta_n \|f q - q\| \\
&\le \max\left\{\|x_n - q\|, \frac{\|f q - q\|}{1-k}\right\}.
\end{aligned} \tag{25}$$

By induction, we get

$$\|x_n - q\| \le \max\left\{\|x_0 - q\|, \frac{\|f q - q\|}{1-k}\right\}, \tag{26}$$

for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is bounded and we also obtain that $\{u_n\}, \{y_n\}, \{f x_n\}$, and $\{z_{n,i}\}$ are bounded. Next, we show that $\lim_{n\to\infty} \operatorname{dist}(u_n, T_i u_n) = 0$ for each $i \in \mathbb{N}$. By Lemma 6, we have

$$\begin{aligned}
&\|y_{n,1} - q\|^2 \\
&= \|a_{n,1} u_n + b_{n,1} x_n + c_{n,1} z_{n,1} - q\|^2 \\
&\le a_{n,1}\|u_n - q\|^2 + b_{n,1}\|x_n - q\|^2 \\
&\quad + c_{n,1}\|z_{n,1} - q\|^2 \\
&\quad - a_{n,1} b_{n,1}\|x_n - u_n\|^2 - a_{n,1} c_{n,1}\|u_n - z_{n,1}\|^2 \\
&= a_{n,1}\|u_n - q\|^2 + b_{n,1}\|x_n - q\|^2 \\
&\quad + c_{n,1}\operatorname{dist}(z_{n,1}, T_1 q)^2 \\
&\quad - a_{n,1} b_{n,1}\|x_n - u_n\|^2 - a_{n,1} c_{n,1}\|u_n - z_{n,1}\|^2 \\
&\le a_{n,1}\|u_n - q\|^2 + b_{n,1}\|x_n - q\|^2 \\
&\quad + c_{n,1} H(T_1 u_n, T_1 q)^2 \\
&\quad - a_{n,1} b_{n,1}\|x_n - u_n\|^2 - a_{n,1} c_{n,1}\|u_n - z_{n,1}\|^2 \\
&\le a_{n,1}\|u_n - q\|^2 + b_{n,1}\|x_n - q\|^2 \\
&\quad + c_{n,1}\|u_n - q\|^2 \\
&\quad - a_{n,1} b_{n,1}\|x_n - u_n\|^2 - a_{n,1} c_{n,1}\|u_n - z_{n,1}\|^2 \\
&\le \|x_n - q\|^2 - a_{n,1} b_{n,1}\|x_n - u_n\|^2 \\
&\quad - a_{n,1} c_{n,1}\|u_n - z_{n,1}\|^2.
\end{aligned} \tag{27}$$

Applying Lemma 6 once more, we have

$$\begin{aligned}
&\|y_{n,2} - q\|^2 \\
&= \|a_{n,2} u_n + b_{n,2} z_{n,1} + c_{n,2} z_{n,2} - q\|^2 \\
&\le a_{n,2}\|u_n - q\|^2 + b_{n,2}\|z_{n,1} - q\|^2 + c_{n,2}\|z_{n,2} - q\|^2 \\
&\quad - a_{n,2} c_{n,2}\|u_n - z_{n,2}\|^2 \\
&= a_{n,2}\|u_n - q\|^2 + b_{n,2}\operatorname{dist}(z_{n,1}, T_1 q)^2 \\
&\quad + c_{n,2}\operatorname{dist}(z_{n,2}, T_2 q)^2 - a_{n,2} c_{n,2}\|u_n - z_{n,2}\|^2 \\
&\le a_{n,2}\|u_n - q\|^2 + b_{n,2} H(T_1 u_n, T_1 q)^2 \\
&\quad + c_{n,2} H(T_1 y_{n,1}, T_2 q)^2 - a_{n,2} c_{n,2}\|u_n - z_{n,2}\|^2 \\
&\le a_{n,2}\|u_n - q\|^2 + b_{n,2}\|u_n - q\|^2 + c_{n,2}\|y_{n,1} - q\|^2 \\
&\quad - a_{n,2} c_{n,2}\|u_n - z_{n,2}\|^2 \\
&\le \|x_n - q\|^2 - a_{n,2} c_{n,2}\|u_n - z_{n,2}\|^2 \\
&\quad - a_{n,1} c_{n,1} c_{n,2}\|u_n - z_{n,1}\|^2 - a_{n,1} b_{n,1} c_{n,2}\|x_n - u_n\|^2.
\end{aligned} \tag{28}$$

By continuing this process we have

$$\begin{aligned}
&\|y_{n,m} - q\|^2 \\
&= \|a_{n,m} u_n + b_{n,m} z_{n,m-1} + c_{n,m} z_{n,m} - q\|^2 \\
&\le a_{n,m}\|u_n - q\|^2 + b_{n,m}\|z_{n,m-1} - q\|^2 + c_{n,m}\|z_{n,m} - q\|^2 \\
&\quad - a_{n,m} c_{n,m}\|u_n - z_{n,m}\|^2 \\
&= a_{n,m}\|u_n - q\|^2 + b_{n,m}\operatorname{dist}(z_{n,m-1}, T_{m-1} q)^2 \\
&\quad + c_{n,m}\operatorname{dist}(z_{n,m}, T_m q)^2 - a_{n,m} c_{n,m}\|u_n - z_{n,m}\|^2 \\
&\le a_{n,m}\|u_n - q\|^2 + b_{n,m} H(T_{m-1} y_{n,m-2}, T_{m-1} q)^2 \\
&\quad + c_{n,m} H(T_m y_{n,m-1}, T_m q)^2 - a_{n,m} c_{n,m}\|u_n - z_{n,m}\|^2 \\
&\le a_{n,m}\|u_n - q\|^2 + b_{n,m}\|y_{n,m-2} - q\|^2 \\
&\quad + c_{n,m}\|y_{n,m-1} - q\|^2 - a_{n,m} c_{n,m}\|u_n - z_{n,m}\|^2 \\
&\le \|u_n - q\|^2 - a_{n,m} c_{n,m}\|u_n - z_{n,m}\|^2 \\
&\quad - a_{n,m-1} c_{n,m-1} c_{n,m}\|u_n - z_{n,m-1}\|^2 \\
&\quad - \cdots - a_{n,1} c_{n,1} c_{n,2} \ldots c_{n,m}\|u_n - z_{n,1}\|^2 \\
&\quad - a_{n,1} b_{n,1} c_{n,2} \ldots c_{n,m}\|u_n - x_n\|^2,
\end{aligned} \tag{29}$$

which implies that

$$\begin{aligned}\|x_{n+1}-q\|^2 &= \|\vartheta_n f x_n + (1-\vartheta_n) y_{n,m} - q\|^2 \\ &\le \vartheta_n \|f x_n - q\|^2 + (1-\vartheta_n)\|y_{n,m}-q\|^2 \\ &\le \vartheta_n \|f x_n - q\|^2 + (1-\vartheta_n)\|u_n-q\|^2 \\ &\quad - (1-\vartheta_n) a_{n,m} c_{n,m} \|u_n - z_{n,m}\|^2 \\ &\quad - (1-\vartheta_n) a_{n,m-1} c_{n,m-1} c_{n,m} \|u_n - z_{n,m-1}\|^2 \\ &\quad - \cdots - (1-\vartheta_n) a_{n,1} c_{n,1} c_{n,2} \ldots c_{n,m} \|u_n - z_{n,1}\|^2 \\ &\quad - (1-\vartheta_n) a_{n,1} b_{n,1} c_{n,2} \ldots c_{n,m} \|u_n - x_n\|^2. \end{aligned} \tag{30}$$

Therefore, we have that

$$\begin{aligned}&(1-\vartheta_n) a_{n,1} b_{n,1} c_{n,2} \ldots c_{n,m} \|u_n - x_n\|^2 \\ &\quad \le \|x_n - q\|^2 - \|x_{n+1}-q\|^2 + \vartheta_n \|\gamma f x_n - q\|. \end{aligned} \tag{31}$$

In order to prove that $x_n \to q$ as $n \to \infty$, we consider the following two cases.

Case 1. Suppose that there exists n_0 such that $\{\|x_n - q\|\}$ is nonincreasing, for all $n \ge n_0$. Boundedness of $\{\|x_n - q\|\}$ implies that $\|x_n - q\|$ is convergent. From (31) and conditions (i), (ii) we have that

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{32}$$

By a similar argument, for $k = 1, 2, \ldots, m$, we obtain that

$$\lim_{n\to\infty} \|u_n - z_{n,k}\| = 0. \tag{33}$$

Hence,

$$\begin{aligned}\lim_{n\to\infty} \operatorname{dist}(u_n, T_1 u_n) &\le \lim_{n\to\infty} \|u_n - z_{n,1}\| = 0, \\ \lim_{n\to\infty} \operatorname{dist}(u_n, T_k y_{n,k-1}) &\le \lim_{n\to\infty} \|u_n - z_{n,k}\| = 0, \\ &\qquad (k = 2, \ldots, m). \end{aligned} \tag{34}$$

Therefore, we have

$$\begin{aligned}\lim_{n\to\infty} \|u_n - y_{n,1}\| &\le \lim_{n\to\infty} b_{n,1} \|u_n - x_n\| \\ &\quad + \lim_{n\to\infty} c_{n,1} \|u_n - z_{n,1}\| = 0. \end{aligned} \tag{35}$$

For $k = 2, \ldots, m$, we have

$$\begin{aligned}\lim_{n\to\infty} \|u_n - y_{n,k}\| &\le \lim_{n\to\infty} b_{n,k} \|u_n - z_{n,k-1}\| \\ &\quad + \lim_{n\to\infty} c_{n,k} \|u_n - z_{n,k}\| = 0. \end{aligned} \tag{36}$$

Using the previous inequality for $k = 2, \ldots, m$, we have

$$\begin{aligned}\operatorname{dist}(u_n, T_k u_n) &\le \operatorname{dist}(u_n, T_k y_{n,k-1}) + H(T_k y_{n,k-1}, T_k u_n) \\ &\le \operatorname{dist}(u_n, T_k y_{n,k-1}) + \mu \operatorname{dist}(y_{n,k-1}, T_k y_{n,k-1}) \\ &\quad + \|y_{n,k-1} - u_n\| \\ &\le (\mu+1) \operatorname{dist}(u_n, T_k y_{n,k-1}) + (\mu+1)\|y_{n,k-1} - u_n\| \\ &\le (\mu+1)\|u_n - z_{n,k}\| + (\mu+1)\|y_{n,k-1} - u_n\| \longrightarrow 0, \\ &\qquad n \longrightarrow \infty. \end{aligned} \tag{37}$$

Next, we show that

$$\limsup_{n\to\infty} \langle q - fq, q - x_n \rangle \le 0, \tag{38}$$

where $q = P_{\mathcal{F}} f(q)$. To show this inequality, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\lim_{i\to\infty} \langle q - fq, q - x_{n_i} \rangle = \limsup_{n\to\infty} \langle q - fq, q - x_n \rangle. \tag{39}$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to v. Without loss of generality, we can assume that x_{n_i} converges weakly to v. Since $\lim_{n\to\infty}\|x_n - u_n\| = 0$, we have u_{n_i} converges weakly to v. We show that $v \in \mathcal{F}$. Let us show $v \in EP(\Psi)$. Since $u_n = S_{r_n} x_n$, we have

$$\Psi(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \ge 0 \quad \forall y \in C. \tag{40}$$

From (A2), we have

$$\frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \ge \Psi(y, u_n). \tag{41}$$

Replacing n with n_i, we have

$$\left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \right\rangle \ge \Psi(y, u_{n_i}). \tag{42}$$

From (A4), we have

$$0 \ge \Psi(y, v), \quad \forall\, y \in C. \tag{43}$$

For $t \in (0,1]$ and $y \in C$, let $y_t = ty + (1-t)v$. Since $y, v \in C$, and C is convex, we have $y_t \in C$ and hence $\Psi(y_t, v) \le 0$. So, from (A1) and (A4) we have

$$0 = \Psi(y_t, y_t) \le t\Psi(y_t, y) + (1-t)\Psi(y_t, v) \le t\Psi(y_t, y), \tag{44}$$

which gives $0 \le \Psi(y_t, y)$. Letting $t \to 0$, we have, for each $y \in C$, $0 \le \Psi(v, y)$ Also, since $u_{n_i} \rightharpoonup v$ and $\lim_{n\to\infty} \operatorname{dist}(u_n, T_i u_n) = 0$, by Lemma 12 we have $v \in \bigcap_{i=1}^m F(T_i)$. Hence, $v \in \mathcal{F}$. Since $q = P_{\mathcal{F}} f(q)$ and $v \in \mathcal{F}$, it follows that

$$\begin{aligned}\limsup_{n\to\infty} \langle q - fq, q - x_n \rangle &= \lim_{i\to\infty} \langle q - fq, q - x_{n_i} \rangle \\ &= \langle q - fq, q - v \rangle \le 0. \end{aligned} \tag{45}$$

By using Lemma 7 and inequality (31) we have

$$\begin{aligned}
&\|x_{n+1}-q\|^2 \\
&\leq \|(1-\vartheta_n)(y_{n,m}-q)\|^2 + 2\vartheta_n \langle fx_n - q, x_{n+1}-q\rangle \\
&\leq (1-\vartheta_n)^2\|y_{n,m}-q\|^2 + 2\vartheta_n \langle fx_n - fq, x_{n+1}-q\rangle \\
&\quad + 2\vartheta_n \langle fq - q, x_{n+1}-q\rangle \\
&\leq (1-\vartheta_n)^2\|x_n-q\|^2 + 2\vartheta_n k \|x_n-q\|\,\|x_{n+1}-q\| \\
&\quad + 2\vartheta_n \langle fq - q, x_{n+1}-q\rangle \\
&\leq (1-\vartheta_n)^2\|x_n-q\|^2 + \vartheta_n k \left(\|x_n-q\|^2 + \|x_{n+1}-q\|^2\right) \\
&\quad + 2\vartheta_n \langle fq - q, x_{n+1}-q\rangle \\
&\leq \left((1-\vartheta_n)^2 + \vartheta_n k\right)\|x_n-q\|^2 + \vartheta_n k\|x_{n+1}-q\|^2 \\
&\quad + 2\vartheta_n \langle fq - q, x_{n+1}-q\rangle .
\end{aligned} \tag{46}$$

This implies that

$$\begin{aligned}
\|x_{n+1}-q\|^2 \leq{}& \left(1 - \frac{2(1-k)\vartheta_n}{1-\vartheta_n k}\right)\|x_n-q\|^2 \\
&+ \frac{\vartheta_n^2}{1-\vartheta_n k}\|x_n-q\|^2 \\
&+ \frac{2\vartheta_n}{1-\vartheta_n k}\langle fq-q, x_{n+1}-q\rangle .
\end{aligned} \tag{47}$$

From Lemma 8, we conclude that the sequence $\{x_n\}$ converges strongly to q.

Case 2. Assume that there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\left\|x_{n_j}-q\right\| < \left\|x_{n_{j+1}}-q\right\|, \tag{48}$$

for all $j \in \mathbb{N}$. In this case, from Lemma 9, there exists a nondecreasing sequence $\{\tau(n)\}$ of $\mathbb{N}$ for all $n \geq n_0$ (for some n_0 large enough) such that $\tau(n) \to \infty$ as $n \to \infty$ and the following inequalities hold for all $n \geq n_0$:

$$\|x_{\tau(n)}-q\| \leq \|x_{\tau(n)+1}-q\|, \qquad \|x_n-q\| \leq \|x_{\tau(n)+1}-q\|. \tag{49}$$

From (31) we obtain $\lim_{n\to\infty}\|u_{\tau(n)} - T_i u_{\tau(n)}\| = 0$, and $\lim_{n\to\infty}\|u_{\tau(n)} - x_{\tau(n)}\| = 0$. Following an argument similar to that in Case 1, we have

$$\lim_{n\to\infty}\|x_{\tau(n)}-q\| = 0, \quad \lim_{n\to\infty}\|x_{\tau(n)+1}-q\| = 0. \tag{50}$$

Thus, by Lemma 9 we have

$$0 \leq \|x_n-q\| \leq \max\{\|x_{\tau(n)}-q\|, \|x_n-q\|\} \leq \|x_{\tau(n)+1}-q\|. \tag{51}$$

Therefore, $\{x_n\}$ converges strongly to $q = P_{\mathcal{F}}f(q)$. This completes the proof. □

Now, we remove the condition that $T(p) = \{p\}$ for all $p \in \mathcal{F}$, and state the following theorem.

Theorem 14. *Let C be a nonempty closed convex subset of a real Hilbert space H and Ψ a bifunction of $C\times C$ into $\mathbb{R}$ satisfying (A1)–(A4). Let, for each $1 \leq i \leq m$, $T_i : C \to P(C)$ be multivalued mappings such that P_{T_i} satisfies the condition (E). Assume that $\mathcal{F} = \bigcap_{i=1}^m F(T_i)\bigcap EP(\Psi) \neq \emptyset$. Let f be a k-contraction of C into itself. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated the following algorithm:*

$$\begin{gathered}
x_0 \in C, \\
u_n \in C \text{ such that } \Psi(u_n, y) + \frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \geq 0, \\
\hfill \forall y \in C \\
y_{n,1} = a_{n,1}u_n + b_{n,1}x_n + c_{n,1}z_{n,1}, \\
y_{n,2} = a_{n,2}u_n + b_{n,2}z_{n,1} + c_{n,2}z_{n,2}, \\
y_{n,3} = a_{n,3}u_n + b_{n,3}z_{n,2} + c_{n,3}z_{n,3} \\
\vdots \\
y_{n,m} = a_{n,m}u_n + b_{n,m}z_{n,m-1} + c_{n,m}z_{n,m}, \\
x_{n+1} = \vartheta_n f x_n + (1-\vartheta_n) y_{n,m}, \quad \forall n \geq 0,
\end{gathered} \tag{52}$$

where $z_{n,1} \in P_{T_1}(u_n)$, $z_{n,k} \in P_{T_k}(y_{n,k-1})$ for $k = 2,\ldots,m$, and $\{a_{n,i}\}$, $\{b_{n,i}\}$, $\{c_{n,i}\}$, $\{\vartheta_n\}$ and, $\{r_n\}$ satisfy the following conditions:

(i) $\{a_{n,i}\},\{b_{n,i}\},\{c_{n,i}\} \subset [a,b] \subset (0,1), a_{n,i}+b_{n,i}+c_{n,i} = 1, (i = 1,2,\ldots,m)$,

(ii) $\{\vartheta_n\} \subset (0,1)$, $\lim_{n\to\infty}\vartheta_n = 0$, $\sum_{n=1}^{\infty}\vartheta_n = \infty$,

(iii) $\{r_n\} \subset (0,\infty)$, *and* $\liminf_{n\to\infty} r_n > 0$.

Then, the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to $q \in \mathcal{F}$, where $q = P_{\mathcal{F}}f(q)$.

Proof. Let $p \in \mathcal{F}$; then $P_{T_i}(p) = \{p\}, (i = 1,2,\ldots,m)$. Now by substituting P_{T_i} instead of T_i, and using a similar argument as in the proof of Theorem 13, the desired result follows. □

As a corollary for single-valued mappings, we obtain the following result.

Corollary 15. *Let C be a nonempty closed convex subset of a real Hilbert space H and Ψ a bifunction of $C\times C$ into $\mathbb{R}$ satisfying (A1)–(A4). Let, for each $1 \leq i \leq m$, $T_i : C \to C$ be a finite family of mappings satisfying condition (E). Assume that $\mathcal{F} = \bigcap_{i=1}^m F(T_i)\bigcap EP(\Psi) \neq \emptyset$. Let f be a k-contraction* *of C into itself. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated the following algorithm:*

$$\begin{aligned}
&x_0 \in C,\\
&u_n \in C \text{ such that } \Psi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C\\
&y_{n,1} = a_{n,1}u_n + b_{n,1}x_n + c_{n,1}T_1u_n,\\
&y_{n,2} = a_{n,2}u_n + b_{n,2}T_1u_n + c_{n,2}T_2y_{n,1}\\
&\vdots\\
&y_{n,m} = a_{n,m}u_n + b_{n,m}T_{m-1}y_{n,m-2} + T_my_{n,m-1},\\
&x_{n+1} = \vartheta_n fx_n + (1-\vartheta_n)y_{n,m}, \quad \forall n \geq 0,
\end{aligned} \tag{53}$$

where $\{a_{n,i}\}$, $\{b_{n,i}\}$,$\{c_{n,i}\}$, $\{\vartheta_n\}$, and $\{r_n\}$ satisfy the following conditions:

(i) $\{a_{n,i}\}, \{b_{n,i}\}, \{c_{n,i}\} \subset [a, b] \subset (0, 1), a_{n,i} + b_{n,i} + c_{n,i} = 1$, $(i = 1, 2, \ldots, m)$,

(ii) $\{\vartheta_n\} \subset (0, 1)$, $\lim_{n\to\infty}\vartheta_n = 0$, $\sum_{n=1}^{\infty}\vartheta_n = \infty$

(iii) $\{r_n\} \subset (0, \infty)$, *and* $\liminf_{n\to\infty} r_n > 0$.

Then, the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to $q \in \mathcal{F}$, where $q = P_{\mathcal{F}}f(q)$.

Remark 16. Our results generalize the corresponding results of S. Takahashi and W. Takahashi [9] from a single valued nonexpansive mapping to a finite family of multivalued mappings satisfying the condition (E). Our results also improve the recent results of Eslamian [16].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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