Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 804152, 9 pages
http://dx.doi.org/10.1155/2013/804152

Research Article

Some Difference Inequalities for Iterated Sums with Applications

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Received 4 September 2013; Revised 13 November 2013; Accepted 28 November 2013

Academic Editor: Jaume Giné

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The main objective of this paper is to establish two new nonlinear sum-difference inequalities with multiple iterated sums. Under several practical assumptions, the inequalities are solved through rigorous analysis, and explicit bounds for the unknown functions are given clearly. These new inequalities can be used as handy tools in the study of the estimation of solutions of difference equations.

1. Introduction

One of the best known and widely used inequalities in the study of nonlinear differential equations is Gronwall-Bellman inequality [1, 2], which can be stated as follows: if u and f are nonnegative continuous functions on an interval $[a,b]$ satisfying

$$u(t) \le c + \int_a^t f(s)\,u(s)\,ds, \quad t \in [a,b], \tag{1}$$

for some constant $c \ge 0$, then

$$u(t) \le c\exp\left(\int_a^t f(s)\,ds\right), \quad t \in [a,b]. \tag{2}$$

It has become one of the very few classic and most influential results in the theory and applications of inequalities. Because of its fundamental importance, over the years, many generalizations and analogous results of (2) have been established, such as [3–14].

Among these references, Baĭnov and Simeonov [4, P. 107] considered the following interesting Gronwall-type inequality:

$$\begin{aligned} u(t) &\le a(t) + \sum_{i=1}^{n}\int_\alpha^t f_1(t,t_1) \\ &\quad\times\left(\int_\alpha^{t_1} f_2(t_1,t_2)\cdots\left(\int_\alpha^{t_{i-1}} f_i(t_{i-1},t_i)\,u(t_i)\,dt_i\right)\cdots\right)dt_1. \end{aligned} \tag{3}$$

Kim [8] considered analogous Gronwall-type integral inequalities involving iterated integrals,

$$\begin{aligned} u(t) &\le a + b(t)\left(\int_\alpha^t f_1(t_1)\,u(t_1)\log u(t_1)\,dt_1\right. \\ &\quad + \sum_{i=2}^{n}\int_\alpha^t g_1(t_1)\left(\int_\alpha^{t_1} g_2(t_2)\right. \\ &\quad\quad \times\left(\cdots\left(\int_\alpha^{t_{i-2}} g_{i-1}(t_{i-1})\right.\right. \end{aligned}$$

$$\times\left(\int_{\alpha}^{t_{i-1}} f_i(t_i)\,u(t_i) \times \log(u(t_i))\,dt_i\right)dt_{i-1}\Bigg) \cdots\Bigg)dt_2\Bigg)dt_1\Bigg). \tag{4}$$

In 2011, Abdeldaim and Yakout [12] studied some new integral inequalities of Gronwall-Bellman-Pachpatte type such as

$$\begin{aligned} u(t) &\le u_0 + \int_0^t f(s)\,u(s) \times\left[u(s) + \int_0^s h(\tau)\left[u(\tau) + \int_0^\tau g(\xi)\,u(\xi)\,d\xi\right]d\tau\right]ds, \\ u(t) &\le u_0 + \int_0^t \left[f(s)\,u(s) + q(s)\right]ds + \int_0^t f(s)\,u(s)\left[u(s) + \int_0^s g(\tau)\,u(\tau)\,d\tau\right]ds. \end{aligned} \tag{5}$$

Along with the development of the theory of integral inequalities and the theory of difference equations, more and more attentions are paid to discrete versions of Gronwall-type inequalities; for detailed information, please refer to the literatures [15–35]. For instance, Pachpatte [19] considered the following discrete inequality:

$$u(n) \le u_0 + \sum_{s=n_0}^{n-1} f(s)\,u(s) + \sum_{s=n_0}^{n-1} g(s) \times\left(\sum_{t=n_0}^{s-1} h(t)\left(\sum_{\tau=n_0}^{t-1} k(\tau)\,u^p(\tau)\right)\right). \tag{6}$$

In 2006, Cheung and Ren [24] studied

$$u^p(m,n) \le c + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} a(s,t)\,u^q(s,t) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t)\,u^q(s,t)\,w(u(s,t)). \tag{7}$$

Later, Zheng et al. [31] discussed the following discrete inequality:

$$u(n) \le a(n) + \sum_{i=1}^{k}\sum_{s=0}^{n-1} f_i(n,s)\,w_i(u(s)). \tag{8}$$

In 2012, Zhou et al. [33] studied the following inequalities:

$$\begin{aligned} u(n) \le{}& a(n) + \sum_{s=n_0}^{n-1} f_1(n,s)\,w(u(s)) + \sum_{s=n_0}^{n-1} f_1(n,s)\,w(u(s)) \sum_{\tau=n_0}^{s-1} f_2(s,\tau)\,w(u(\tau)) \\ &+ \sum_{s=n_0}^{n-1} f_1(n,s)\,w(u(s)) \times \sum_{\tau=n_0}^{s-1} f_2(s,\tau) \sum_{\xi=n_0}^{\tau-1} f_3(\tau,\xi)\,w(u(\xi)), \\ u(n) \le{}& a(n) + \sum_{s=n_0}^{n-1} f_1(n,s)\,w_1(u(s)) + \sum_{s=n_0}^{n-1} f_1(n,s)\,w_1(u(s)) \sum_{\tau=n_0}^{s-1} f_2(s,\tau)\,w_2(u(\tau)) \\ &+ \sum_{s=n_0}^{n-1} f_1(n,s)\,w_1(u(s)) \times \sum_{\tau=n_0}^{s-1} f_2(s,\tau)\,w_2(u(\tau)) \sum_{\xi=n_0}^{\tau-1} f_3(\tau,\xi)\,w_3(u(\xi)). \end{aligned} \tag{9}$$

However, the above results are not applicable to some certain inequalities with multiple iterated sums. Hence, it is desirable to consider more general difference inequalities of these extended types. They can be used in the study of certain classes of difference equations or applied in many practical engineering problems.

Motivated by the results given in [7, 8, 12, 19, 24, 25, 29, 33], in this paper we discuss the following two types of inequalities:

$$u(n) \le a(n) + \sum_{t_1=n_0}^{n-1} f_1(n,t_1) \times\left(\sum_{t_2=n_0}^{t_1-1} f_2(t_1,t_2)\cdots\left(\sum_{t_k=n_0}^{t_{k-1}} f_k(t_{k-1},t_k)\,u^p(t_k)\right)\cdots\right), \tag{10}$$

$$u(n) \le a(n) + c(n)\left[\sum_{t_1=n_0}^{n-1} f_1(n,t_1)\, g(u(t_1)) + \sum_{i=2}^{k}\sum_{t_1=n_0}^{n-1} f_1(n,t_1) \times \left(\sum_{t_2=n_0}^{t_1-1} f_2(t_1,t_2)\cdots \times \left(\sum_{t_i=n_0}^{t_{i-1}} f_i(t_{i-1},t_i) \times g(u(t_i))\right)\cdots\right)\right]. \tag{11}$$

All the assumptions on (10) and (11) are given in the next sections. The inequalities (10) and (11) consist of multiple iterated sums. Under several practical assumptions, the inequalities are solved through rigorous analysis, and explicit bounds for the unknown functions are given clearly. Further, the derived results are applied to study the estimation of solutions of difference equations.

2. Main Results

Throughout this paper, let $\mathbb{N}_{n_0} := \{n_0, n_0+1, n_0+2, \ldots\}$ and $\mathbb{N}_{n_0}^{b} := \{n_0, n_0+1, n_0+2, \ldots, n_0+n = b\}$ ($n_0 \in \mathbb{N}_0, n, b \in \mathbb{N}$). For function $u(n)$, its difference is defined by $\Delta u(n) = u(n+1) - u(n)$. Obviously, the linear difference equation $\Delta u(n) = f(n)$ with the initial condition $u(n_0) = 0$ has the solution $u(n) = \sum_{s=n_0}^{n-1} f(s)$. For convenience, in the sequel we complementarily define that $\sum_{s=n_0}^{n_0-1} f(s) = 0$.

Lemma 1. *Let $u(n)$, $a(n)$ and $c(n)$ be real-valued nonnegative functions defined on $\mathbb{N}_0$ and satisfy the inequality*

$$\Delta u(n) \le a(n)\,u(n) + c(n), \quad \forall n \in \mathbb{N}_{n_0}, \tag{12}$$

where u_{n_0} is a nonnegative constant. Then,

$$u(n) \le \left(u_{n_0} + \sum_{s=n_0}^{n-1} c(s) \prod_{t=n_0}^{s} (1+a(t))^{-1}\right) \times \prod_{s=n_0}^{n-1} (1+a(s)), \quad \forall n \in \mathbb{N}_{n_0}. \tag{13}$$

Proof. From (12), we have

$$u(n+1) - (1+a(n))\,u(n) \le c(n), \quad \forall n \in \mathbb{N}_{n_0}. \tag{14}$$

Multiplying by $\prod_{s=n_0}^{n-1}(1+a(s))^{-1}$ on both sides of the above inequality (14) and summing up both sides from n_0 to $n-1$, we obtain

$$u(n)\prod_{s=n_0}^{n-1}(1+a(s))^{-1} - u_{n_0} \le \sum_{s=n_0}^{n-1} c(s) \prod_{t=n_0}^{s} (1+a(t))^{-1}, \quad \forall n \in \mathbb{N}_{n_0}. \tag{15}$$

From (15), we obtain the desired estimate (13). □

Theorem 2. *Let $u(n)$ and $a(n)$ be nonnegative functions defined on $\mathbb{N}_{n_0}$ with $a(n)$ nondecreasing on $\mathbb{N}_{n_0}$. Moreover, let $f_i(n,s)$, $i = 1, 2, \ldots, k$, be nonnegative functions for $n_0 \le s \le n$ ($n_0, n, s \in \mathbb{N}_{n_0}$) and nondecreasing in n for fixed $s \in \mathbb{N}_{n_0}$. If $p \ge 0$ and p is not equal to 1, then the discrete inequality* (10) *gives*

$$u(n) \le V_1(n,n), \quad \forall n \in \mathbb{N}_{n_0}^{b_1}, \tag{16}$$

where $V_1(n,n)$ can be successively determined from the formulas

$$V_k(M,n) = \exp\left(W_1^{-1}\left(W_1\left(\ln(a(M)) + \sum_{s=n_0}^{M-1}\left(\sum_{i=1}^{k-1} f_i(M,s)\right)\right) + \sum_{s=n_0}^{n-1} f_k(M,s)\right)\right), \tag{17}$$

$$V_j(M,n) \le \left(a(M) + \sum_{s=n_0}^{n-1} f_j(M,s)\, V_{j+1}(M,s) \times \prod_{t=n_0}^{s}\left(1 + \left(\sum_{i=1}^{j-1} f_i(M,t) - f_j(M,t)\right)\right)^{-1}\right) \times \prod_{s=n_0}^{n-1}\left(1 + \left(\sum_{i=1}^{j-1} f_i(M,s) - f_j(M,s)\right)\right) = V_j(M,n), \tag{18}$$

for $j = k-1, \ldots, 2, 1$, $n \in \mathbb{N}_{n_0}^{M}$,

$$W_1(x) = \int_{x_0}^{x} \frac{ds}{\exp((p-1)s)}, \quad x_0 > 0, \tag{19}$$

where W_1^{-1} is the inverse functions of W_1, $M \in \mathbb{N}_{n_0}$, $M \le b_1$ is chosen arbitrarily, and b_1 is the largest natural number such that

$$W_1\left(\ln(a(b_1)) + \sum_{s=n_0}^{b_1-1}\left(\sum_{i=1}^{k-1} f_i(b_1,s)\right)\right) + \sum_{s=n_0}^{b_1-1} f_k(b_1,s) \in \operatorname{Dom}\left(W_1^{-1}\right). \tag{20}$$

Remark 3. Firstly, from (17) and (18), we obtain $V_1(M,n)$; then let $M = n$, and we get $V_1(n,n)$ since M is chosen arbitrarily.

Remark 4. We can obtain b_1 using MATLAB program: firstly let $b_1 = n_0$, when $W_1(\ln(a(b_1)) + \sum_{s=n_0}^{b_1-1}(\sum_{i=1}^{k-1} f_i(b_1,s))) +$

$\sum_{s=n_0}^{b_1-1} f_k(b_1, s) < W_1(\infty)$; let $b_1 = b_1 + 1$, when $W_1(\ln(a(b_1))+\sum_{s=n_0}^{b_1-1}(\sum_{i=1}^{k-1} f_i(b_1, s))) + \sum_{s=n_0}^{b_1-1} f_k(b_1, s) < W_1(\infty)$; let $b_1 = b_1 + 1$, and so on until $W_1(\ln(a(b_1)) + \sum_{s=n_0}^{b_1-1}(\sum_{i=1}^{k-1} f_i(b_1, s))) + \sum_{s=n_0}^{b_1-1} f_k(b_1, s) \geq W_1(\infty)$. If $W_1(\ln(a(b_1)) + \sum_{s=n_0}^{b_1-1}(\sum_{i=1}^{k-1} f_i(b_1, s))) + \sum_{s=n_0}^{b_1-1} f_k(b_1, s) < W_1(\infty)$, for all $b_1 \in \mathbb{N}_{n_0}$, then $b_1 = \infty$.

Proof. Fix $M \in \mathbb{N}_{n_0}^{b_1}$, where M is chosen arbitrarily and b_1 is defined by (20). For $n \in \mathbb{N}_{n_0}^{M}$, from (10) we have

$$u(n) \leq a(M) + \sum_{t_1=n_0}^{n-1} f_1(M, t_1) \times \left(\sum_{t_2=n_0}^{t_1-1} f_2(M, t_2) \cdots \left(\sum_{t_k=n_0}^{t_{k-1}} f_k(M, t_k) u^p(t_k)\right) \cdots\right). \tag{21}$$

Now we introduce the functions

$$v_1(n) = a(M) + \sum_{t_1=n_0}^{n-1} f_1(M, t_1) \times \left(\sum_{t_2=n_0}^{t_1-1} f_2(M, t_2) \cdots \left(\sum_{t_k=n_0}^{t_{k-1}} f_k(M, t_k) u^p(t_k)\right) \cdots\right), \tag{22}$$

$$v_j(n) = v_{j-1}(n) + \sum_{t_j=n_0}^{n-1} f_j(M, t_j) \times \left(\sum_{t_{j+1}=n_0}^{t_j-1} f_{j+1}(M, t_{j+1}) \cdots \times \left(\sum_{t_k=n_0}^{t_{k-1}} f_k(M, t_k) u^p(t_k)\right) \cdots\right). \tag{23}$$

For $n \in \mathbb{N}_{n_0}^{M}$ and $j = 2, 3, \ldots, k$, then v_j, $j = 1, 2, \ldots, k$, are all positive and nondecreasing functions on $\mathbb{N}_{n_0}^{M}$ with $v_j(n_0) = a(M)$, $j = 1, 2, \ldots, k$, and the inequalities (22) and (23) imply that

$$u(n) \leq v_1(n) \leq v_2(n) \leq \cdots \leq v_k(n), \quad \forall n \in \mathbb{N}_{n_0}^{M}. \tag{24}$$

From (22), we observe that

$$\begin{aligned}\Delta v_1(n) &= v_1(n+1) - v_1(n) \\ &= f_1(M, n) \times \left(\sum_{t_2=n_0}^{n-1} f_2(M, t_2) \cdots \left(\sum_{t_k=n_0}^{t_{k-1}} f_k(M, t_k) u^p(t_k)\right) \cdots\right) \\ &\leq f_1(M, n)(v_2(n) - v_1(n)) \\ &= -f_1(M, n) v_1(n) + f_1(M, n) v_2(n), \quad \forall n \in \mathbb{N}_{n_0}^{M}.\end{aligned} \tag{25}$$

We claim that

$$\Delta v_j(n) \leq \left(\sum_{i=1}^{j-1} f_i(M, n) - f_j(M, n)\right) v_j(n) + f_j(M, n) v_{j+1}(n), \tag{26}$$

$$\Delta v_k(n) \leq \left(\sum_{i=1}^{k-1} f_i(M, n)\right) v_k(n) + f_k(M, n) v_k^p(n), \tag{27}$$

for $n \in \mathbb{N}_{n_0}^{M}$, $j = 2, 3, \ldots, k-1$.

Now we prove (26) and (27) by induction. Obviously, (26) is true for $j = 1$ by (26). We make the inductive assumption that (26) is true for $j-1$. By the inductive assumption and (24), from (23) we obtain

$$\begin{aligned}\Delta v_j(n) &\leq \Delta v_{j-1}(n) + f_j(M, n) \times \left(\sum_{t_{j+1}=n_0}^{t_j-1} f_{j+1}(M, t_{j+1}) \cdots \times \left(\sum_{t_k=n_0}^{t_{k-1}} f_k(M, t_k) u^p(t_k)\right) \cdots\right) \\ &\leq \left(\sum_{i=1}^{j-2} f_i(M, n) - f_{j-1}(M, n)\right) v_{j-1}(n) + f_{j-1}(M, n) v_j(n) + f_j(M, n)\left(v_{j+1}(n) - v_j(n)\right) \\ &\leq \left(\sum_{i=1}^{j-2} f_i(M, n)\right) v_j(n) + f_{j-1}(M, n) v_j(n) + f_j(M, n) v_{j+1}(n) - f_j(M, n) v_j(n) \\ &= \left(\sum_{i=1}^{j-1} f_i(M, n) - f_j(M, n)\right) v_j(n) + f_j(M, n) v_{j+1}(n), \quad \forall n \in \mathbb{N}_{n_0}^{M}.\end{aligned} \tag{28}$$

It actually proves (26) by induction. From (23) and (26), we have

$$\begin{aligned}\Delta v_k(n) &= \Delta v_{k-1}(n) + f_k(M,n)\,u^p(n)\\ &\le \left(\sum_{i=1}^{k-2} f_i(M,n) - f_{k-1}(M,n)\right) v_{k-1}(n)\\ &\quad + f_{k-1}(M,n)\,v_k(n) + f_k(M,n)\,v_k^p(n)\\ &\le \left(\sum_{i=1}^{k-1} f_i(M,n)\right) v_k(n) + f_k(M,n)\,v_k^p(n),\\ &\qquad \forall n \in \mathbb{N}_{n_0}^M.\end{aligned} \tag{29}$$

It proves (27). From (27), we have

$$\frac{\Delta v_k(n)}{v_k(n)} \le \left(\sum_{i=1}^{k-1} f_i(M,n)\right) + f_k(M,n)\,v_k^{p-1}(n), \qquad \forall n \in \mathbb{N}_{n_0}^M. \tag{30}$$

On the other hand, by the mean-value theorem for integrals, for arbitrarily given integers $n, n+1 \in \mathbb{N}_{n_0}^M$, there exists ξ in the open interval $(v_k(n), v_k(n+1))$ such that

$$\begin{aligned}\ln(v_k(n+1)) - \ln(v_k(n)) &= \int_{v_k(n)}^{v_k(n+1)} \frac{ds}{s}\\ &= \frac{\Delta v_k(n)}{\xi} \le \frac{\Delta v_k(n)}{v_k(n)},\\ &\qquad \forall n \in \mathbb{N}_{n_0}^M.\end{aligned} \tag{31}$$

By setting $n = s$ in (31) and substituting $s = n_0, n_0+1, n_0+2, \ldots, n-1$, successively, we obtain

$$\begin{aligned}\ln(v_k(n)) &\le \ln(v_k(n_0)) + \sum_{s=n_0}^{n-1}\left(\sum_{i=1}^{k-1} f_i(M,s)\right)\\ &\quad + \sum_{s=n_0}^{n-1} f_k(M,s)\,v_k^{p-1}(s)\\ &\le \ln(v_k(n_0)) + \sum_{s=n_0}^{M-1}\left(\sum_{i=1}^{k-1} f_i(M,s)\right)\\ &\quad + \sum_{s=n_0}^{n-1} f_k(M,s)\,v_k^{p-1}(s), \quad \forall n \in \mathbb{N}_{n_0}^M.\end{aligned} \tag{32}$$

Let $w_1(n)$ denote the right-hand side of (32), which is a positive and nondecreasing function on $\mathbb{N}_{n_0}^M$ with

$$w_1(n_0) = \ln(v_k(n_0)) + \sum_{s=n_0}^{M-1}\left(\sum_{i=1}^{k-1} f_i(M,s)\right). \tag{33}$$

Then, (32) is equivalent to

$$v_k(n) \le \exp(w_1(n)), \quad \forall n \in \mathbb{N}_{n_0}^M. \tag{34}$$

By the definition of w_1, we obtain

$$\begin{aligned}\Delta w_1(n) &= f_k(M,n)\,v_k^{p-1}(n)\\ &\le f_k(M,n)\exp((p-1)\,w_1(n)),\\ &\qquad \forall n \in \mathbb{N}_{n_0}^M.\end{aligned} \tag{35}$$

From (34) and (35), we get

$$\frac{\Delta w_1(n)}{\exp((p-1)\,w_1(n))} \le f_k(M,n), \quad \forall n \in \mathbb{N}_{n_0}^M. \tag{36}$$

Once again, performing the similar procedure from (30) to (32), (36) gives

$$W_1(w_1(n)) \le W_1(w_1(n_0)) + \sum_{s=n_0}^{n-1} f_k(M,s), \qquad \forall n \in \mathbb{N}_{n_0}^M, \tag{37}$$

where W_1 is defined by (19). By combining (33), (34), and (37), we can obtain that

$$\begin{aligned}v_k(n) &\le \exp(w_1(n))\\ &\le \exp\left(W_1^{-1}\left(W_1(w_1(n_0)) + \sum_{s=n_0}^{n-1} f_k(M,s)\right)\right)\\ &\le \exp\left(W_1^{-1}\left(W_1\left(\ln(a(M)) + \sum_{s=n_0}^{M-1}\left(\sum_{i=1}^{k-1} f_i(M,s)\right)\right) + \sum_{s=n_0}^{n-1} f_k(M,s)\right)\right)\\ &= V_k(M,n), \quad \forall n \in \mathbb{N}_{n_0}^M,\end{aligned} \tag{38}$$

where $V_k(M,n)$ is defined by (17). Applying Lemma 1 to (26) for $j = k-1, \ldots, 2, 1$, we have

$$\begin{aligned}v_j(n) &\le \left(a(M) + \sum_{s=n_0}^{n-1} f_j(M,s)\,V_{j+1}(M,s) \times \prod_{t=n_0}^{s}\left(1 + \left(\sum_{i=1}^{j-1} f_i(M,t) - f_j(M,t)\right)\right)^{-1}\right)\\ &\quad \times \prod_{s=n_0}^{n-1}\left(1 + \left(\sum_{i=1}^{j-1} f_i(M,s) - f_j(M,s)\right)\right)\\ &= V_j(M,n), \quad \forall n \in \mathbb{N}_{n_0}^M,\end{aligned} \tag{39}$$

where $V_j(M,n)$ is defined by (18). From (24) and (39), we have

$$u(n) \le v_1(n) \le V_1(M,n), \quad \forall n \in \mathbb{N}_{n_0}^M. \tag{40}$$

Since $M \in \mathbb{N}_{n_0}$ is arbitrary, from (40), we get the required estimate

$$u(n) \le V_1(n,n), \quad \forall n \in \mathbb{N}_{n_0}^{b_1}, \tag{41}$$

where b_1 is defined by (20). Theorem 2 is proved. □

Theorem 5. *Let $u(n)$, $a(n)$, and $c(n)$ be nonnegative functions defined on $\mathbb{N}_{n_0}$ with $a(n)$ and $c(n)$ nondecreasing on $\mathbb{N}_{n_0}$, and let $f_i(n,s)$, $i = 1,2,\ldots,k$, be nonnegative functions for $n, s \in \mathbb{N}_{n_0}$, $n_0 \le s \le n$, which are nondecreasing in n for fixed $s \in \mathbb{N}_{n_0}$. Suppose that $g(u)$ is a nondecreasing continuous function on $[0,\infty)$ with $g(u) > 0$ for $u > 0$. The inequality (11) implies that*

$$u(n) \le G^{-1}\left(G(a(n)) + c(n)\sum_{s=n_0}^{n-1} f_1(n,s)E(n,s)\right), \quad \forall n \in \mathbb{N}_{n_0}^{b_2}, \tag{42}$$

where G^{-1} is the inverse function of G,

$$G(u) = \int_{u_0}^{u}\frac{ds}{g(s)}, \quad u_0 > 0, \tag{43}$$

$$\begin{aligned} E(n,s) &:= \{1 + f_2(n,s)[1 + f_3(n,s) \\ &\quad \times (\cdots (1 + f_{k-1}(n,s)\cdots \\ &\quad \times (1 + f_k(n,s)))\cdots)]\}, \end{aligned} \tag{44}$$

and b_2 is the largest natural number such that

$$G(a(b_2)) + c(b_2)\sum_{s=n_0}^{b_2-1} f_1(b_2,s)E(b_2,s) \in \operatorname{Dom}(G^{-1}). \tag{45}$$

Remark 6. We can obtain b_2 using Matlab program similar to Remark 4.

Proof. Let the function $a(n)$ be positive. Fix $M \in \mathbb{N}_{n_0}^{b_2}$, where M is chosen arbitrarily and b_2 is defined by (45). For $n \in \mathbb{N}_{n_0}^{M}$, from (11) we have

$$\begin{aligned} u(n) &\le a(M) + c(M)\Bigg[\sum_{t_1=n_0}^{n-1} f_1(M,t_1)g(u(t_1)) \\ &\quad + \sum_{i=2}^{k}\sum_{t_1=n_0}^{n-1} f_1(M,t_1) \\ &\quad \times \left(\sum_{t_2=n_0}^{t_1-1} f_2(M,t_2)\cdots \right.\\ &\quad \left.\times\left(\sum_{t_i=n_0}^{t_{i-1}} f_i(M,t_i)g(u(t_i))\right)\cdots\right)\Bigg]. \end{aligned} \tag{46}$$

We denote the right-hand side of (46) by $y(n)$ for $n \in \mathbb{N}_{n_0}^{M}$. Then $y(n_0) = a(M)$, the function $y(n)$ is positive and nondecreasing in $n \in \mathbb{N}_{n_0}^{M}$, $u(n) \le y(n)$, and

$$\begin{aligned} \Delta y(n) &= c(M)\Bigg[f_1(M,n)g(u(n)) + f_1(M,n) \\ &\quad \times\left(\sum_{t_2=n_0}^{n-1} f_2(M,t_2)g(u(t_2))\right) + \sum_{i=3}^{k} f_1(M,n) \\ &\quad \times\left(\sum_{t_2=n_0}^{n-1} f_2(M,t_2)\cdots\right. \\ &\quad \left.\times\left(\sum_{t_i=n_0}^{t_{i-1}} f_i(M,t_i)g(u(t_i))\right)\cdots\right)\Bigg] \\ &= c(M)f_1(M,n)\Bigg[g(u(n)) + \sum_{t_2=n_0}^{n-1} f_2(M,t_2)g(u(t_2)) \\ &\quad + \sum_{i=3}^{k}\left(\sum_{t_2=n_0}^{n-1} f_2(M,t_2)\cdots\right. \\ &\quad \left.\times\left(\sum_{t_i=n_0}^{t_{i-1}} f_i(M,t_i)g(u(t_i))\right)\cdots\right)\Bigg], \quad \forall n \in \mathbb{N}_{n_0}^{M}. \end{aligned} \tag{47}$$

Define a function $y_1(n)$ by

$$\begin{aligned} y_1(n) &= \sum_{t_2=n_0}^{n-1} f_2(M,t_2)g(u(t_2)) \\ &\quad + \sum_{i=3}^{k}\left(\sum_{t_2=n_0}^{n-1} f_2(M,t_2)\cdots\right. \\ &\quad \left.\times\left(\sum_{t_i=n_0}^{t_{i-1}} f_i(M,t_i)g(u(t_i))\right)\cdots\right), \end{aligned} \tag{48}$$

for all $n \in \mathbb{N}_{n_0}^{M}$. From (47) and (48), we have

$$\Delta y(n) = c(M)f_1(M,n)[g(u(n)) + y_1(n)], \quad \forall n \in \mathbb{N}_{n_0}^{M}. \tag{49}$$

From (48), we have

$$\Delta y_1(n) = f_2(M,n)\left[g(u(n)) + \sum_{t_3=n_0}^{n-1} f_3(M,t_3)\, g(u(t_3)) + \sum_{i=4}^{k}\left(\sum_{t_3=n_0}^{n-1} f_3(M,t_3)\cdots \times\left(\sum_{t_i=n_0}^{t_{i-1}} f_i(M,t_i)\, g(u(t_i))\right)\cdots\right)\right], \quad \forall n \in \mathbb{N}_{n_0}^{M}. \tag{50}$$

From (50), we get

$$\Delta y_1(n) = f_2(M,n)\left[g(u(n)) + y_2(n)\right], \quad \forall n \in \mathbb{N}_{n_0}^{M}, \tag{51}$$

where

$$y_2(n) = \sum_{t_3=n_0}^{n-1} f_3(M,t_3)\, g(u(t_3)) + \sum_{i=4}^{k}\left(\sum_{t_2=n_0}^{n-1} f_3(M,t_3)\cdots\left(\sum_{t_i=n_0}^{t_{i-1}} f_i(M,t_i)\, g(u(t_i))\right)\cdots\right), \tag{52}$$

for all $n \in \mathbb{N}_{n_0}^{M}$.

Continuing in this way, we obtain

$$\Delta y_{k-2}(n) = f_{k-1}(M,n)\left[g(u(n)) + y_{k-1}(n)\right], \quad \forall n \in \mathbb{N}_{n_0}^{M}, \tag{53}$$

where

$$y_{k-1}(n) = \sum_{t_n=n_0}^{n-1} f_k(M,t_k)\, g(u(t_k)), \quad \forall n \in \mathbb{N}_{n_0}^{M}. \tag{54}$$

From (54) and the inequality $u(n) \le y(n)$, we have

$$\frac{\Delta y_{k-1}(n)}{g(y(n))} \le f_k(M,n), \quad \forall n \in \mathbb{N}_{n_0}^{M}. \tag{55}$$

We define the functions $\tilde{y}(s), \tilde{y}_i(s)$ $(i = 1,2\ldots,k-1)$, which are nondecreasing and continuously differentiable on $[n_0,\infty)$ with $\tilde{y}(n) = y(n)$, $\tilde{y}_i(n) = y_i(n)$ $(i = 1,2\ldots,k-1)$ on $\mathbb{N}_{n_0}^{M}$.

On the other hand, by the formula of partial integration, we have

$$\int_n^{n+1} \frac{\tilde{y}_{k-1}'(s)}{g(\tilde{y}(s))}ds = \frac{y_{k-1}(n)}{g(y(n))} + \int_n^{n+1} \frac{\tilde{y}_{k-1}(s)\, g'(\tilde{y}(s))\, \tilde{y}'(s)}{g^2(\tilde{y}(s))}ds, \quad \forall n \in \mathbb{N}_{n_0}^{M}. \tag{56}$$

By the monotonicity of g, y, from (56) we have

$$\int_n^{n+1} \frac{\tilde{y}_{k-1}'(s)}{g(\tilde{y}(s))}ds \ge \frac{y_{k-1}(n)}{g(y(n))}, \quad \forall n \in \mathbb{N}_{n_0}^{M}. \tag{57}$$

By the mean-value theorem for integrals, for arbitrarily given integers $n, n+1 \in \mathbb{N}_{n_0}^{M}$, there exists ξ in the open interval $(n, n+1)$ such that

$$\int_n^{n+1} \frac{\tilde{y}_{k-1}'(s)}{g(\tilde{y}(s))}ds = \int_n^{n+1} \frac{d(\tilde{y}_{k-1}(s))}{g(\tilde{y}(s))} = \frac{1}{g(y(\xi))}\int_n^{n+1} d(\tilde{y}_{k-1}(s)) \le \frac{\Delta y_{k-1}(n)}{g(y(n))}, \quad \forall n \in \mathbb{N}_{n_0}^{M}. \tag{58}$$

From (55), (57), and (58), we have

$$\frac{y_{k-1}(n)}{g(y(n))} \le \frac{\Delta y_{k-1}(n)}{g(y(n))} \le f_k(M,n), \quad \forall n \in \mathbb{N}_{n_0}^{M}. \tag{59}$$

Next, from the inequalities (53) and (59), we have

$$\frac{\Delta y_{k-2}(n)}{g(y(n))} \le f_{k-1}(M,n)\left[1 + f_k(M,n)\right], \quad \forall n \in \mathbb{N}_{n_0}^{M}. \tag{60}$$

Once again, applying the same procedure from (56) to (59) to the inequality (60), we get

$$\frac{y_{k-2}(n)}{g(y(n))} \le \frac{\Delta y_{k-2}(n)}{g(y(n))} \le f_{k-1}(M,n)\left[1 + f_k(M,n)\right], \quad \forall n \in \mathbb{N}_{n_0}^{M}. \tag{61}$$

Proceeding in this way, we obtain

$$\frac{y_1(n)}{g(y(n))} \le \frac{\Delta y_1(n)}{g(y(n))} \le f_2(M,n) \times \{1 + f_3(M,n) \times [1 + f_4(M,n) \times (\cdots(1 + f_{k-1}(M,n) \times (1 + f_k(M,n)))\cdots)]\}, \quad \forall n \in \mathbb{N}_{n_0}^{M}. \tag{62}$$

Using the inequalities (49) and (62), we have

$$\frac{\Delta y(n)}{g(y(n))} \leq c(M) f_1(M,n) \times \{1 + f_2(M,n)[1 + f_3(M,n) \times (\cdots(1+f_{k-1}(M,n) \times 1 + f_k(M,n))\cdots)]\} = c(M) f_1(M,n) E(M,n), \quad \forall n \in \mathbb{N}_{n_0}^M, \tag{63}$$

where $E(M,n)$ is defined by (44).

Once again, using the mean-value theorem for integrals, for arbitrarily given integers $n, n+1 \in \mathbb{N}_{n_0}^M$, there exists ξ in the open interval $(y(n), y(n+1))$ such that

$$G(y(n+1)) - G(y(n)) = \int_{y(n)}^{y(n+1)} \frac{ds}{g(s)} = \frac{\Delta y(n)}{g(\xi)} \leq \frac{\Delta y(n)}{g(y(n))}, \quad \forall n \in \mathbb{N}_{n_0}^M, \tag{64}$$

where G is defined by (43). Using (63), (64), and $y(n_0) = a(M)$, we obtain

$$u(n) \leq y(n) \leq G^{-1}\left(G(a(M)) + c(M)\sum_{s=n_0}^{n-1} f_1(M,s) E(M,s)\right), \quad \forall n \in \mathbb{N}_{n_0}^M. \tag{65}$$

In (65), let $n = M$; we have

$$u(M) \leq G^{-1}\left(G(a(M)) + c(M)\sum_{s=n_0}^{M-1} f_1(M,s) E(M,s)\right), \quad \forall n \in \mathbb{N}_{n_0}^M. \tag{66}$$

Due to the randomness of T, (42) is achieved immediately from (66). □

3. Application

In this section, we apply Theorem 5 to the following difference equation:

$$\Delta x(n) = F\left(n, x(n), \sum_{s=n_0}^{n-1} z(s, x(s))\right), \quad \forall n \in \mathbb{N}_{n_0}. \tag{67}$$

Corollary 7. *Assume that F is defined on $\mathbb{N}_{n_0} \times [0,\infty) \times [0,\infty)$, and there exist nonnegative functions $d_i(n)$, $i = 1, 2$, such that*

$$\begin{aligned} |F(n,x,y)| &\leq d_1(n) g(|x|) + d_1(n) y, \\ |z(s,x)| &\leq d_2(n) g(|x|), \end{aligned} \tag{68}$$

where the function g is as in Theorem 5. If $x(n)$ is any solution of the problem (67), then

$$|x(n)| \leq G^{-1}\left(G(|x(n_0)|) + \sum_{s=n_0}^{n-1} d_1(s) E(s)\right), \quad \forall n \in \mathbb{N}_{n_0}^{b_3}, \tag{69}$$

where the functions G, G^{-1} are as in Theorem 5,

$$E(n) = 1 + d_1(n)(1 + d_2(n)), \tag{70}$$

and b_3 is the largest natural number such that

$$G(|x(n_0)|) + \sum_{s=n_0}^{b_3-1} d_1(s) E(s) \in \operatorname{Dom}(G^{-1}). \tag{71}$$

Proof. The difference equation (67) is equivalent to

$$x(n) = x(n_0) + \sum_{s=n_0}^{n-1} F\left(s, x(s), \sum_{t=n_0}^{s-1} z(t, x(t))\right), \quad \forall n \in \mathbb{N}_{n_0}. \tag{72}$$

Using (68), from (72), we have

$$|x(n)| \leq |x(n_0)| + \sum_{s=n_0}^{n-1} d_1(s) g(|x(s)|) + \sum_{s=n_0}^{n-1} d_1(s)\left(\sum_{t=n_0}^{s-1} d_2(t) g(|x(s)|)\right), \quad \forall n \in \mathbb{N}_{n_0}. \tag{73}$$

Notice that, by the assumption, all functions in (73) satisfy the conditions of Theorem 5. Applying Theorem 5 to the inequality (73), (69) is immediately derived. This completes the proof of Corollary 7. □

Conflict of Interests

The authors declare that they have no competing interests.

Acknowledgments

This research was supported, in part, by the National Natural Science Foundation of China under Grant 11161018 by the Guangxi Natural Science Foundation of China under Grant 2012GXNSFAA053009, and by the Natural Science Foundation of Fujian province of China under Grant 2012J01014. The authors are grateful to the anonymous referees for their careful comments and valuable suggestions on this paper.

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