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Research Article

Cucker-Smale Flocking with Bounded Cohesive and Repulsive Forces

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This paper proposes two Cucker-Smale-type flocking models by introducing both cohesive and repulsive forces to second-order multiagent systems. Under some mild conditions on the initial state of the flocking system, it is shown that the velocity consensus of the agents can be reached independent of the parameter which describes the decay of communication rates. In particular, the collision between any two agents can always be avoided by designing an appropriate bounded repulsive function based on the initial energy of the flock. Numerical examples are given to demonstrate the effectiveness of the theoretical analysis.

1. Introduction

Over the past few decades, the flocking problem of multiagent systems has attracted increasing attention due to its wide applications in many fields such as unmanned air vehicles, mobile robots, and sensor networks. To simulate the collective behaviors of birds and fish, Reynolds [1] proposed three well-known flocking rules for multiagent systems, that is, center cohesion, collision avoidance, and velocity consensus. In [2], Vicsek et al. introduced a simple discrete time flocking model to study the emergence of autonomous agents moving in the plane with the same speed but with different headings. By using nonnegative matrix and algebraic graph theories, Jadbabaie et al. [3] provided the theoretical analysis for Vicsek's flocking model. Olfati-Saber [4] presented a systematic framework to design distributed flocking algorithms for multiagent systems. Tanner et al. [5] investigated the flocking behaviors of multiagent systems with fixed and switching network topologies. In [6], both theoretical and experimental results were given for a flock of mobile robots. In [7–10], some strategies were developed to ensure the connectivity of the time-varying communication topology.

In 2007, Cucker and Smale [11] proposed a flocking model to investigate the emergence behavior in multiagent systems. For a network of N autonomous agents, the continuous-time version of the Cucker-Smale flocking model is described as follows [11]:

$$\begin{aligned} \dot{x}_i(t) &= v_i(t), \quad i = 1, \ldots, N, \\ \dot{v}_i(t) &= \sum_{j=1, j\neq i}^{N} a_{ij}(x)\left(v_j - v_i\right), \end{aligned} \tag{1}$$

where $x_i(t)$ and $v_i(t) \in \mathbb{R}^n$ are the position and velocity states of the ith agent, respectively, $a_{ij}(x)$ is called the communication rate defined as

$$\begin{aligned} a_{ij}(x) &= \frac{H}{\left(1 + \left\|x_i - x_j\right\|^2\right)^{\beta}}, \quad i \neq j, \\ a_{ii}(x) &= 0, \end{aligned} \tag{2}$$

in which $\beta \geq 0$ determines the decay of $a_{ij}(x)$, $H > 0$, and $x = (x_1^T, \ldots, x_N^T)^T$.

In [11], Cucker and Smale studied the flocking behavior of model (1) and showed that the flock converges to a common velocity unconditionally when $\beta < 1/2$, while the stability of the flock depends on the initial positions and velocities when $\beta \geq 1/2$. The Cucker-Smale flocking model (1) has attracted much attention in recent years. Shen [12] investigated the flocking behavior of an extended Cucker-Smale model with hierarchical leadership. Ha and Liu [13] provided a simple proof for the Cucker-Smale model (1) and derived some conditions for reaching exponential flocking. Ha et al. [14] presented a Cucker-Smale-type model with nonlinear velocity couplings. Park et al. [15] proposed an augmented Cucker-Smale model by introducing interagent bonding forces. Perea et al. [16] successfully applied Cucker-Smale model (1) to the real-flight formation control in the Darwin space mission.

According to the flocking rules of Reynolds, one knows that the collision avoidance is not considered in the original Cucker-Smale model (1) and its variations [12–16]. To fix this drawback, Cucker and Dong [17] developed the following extended Cucker-Smale-type model with repelling forces:

$$\begin{aligned}
\dot{x}_i(t) &= v_i(t), \quad i = 1, \ldots, N, \\
\dot{v}_i(t) &= \sum_{j=1, j\neq i}^{N} a_{ij}(x)\left(v_j - v_i\right) \\
&\quad + \left(\frac{1}{N}\sum_{i=1}^{N}\sum_{i>j}^{N}\left\|v_i - v_j\right\|^2\right)^{1/2} \\
&\quad \times \sum_{j=1, j\neq i}^{N} f\left(\left\|x_i - x_j\right\|^2\right)\left(x_i - x_j\right),
\end{aligned} \tag{3}$$

where the differentiable function $f : (d_0, \infty) \rightarrow (0, \infty)$ satisfies $\int_{d_0}^{d_0+1} f(r)dr = \infty$ and $\int_{d_0+1}^{\infty} f(r)dr < \infty$ to ensure the collision avoidance among the agents, in which $d_0 > 0$ is a specified distance. Assuming that the initial positions of model (3) satisfy $\|x_i(0) - x_j(0)\| > d_0$ for all $i \neq j$, Cucker and Dong [17] showed that the flock asymptotically converges to a common velocity and the distance between any two agents is greater than d_0 when $\beta \leq 1/2$, while the convergence of the flock relies on the initial positions and velocities of the agents when $\beta > 1/2$. Note that when $\|x_i - x_j\| \rightarrow d_0^+$, one has $f(\|x_i - x_j\|^2) \rightarrow \infty$, which means that the repulsive force in model (3) is unbounded. However, in many practical cases, the actuator of an agent can only handle finite forces or torques due to its saturation. Therefore, it is imperative to design some bounded repulsive functions for Cucker-Smale-type flocking models.

Motivated by the above discussions, we propose two improved Cucker-Smale-type flocking models in this paper. The main contribution of this paper is three-fold. First, inspired by the flocking model developed by Park et al. [15] and the aggregation techniques proposed by Gazi and Passino [18, 19], this paper presents two control strategies for the Cucker-Smale model (1) to ensure the cohesiveness of the flocking system. Second, to overcome the actuator saturation of multiagent systems, this paper designs a class of bounded repulsive functions for Cucker-Smale model (1) such that the collision-free motion of the flock can always be guaranteed. Third, it is shown that the velocity convergence of two proposed flocking models in this paper can be achieved independent of the parameter β, which describes the decay of communication rate $a_{ij}(x)$. Due to these distinguishing features, the proposed flocking models in this paper significantly improve some previous work related to Cucker-Smale-type flocking.

The remainder of this paper is organized as follows. Section 2 formulates the asymptotical flocking problem and defines bounded repulsive functions. Section 3 proposes two improved Cucker-Smale models and investigates their flocking behaviors. Numerical examples are given in Section 4 to verify the theoretical analysis. Finally, some concluding remarks and future trends are stated in Section 5.

Notations. The standard notations are used throughout the paper. The superscript "T" represents the transpose of a vector or a matrix. $\mathbb{R}$ denotes the set of real numbers. For $x \in \mathbb{R}^n$, let $\|x\|$ be its Euclidean norm and let $\min_i\{x_i\}$ be its minimal element, and let $\langle \cdot, \cdot \rangle$ denote the inner product. Let $0_n = (0, \ldots, 0)^T \in \mathbb{R}^n$ be a vector with all zero entries. The symbol "∇" is the gradient operator.

2. Preliminaries

This section formulates the flocking problem of second-order multiagent systems, and defines a class of bounded repulsive functions to avoid interagent collisions.

2.1. Problem Formulation. Consider a second-order multiagent system consisting of N agents described by

$$\begin{aligned}
\dot{x}_i(t) &= v_i, \quad i = 1, \ldots, N, \\
\dot{v}_i(t) &= u_i,
\end{aligned} \tag{4}$$

where $x_i(t)$ and $v_i(t) \in \mathbb{R}^n$ are the position and velocity states of the ith agent, respectively, and $u_i \in \mathbb{R}^n$ is the control input for the ith agent.

Definition 1. According to the flocking rules proposed by Reynolds [1], multiagent system (4) is said to achieve asymptotical flocking if its solution satisfies the following conditions for all $1 \leq i, j \leq N, i \neq j$, and $t \geq 0$:

(i) flock cohesion: $\sup_{0\leq t<\infty}\|x_i(t) - x_j(t)\| < \infty$;

(ii) collision avoidance: $\|x_i(t) - x_j(t)\| > 0$;

(iii) velocity consensus: $\lim_{t\rightarrow\infty}\|v_i(t) - v_j(t)\| = 0$.

Remark 2. The first condition of the asymptotical flocking in Definition 1 indicates that each agent should stay close to the nearby flockmates to ensure the cohesiveness of the flock.

Remark 3. The collision in a multiagent system means that there exist at least two agents occupying the same space [4, 5, 7–9]. To ensure the collision-free motion of the flock, the minimal interagent distance in multiagent system (4) should be always greater than zero.

2.2. Bounded Repulsive Functions. To avoid collisions in a multiagent system, one can design some artificial potential functions to create interagent repulsive forces [4–8, 17–19]. Up to date, most researchers utilize the negative gradients of unbounded potential functions to avoid collisions among agents [5, 7, 8, 17–19]. However, in many practical situations, the control input of a multiagent system should be bounded because no actuator could provide an infinite control force. In this paper, we will show that interagent collisions can be avoided by designing some appropriate bounded repulsive functions. Below we define a class of bounded repulsive functions.

Definition 4. For multiagent system (4), assume that the initial positions of the agents satisfy $\|x_i(0) - x_j(0)\| > 0$ for all $1 \le i, j \le N, i \neq j$. Let $\Psi : (0, \infty) \to (0, \infty)$ be a differentiable, nonnegative, and decreasing function with respective to the distance $\|x_i - x_j\|$ between agents i and j, such that

(i) $\nabla_{x_i}\Psi(\|x_i - x_j\|) = -g_r(\|x_i - x_j\|)(x_i - x_j)$. When $\|x_i - x_j\| \to 0^+$, $g_r(\|x_i - x_j\|) = P$; otherwise, $g_r(\|x_i - x_j\|) < P$,

(ii) $\lim_{\|x_i - x_j\| \to 0^+} \Psi(\|x_i - x_j\|) = Q$,

where $P, Q > 0$, and $g_r(\|x_i - x_j\|) > 0$ is a smooth and decreasing function in $(0, \infty)$. If $P < \infty$ and $Q < \infty$, the potential function Ψ is called a bounded repulsive function.

Remark 5. If $\lim_{\|x_i - x_j\| \to 0^+} \Psi(\|x_i - x_j\|) = \infty$, that is, $Q = \infty$, the repulsive function Ψ is unbounded.

Remark 6. A possible bounded repulsive function to satisfy Definition 4 can be chosen as follows:

$$\Psi\left(\left\|x_i - x_j\right\|\right) = Q \exp\left(-\mu \left\|x_i - x_j\right\|^2\right) \quad \text{for } \left\|x_i - x_j\right\| > 0, \tag{5}$$

where $0 < Q < \infty$ and $\mu > 0$.

Remark 7. Considering the symmetry of the potential function Ψ in Definition 4, that is, $\Psi(\|x_i - x_j\|) = \Psi(\|x_j - x_i\|)$, we have [5]

$$\begin{aligned} \nabla_{x_i}\Psi\left(\left\|x_i - x_j\right\|\right) &= -\nabla_{x_j}\Psi\left(\left\|x_i - x_j\right\|\right), \\ \frac{d}{dt}\sum_{i=1}^{N}\sum_{j=1, j\neq i}^{N}\frac{1}{2}\Psi\left(\left\|x_i - x_j\right\|\right) & \\ = \sum_{i=1}^{N} v_i^T \sum_{j=1, j\neq i}^{N} \nabla_{x_i}\Psi\left(\left\|x_i - x_j\right\|\right). & \end{aligned} \tag{6}$$

3. Main Results

This section proposes two improved Cucker-Smale flocking models and investigates their flocking behaviors. In particular, some conditions are derived to ensure the collision-free motion of the flock.

3.1. Cucker-Smale Flocking with Bonding and Repulsive Forces. Recently, Park et al. [15] proposed an extended Cucker-Smale model as follows:

$$\begin{aligned} \dot{x}_i(t) &= v_i(t), \quad i = 1, \ldots, N, \\ \dot{v}_i(t) &= \frac{\lambda}{N}\sum_{j=1, j\neq i}^{N}\psi\left(r_{ij}\right)\left(v_j - v_i\right) \\ &\quad + \frac{\sigma}{N}\sum_{j=1, j\neq i}^{N}\frac{K_1}{2r_{ij}^2} \\ &\quad \times \left\langle v_i - v_j, x_i - x_j\right\rangle\left(x_j - x_i\right) \\ &\quad + \frac{\sigma}{N}\sum_{j=1, j\neq i}^{N}\frac{K_2}{2r_{ij}}\left(r_{ij} - 2R\right)\left(x_j - x_i\right), \end{aligned} \tag{7}$$

where $\lambda, \sigma, K_1, K_2$, and R are positive parameters, $r_{ij} = \|x_i - x_j\|$, and the function $\psi(\cdot)$ is nonnegative.

In [15], it was shown that model (7) can exhibit cohesive flocking due to the bonding forces among the agents. However, the collision avoidance in model (7) was not addressed and thus should be resolved by adopting some new techniques. To improve model (7), we consider the following extended Cucker-Smale model with bonding and repulsive forces:

$$\begin{aligned} \dot{x}_i(t) &= v_i(t), \quad i = 1, \ldots, N, \\ \dot{v}_i(t) &= \sum_{j=1, j\neq i}^{N} a_{ij}(x)\left(v_j - v_i\right) \\ &\quad + k_a\sum_{j=1, j\neq i}^{N}\frac{r_{ij} - \eta}{r_{ij}}\left(x_j - x_i\right) \\ &\quad - \sum_{j=1, j\neq i}^{N}\nabla_{x_i}\Psi\left(\left\|x_i - x_j\right\|\right), \end{aligned} \tag{8}$$

where $k_a > 0$, $\eta > 0$, $r_{ij} = \|x_i - x_j\|$, and $\Psi(\|x_i - x_j\|)$ is a bounded repulsive function described in Definition 4.

Remark 8. We will show that the terms $k_a \sum_{j=1, j\neq i}^{N}((r_{ij} - \eta)/r_{ij})(x_j - x_i)$ and $-\sum_{j=1, j\neq i}^{N}\nabla_{x_i}\Psi(\|x_i - x_j\|)$ in multiagent system (8) can yield bounded interagent bonding and repulsive forces, respectively.

The following result is very useful to derive the main results of this paper.

Lemma 9 (see [17]). *For $z_i \in \mathbb{R}$, $i = 1, \ldots, N$, let $\bar{z} = \sum_{i=1}^{N} z_i / N$ and $\hat{z}_i = z_i - \bar{z}$. One has $\sum_{i=1}^{N-1} \sum_{j>i}^{N} \|\hat{z}_j - \hat{z}_i\|^2 = N \sum_{i=1}^{N} \|\hat{z}_i\|^2$.*

Let x_c and v_c be the position and velocity of the mass center of system (8), respectively as follows:

$$x_c = \frac{\sum_{i=1}^{N} x_i}{N}, \qquad v_c = \frac{\sum_{i=1}^{N} v_i}{N}. \tag{9}$$

In view of $a_{ij}(x) = a_{ji}(x)$, $r_{ij} = r_{ji}$, and Remark 7, we can obtain $\dot{v}_c(t) = 0$. It follows that the velocity of the centroid of multiagent system (8) is a constant; that is, $v_c(t) = v_c(0)$ for all $t \geq 0$. Let $\hat{x}_i = x_i - x_c$ and $\hat{v}_i = v_i - v_c$. It is obvious to see that $\sum_{i=1}^{N} \hat{x}_i = 0$ and $\sum_{i=1}^{N} \hat{v}_i = 0$. Also, we have $\hat{x}_i - \hat{x}_j = x_i - x_j$ and $\hat{v}_i - \hat{v}_j = v_i - v_j$. Then, it follows that $\Psi(\|\hat{x}_i - \hat{x}_j\|) = \Psi(\|x_i - x_j\|)$ and $\nabla_{\hat{x}_i} \Psi(\|\hat{x}_i - \hat{x}_j\|) = \nabla_{x_i} \Psi(\|x_i - x_j\|)$. Moreover, we have $r_{ij} = \|\hat{x}_i - \hat{x}_j\|$, whose time derivative is

$$\dot{r}_{ij}(t) = \frac{(\hat{v}_i - \hat{v}_j)^T (\hat{x}_i - \hat{x}_j)}{r_{ij}}. \tag{10}$$

Considering $\dot{x}_c = v_c$ and $\dot{v}_c = 0$, from (8) we have the following error system:

$$\begin{aligned} \dot{\hat{x}}_i(t) &= \hat{v}_i(t), \quad i = 1, \ldots, N, \\ \dot{\hat{v}}_i(t) &= \sum_{j=1, j\neq i}^{N} a_{ij}(\hat{x})(\hat{v}_j - \hat{v}_i) \\ &\quad + k_a \sum_{j=1, j\neq i}^{N} \frac{r_{ij} - \eta}{r_{ij}} (\hat{x}_j - \hat{x}_i) \\ &\quad - \sum_{j=1, j\neq i}^{N} \nabla_{\hat{x}_i} \Psi(\|\hat{x}_i - \hat{x}_j\|), \end{aligned} \tag{11}$$

where $a_{ij}(\hat{x}) = a_{ij}(x)$ and $\hat{x} = (\hat{x}_1^T(t), \ldots, \hat{x}_N^T(t))^T$.

To investigate the flocking behavior of multiagent system (8), we define the following energy function:

$$\begin{aligned} W(t) &= \frac{1}{2} \sum_{i=1}^{N} \hat{v}_i^T \hat{v}_i + \frac{k_a}{4} \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} (r_{ij} - \eta)^2 \\ &\quad + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} \Psi(\|\hat{x}_i - \hat{x}_j\|). \end{aligned} \tag{12}$$

Theorem 10. *For a flock of N agents with dynamics* (8), *assume that the initial energy defined by* (12) *is finite, the initial positions satisfy $\|x_i(0) - x_j(0)\| > 0$ for all $i \neq j$, and the repulsive function Ψ is chosen to be bounded according to Definition 4. Then, multiagent system* (8) *exhibits asymptotical flocking with the common velocity $v_c(0)$; that is, the velocity of the mass center, and the collision between any two agents can always be avoided if the repulsive function Ψ satisfies $W(0) < \Psi(0)$.*

Proof. Considering (10) and Remark 7, we compute the time derivative of $W(t)$ in (12) along the trajectory of (11) as follows:

$$\begin{aligned} \dot{W}(t) &= \sum_{i=1}^{N} \hat{v}_i^T \Bigg[\sum_{j=1, j\neq i}^{N} a_{ij}(\hat{x})(\hat{v}_j - \hat{v}_i) \\ &\qquad + k_a \sum_{j=1, j\neq i}^{N} \frac{r_{ij} - \eta}{r_{ij}} (\hat{x}_j - \hat{x}_i) \\ &\qquad - \sum_{j=1, j\neq i}^{N} \nabla_{\hat{x}_i} \Psi(\|\hat{x}_i - \hat{x}_j\|) \Bigg] \\ &\quad + \frac{k_a}{2} \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} \frac{r_{ij} - \eta}{r_{ij}} (\hat{v}_i - \hat{v}_j)^T (\hat{x}_i - \hat{x}_j) \\ &\quad + \sum_{i=1}^{N} \hat{v}_i^T \sum_{j=1, j\neq i}^{N} \nabla_{\hat{x}_i} \Psi(\|\hat{x}_i - \hat{x}_j\|) \\ &= \sum_{i=1}^{N-1} \sum_{j>i}^{N} \Big[a_{ij}(\hat{x}) \hat{v}_i^T (\hat{v}_j - \hat{v}_i) \\ &\qquad + a_{ji}(\hat{x}) \hat{v}_j^T (\hat{v}_i - \hat{v}_j) \Big] \\ &\quad + k_a \sum_{i=1}^{N-1} \sum_{j>i}^{N} \Bigg[\frac{r_{ij} - \eta}{r_{ij}} \hat{v}_i^T (\hat{x}_j - \hat{x}_i) \\ &\qquad + \frac{r_{ji} - \eta}{r_{ji}} \hat{v}_j^T (\hat{x}_i - \hat{x}_j) \Bigg] \\ &\quad + \frac{k_a}{2} \sum_{i=1}^{N-1} \sum_{j>i}^{N} \Bigg[\frac{r_{ij} - \eta}{r_{ij}} (\hat{v}_i - \hat{v}_j)^T (\hat{x}_i - \hat{x}_j) \\ &\qquad + \frac{r_{ji} - \eta}{r_{ji}} (\hat{v}_j - \hat{v}_i)^T (\hat{x}_j - \hat{x}_i) \Bigg] \\ &= -\sum_{i=1}^{N-1} \sum_{j>i}^{N} a_{ij}(\hat{x}) (\hat{v}_j - \hat{v}_i)^T (\hat{v}_j - \hat{v}_i) \\ &\leq -\min_{j>i} a_{ij}(\hat{x}) \sum_{i=1}^{N-1} \sum_{j>i}^{N} (\hat{v}_j - \hat{v}_i)^T (\hat{v}_j - \hat{v}_i) \\ &= -N \min_{j>i} a_{ij}(\hat{x}) \sum_{i=1}^{N} \|\hat{v}_i\|^2 \leq 0, \end{aligned} \tag{13}$$

where we use $a_{ij}(\hat{x}) = a_{ji}(\hat{x})$ and $r_{ij} = r_{ji}$, and the fact that $\sum_{i=1}^{N-1} \sum_{j>i}^{N} \|\hat{v}_j - \hat{v}_i\|^2 = N \sum_{i=1}^{N} \|\hat{v}_i\|^2$ (see Lemma 9) to obtain the third and the last equalities, respectively.

By (13), we have $\dot{W}(t) \leq 0$, which means that $W(t) \leq W(0) < \infty$ holds for all $t \geq 0$. Then, it follows from (12) that

$$\frac{k_a}{4} \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} (r_{ij} - \eta)^2 \leq W(0). \tag{14}$$

Since $r_{ij} = r_{ji}$, from (14) we obtain $|r_{ij} - \eta| \leq \sqrt{2W(0)/k_a}$, which means that

$$\left\|x_i(t) - x_j(t)\right\| \leq \eta + \sqrt{\frac{2W(0)}{k_a}} < \infty, \quad i \neq j, \ t \geq 0. \quad (15)$$

Hence, the cohesion condition of Definition 1 is satisfied.

By (2) and (15), we have $a_{ij}(\widehat{x}) > 0$ for all $i \neq j$. In view of (13), we have $\dot{W}(t) \equiv 0$ if and only if $\widehat{v}_i(t) = 0_n, i = 1, \ldots, N$. Let $\widehat{v}(t) = (\widehat{v}_1^T(t), \ldots, \widehat{v}_N^T(t))^T$. Thus, the set $\mathscr{M} = \{\widehat{v}(t) \mid \widehat{v}(t) = 0_{Nn}\}$ is the largest invariant set contained in the set $\mathscr{D} = \{\widehat{v}(t) \mid \dot{W}(t) \equiv 0\}$ for error system (11). According to LaSalle's invariance principle [20], starting from any initial condition, every solution of system (11) approaches $\mathscr{M}$ as $t \to \infty$, that is, $\widehat{v}_i(t) \to 0_n, i = 1, \ldots, N$. So, the flock with dynamics (8) converges to a common velocity $v_c(0)$, that is, the velocity of the mass center.

Now we show that the collision between any two agents in multiagent system (8) can always be avoided if $W(0) < \Psi(0)$ is satisfied. Suppose that agents i and j collides at some time instant $t_1 > 0$. Then, we obtain $\Psi(\|\widehat{x}_i(t_1) - \widehat{x}_j(t_1)\|) = \Psi(\|x_i(t_1) - x_j(t_1)\|) = \Psi(0)$. From (12), we have $W(t_1) \geq \Psi(0)$. Considering the assumption of $\Psi(0) > W(0)$, we have $W(t_1) > W(0)$. However, $W(t_1) \leq W(0)$ always holds since $\dot{W}(t) \leq 0$. Hence, a contradiction occurs. Then, the distance between any two agents in multiagent system (8) should always be greater than zero, which ensures the collision-free motion of the flocking model (8).

Now, all three conditions of asymptotical flocking in Definition 1 hold. This completes the proof. □

Remark 11. From Theorem 10, we know that the velocity convergence of flocking model (8) is independent of the parameter β, which improves the original Cucker-Smale model and its existing variations.

Remark 12. The flocking model (8) theoretically ensures that any two agents will never collide during the time evolution of the system thus improving the augmented Cucker-Smale model (7).

Remark 13. From the proof of Theorem 10, we have $|r_{ij} - \eta| \leq \sqrt{2W(0)/k_a}$. Then, it follows that the term $k_a \sum_{j=1,j\neq i}^{N}((r_{ij} - \eta)/r_{ij})(x_j - x_i)\sqrt{2W(0)/k_a}$ in multiagent system (8) yields bounded bonding forces among agents satisfying $\|k_a \sum_{j=1,j\neq i}^{N}((r_{ij} - \eta)/r_{ij})(x_j - x_i)\| \leq (N-1)\sqrt{2k_a W(0)}$. If the repulsive function Ψ is bounded, the repulsive force in (8) is also bounded satisfying $\|\sum_{j=1,j\neq i}^{N} \nabla_{x_i}\Psi(\|x_i - x_j\|)\| \leq (N-1)\max_{i\neq j} g_r(\|x_i - x_j\|)\sup_{i\neq j}\|x_i - x_j\| \leq (N-1)P(\eta + \sqrt{2W(0)/k_a})$ by Definition 4 and (15).

Remark 14. From Theorem 10, we know that the bounded repulsive function Ψ for the flocking model (8) can be designed based on Definition 4 and the initial energy of the flock.

3.2. Cucker-Smale Flocking with Attractive and Repulsive Forces. For a network of agents, Gazi and Passino [18] proposed the following swarming model:

$$\begin{aligned} \dot{x}_i(t) = &- a \sum_{j=1, j\neq i}^{N} \left(x_i - x_j\right) \\ &+ b \sum_{j=1, j\neq i}^{N} \left(x_i - x_j\right) \exp\left(-\frac{\left\|x_i - x_j\right\|^2}{c}\right), \end{aligned} \quad (16)$$

where a, b and c are positive parameters, the first and the second terms in the right-hand side describe the attraction and repulsion among the agents, respectively. Motivated by the aggregation technique in the swarming model (16), we introduce both attractive and repulsive forces to the original Cucker-Smale model (1) as follows:

$$\begin{aligned} \dot{x}_i(t) &= v_i(t), \quad i = 1, \ldots, N, \\ \dot{v}_i(t) &= \sum_{j=1, j\neq i}^{N} a_{ij}(x)\left(v_j - v_i\right) \\ &\quad + \frac{k_a}{N} \sum_{j=1, j\neq i}^{N} \left(x_j - x_i\right) \\ &\quad - \sum_{j=1, j\neq i}^{N} \nabla_{x_i}\Psi\left(\left\|x_i - x_j\right\|\right), \end{aligned} \quad (17)$$

where $k_a > 0$, and $\Psi(\|x_i - x_j\|)$ is a bounded repulsive function described in Definition 4.

Remark 15. Later, we will show that the terms $(k_a/N)\sum_{j=1,j\neq i}^{N}(x_j - x_i)$ and $-\sum_{j=1,j\neq i}^{N} \nabla_{x_i}\Psi(\|x_i - x_j\|)$ in (17) can yield bounded attractive and repulsive forces among the agents, respectively.

Let x_c and v_c be the position and the velocity of the mass center of multiagent system (17), respectively. It is easy to verify $\dot{v}_c = 0$, which means that the velocity of the centroid is constant. Let $\widehat{x}_i = x_i - x_c$ and let $\widehat{v}_i = v_i - v_c$. Note that $\sum_{j=1,j\neq i}^{N}(x_j - x_i) = N(x_c - x_i) = -N\widehat{x}_i$. From (17), we obtain the following error system:

$$\begin{aligned} \dot{\widehat{x}}_i(t) &= \widehat{v}_i(t), \quad i = 1, \ldots, N, \\ \dot{\widehat{v}}_i(t) &= \sum_{j=1, j\neq i}^{N} a_{ij}(\widehat{x})\left(\widehat{v}_j - \widehat{v}_i\right) - k_a \widehat{x}_i \\ &\quad - \sum_{j=1, j\neq i}^{N} \nabla_{\widehat{x}_i}\Psi\left(\left\|\widehat{x}_i - \widehat{x}_j\right\|\right). \end{aligned} \quad (18)$$

Construct the following energy function to study the flocking behavior of multiagent system (17):

$$W(t) = \frac{1}{2}\sum_{i=1}^{N}\widehat{v}_i^T\widehat{v}_i + \frac{k_a}{2}\sum_{i=1}^{N}\widehat{x}_i^T\widehat{x}_i + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}\Psi\left(\left\|\widehat{x}_i - \widehat{x}_j\right\|\right). \tag{19}$$

Theorem 16. *For flocking system* (17)*, assume that the initial energy defined by* (19) *is finite, the initial positions satisfy* $\|x_i(0) - x_j(0)\| > 0$ *for all* $i \neq j$*, and the repulsive function* Ψ *is designed to be bounded based on Definition 4. Then, the flocking system* (17) *reaches velocity consensus on the common value* $v_c(0)$*, and the collision-free motion of the flock can be ensured if* $W(0) < \Psi(0)$ *holds.*

Proof. The proof can be carried out following the similar procedure for the proof of Theorem 10. The time derivative of $W(t)$ in (19) along the trajectory of (18) yields

$$\begin{aligned}
\dot{W}(t) &= \sum_{i=1}^{N}\widehat{v}_i^T\left[\sum_{j=1,j\neq i}^{N}a_{ij}(\widehat{x})\left(\widehat{v}_j - \widehat{v}_i\right) - k_a\widehat{x}_i - \sum_{j=1,j\neq i}^{N}\nabla_{\widehat{x}_i}\Psi\left(\left\|\widehat{x}_i - \widehat{x}_j\right\|\right)\right] \\
&\quad + k_a\sum_{i=1}^{N}\widehat{v}_i^T\widehat{x}_i + \sum_{i=1}^{N}\widehat{v}_i^T\sum_{j=1,j\neq i}^{N}\nabla_{\widehat{x}_i}\Psi\left(\left\|\widehat{x}_i - \widehat{x}_j\right\|\right) \\
&= \sum_{i=1}^{N-1}\sum_{j>i}^{N}\left[a_{ij}(\widehat{x})\widehat{v}_i^T\left(\widehat{v}_j - \widehat{v}_i\right) + a_{ji}(\widehat{x})\widehat{v}_j^T\left(\widehat{v}_i - \widehat{v}_j\right)\right] \\
&= -\sum_{i=1}^{N-1}\sum_{j>i}^{N}a_{ij}(\widehat{x})\left(\widehat{v}_j - \widehat{v}_i\right)^T\left(\widehat{v}_j - \widehat{v}_i\right) \\
&\leq -\min_{j>i}a_{ij}(\widehat{x})\sum_{i=1}^{N-1}\sum_{j>i}^{N}\left(\widehat{v}_j - \widehat{v}_i\right)^T\left(\widehat{v}_j - \widehat{v}_i\right) \\
&= -N\min_{j>i}a_{ij}(\widehat{x})\sum_{i=1}^{N}\left\|\widehat{v}_i\right\|^2.
\end{aligned} \tag{20}$$

By (20), we have $\dot{W}(t) \leq 0$. Thus, $W(t) \leq W(0) < \infty$ for all $t \geq 0$. From (19), we obtain

$$\frac{k_a}{2}\sum_{i=1}^{N}\widehat{x}_i(t)^T\widehat{x}_i(t) \leq W(0). \tag{21}$$

Then, it follows that $\|\widehat{x}_i\| = \|x_i - x_c\| \leq \sqrt{2W(0)/k_a}$, $i = 1,\ldots,N$, which indicates that the term $(k_a/N)\sum_{j=1,j\neq i}^{N}(x_j - x_i) = -k_a\widehat{x}_i$ in (17) creates attractive forces among agents such that the positions of all agents remain bounded in a ball centered at x_c with radius $R = \sqrt{2W(0)/k_a}$. Then, the distance between any two agents satisfies

$$\left\|x_i(t) - x_j(t)\right\| \leq 2R < \infty, \quad i \neq j,\ t \geq 0. \tag{22}$$

Since $\|x_i(t) - x_j(t)\| \leq 2R < \infty$, we have $a_{ij}(\widehat{x}) > 0$ for all $i \neq j$ by (2). Then, in view of (20), we have $\dot{W}(t) \equiv 0$ if and only if $\widehat{v}_i(t) = 0_n, i = 1,\ldots,N$. Following the similar lines in the proof of Theorem 10, we can show that the velocities of all agents in system (17) converge to the common velocity $v_c(0)$, and the distance between any two agents is always greater than zero if the bounded repulsive function Ψ satisfies $W(0) < \Psi(0)$. □

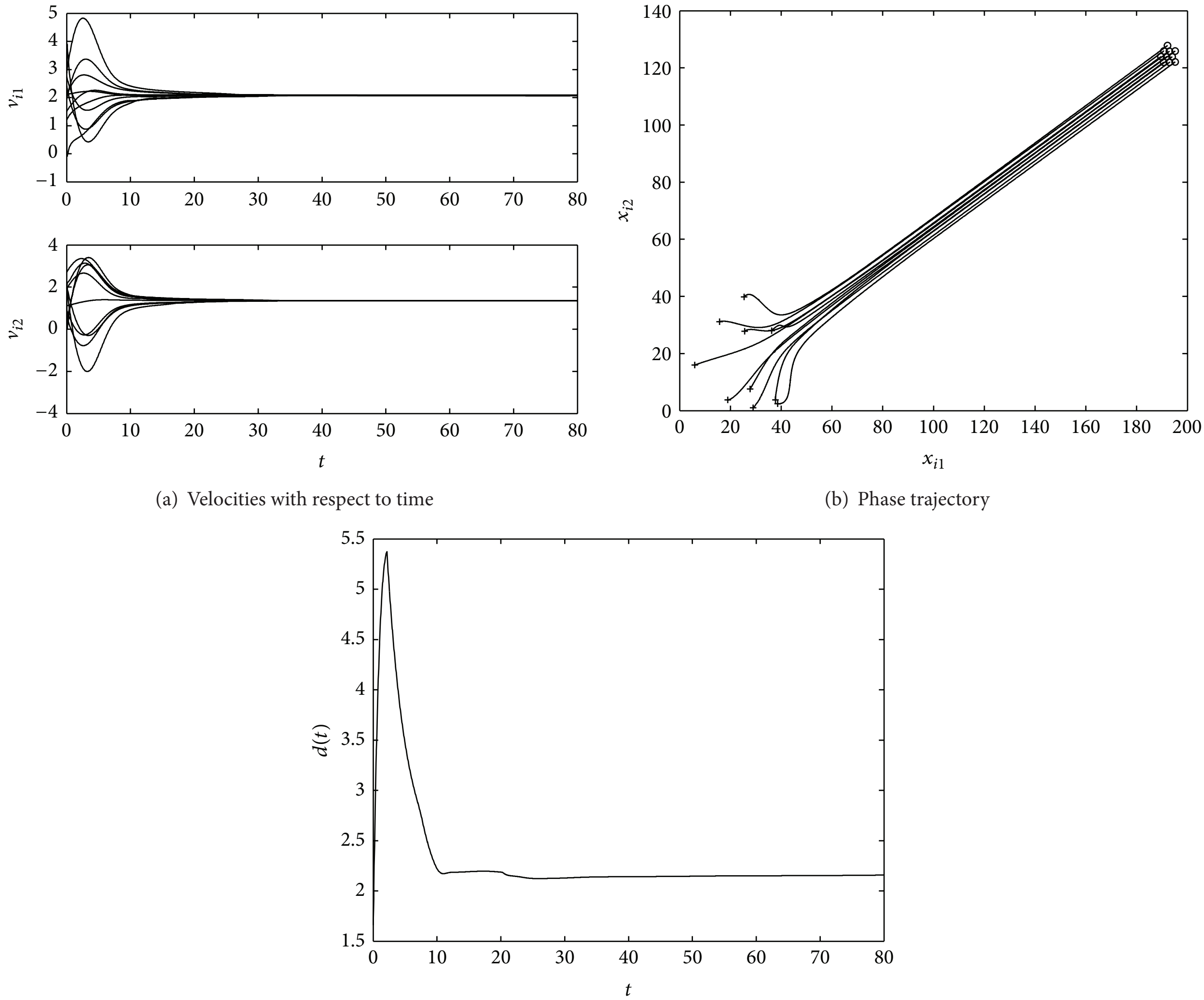
Remark 17. From the proof of Theorem 16, we have $\|\widehat{x}_i\| \leq \sqrt{2W(0)/k_a}$. Then, it follows that the term $(k_a/N)\sum_{j=1,i\neq j}^{N}(x_j - x_i) = -k_a\widehat{x}_i$ in (17) yields a bounded attractive force, that is, $k_a\|x_i\| \leq \sqrt{2k_aW(0)}$. If the repulsive function Ψ is bounded, the repulsive force in (17) is also bounded satisfying $\|\sum_{j=1,j\neq i}^{N}\nabla_{x_i}\Psi(\|x_i - x_j\|)\| \leq (N-1)\max_{i\neq j}g_r(\|x_i - x_j\|)\sup_{i\neq j}\|x_i - x_j\| \leq 2(N-1)P\sqrt{2W(0)/k_a}$ by Definition 4 and (22).

4. Numerical Results

This section provides two simulation examples to verify our theoretical results. The bounded repulsive function (5), that is, $\Psi(\|x_i - x_j\|) = Q\exp(-\mu\|x_i - x_j\|^2)$ for $\|x_i - x_j\| > 0$, is adopted to avoid interagent collisions. Define $d(t) = \min_{i\neq j}\|x_i(t) - x_j(t)\|$ to measure the proximity of the agents [17].

Example 18. Consider flocking model (8) with ten two-dimensional agents. Initially, the minimum distance between any two agents is 1.6584. The system parameters are chosen to be $H = 1$, $\beta = 0.5$, $k_a = 0.01$, and $\eta = 1.5$. The parameters of the repulsive function are taken as $Q = 140$ and $\mu = 3$. After some simple calculation using (12), we obtain $W(0) = 131.5835$ which is the initial energy of the flock. Considering $W(0) < \Psi(0) = Q = 140$, we know that the condition on the bounded repulsive function in Theorem 10 is satisfied. From Figure 1(a), we can see that the flock converges to a common velocity. The cohesiveness among agents is clearly shown in Figure 1(b), in which the symbols "+" and "o" denote the starting and ending points of the trajectory of an agent, respectively. In particular, from Figure 1(c), we note that collision avoidance in the flock is ensured because the minimal interagent distance is always greater than zero.

Example 19. Consider a multiagent system composed of ten two-dimensional agents described by dynamics (17) with $H = 1$, $\beta = 0.95$, and $k_a = 0.01$. The initial positions of the agents satisfy $\min_{i\neq j}\|x_i(0) - x_j(0)\| = 2.4876$. The parameters of the repulsive function are given by $Q = 40$ and $\mu = 1.8$. By (19), the initial energy of the flock is determined to be $W(0) = 34.0842$. Hence, the condition



(a) Velocities with respect to time

(b) Phase trajectory

(c) Proximity $d(t)$ with respect to time

Figure 1: Flocking behavior of multiagent system (8) with ten two-dimensional agents.

for the bounded repulsive function in Theorem 16 holds in view of $W(0) < \Psi(0) = Q = 40$. Figure 2(a) shows that the flock achieves velocity consensus. The cohesive behavior of the flock is demonstrated by Figure 2(b). Figure 2(c) indicates that the interagent distance is always greater than zero which guarantees the collision-free motion of the agents.

5. Conclusions and Future Work

In this paper, we have proposed two Cucker-Smale-type flocking models with both cohesive and repulsive forces to improve the original Cucker-Smale model and its existing variations. It is shown that the velocity convergence of the flocking system is independent of the parameter β describing the decay of communication rates. We have also proved that the collision-free motion of the multiagent system can always be guaranteed by choosing an appropriate bounded repulsive function according to the initial energy of the flock. Up to date, the Cucker-Smale-type flocking models assume that each agent has the same sensing radius, which implies that the network topology is undirected. However, the sensing radii of the agents in a multiagent system may be different in many practical cases. In our future work, we will consider the Cucker-Smale flocking problem for multiagent systems with directed topologies.

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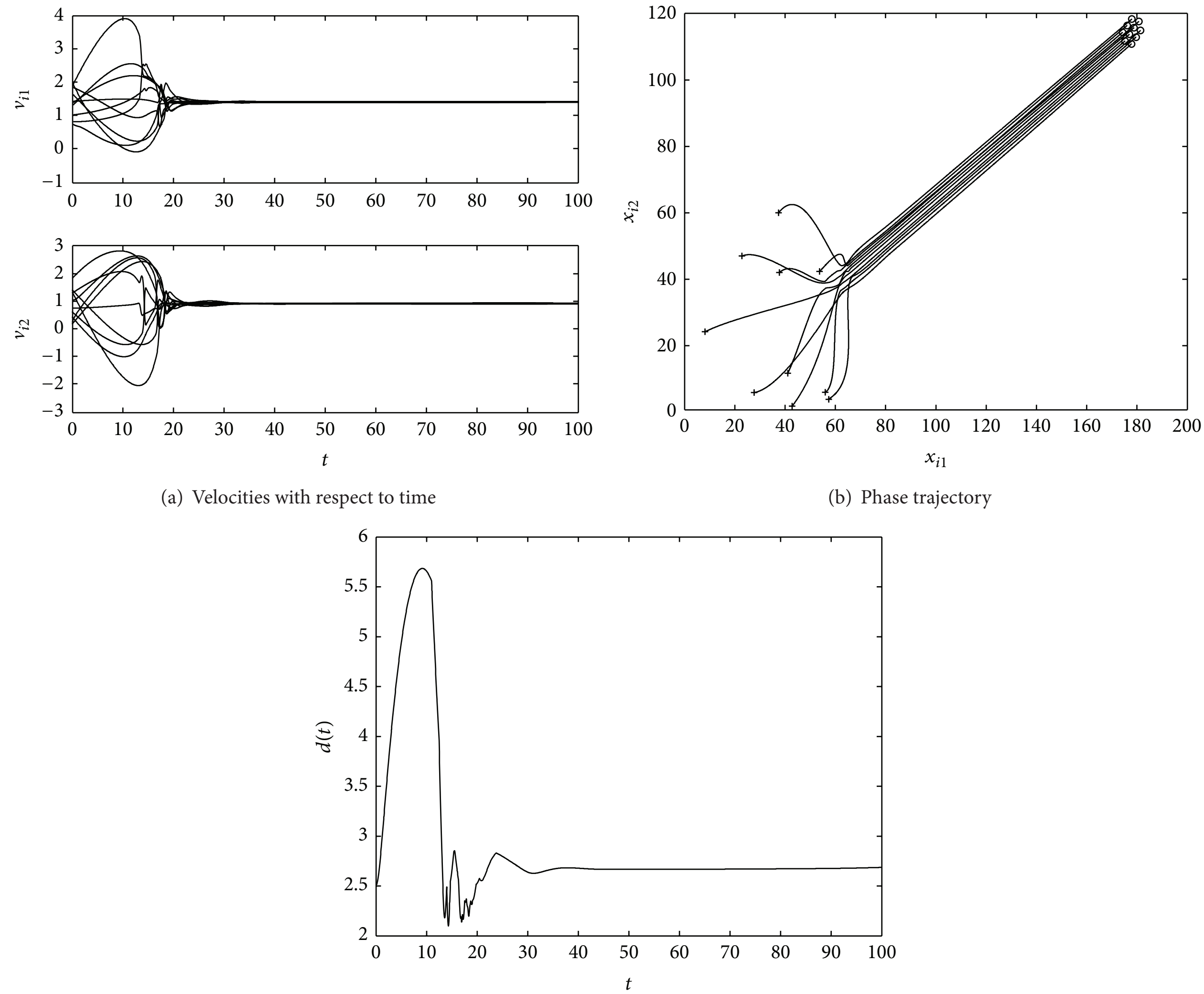

(a) Velocities with respect to time

(b) Phase trajectory

(c) Proximity $d(t)$ with respect to time

Figure 2: Flocking behavior of multiagent system (17) with ten two-dimensional agents.

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