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Research Article

Minimax Results with Respect to Different Altitudes in the Situation of Linking

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Consider a continuous function on a metric space. In the presence of linking between a compact pair and a closed set, depending on the different behaviors of the function on the linking sets, we establish minimax results guaranteeing existence of Palais-Smale sequences or providing gradient estimates. Our approach relies on deformation techniques.

1. Introduction and Statement of Main Results

Minimax theorems play a central role in critical point theory: in this respect, we refer to celebrated minimax results as the mountain pass theorem (see Ambrosetti and Rabinowitz [1]) and the saddle point theorem (see Rabinowitz [2]). A practical way to define minimax values is by means of a linking condition in a topological space adapted to the problem. In the above-cited results, there is a specific linking based on the geometry of the involved problem. In this paper, we provide linking results for continuous functions on metric spaces, which extend the results in [3, 4] in this setting. Beyond this general framework, the main novelty that we emphasize with our approach is that we provide a systematic study related to the involved geometry, being concerned with situations which are not covered by the "classical geometry." This approach is original in the case of Banach spaces and smooth functions too.

Recall that when $(X, \|\cdot\|)$ is a Banach space and $f : X \to \mathbb{R}$ is continuously differentiable, an element $u \in X$ is called a *critical point* of f if $f'(u) = 0$. In this paper, more generally, we assume that X is a metric space endowed with the metric d and f is a continuous function. In this context, the following notion of critical point has been introduced by Degiovanni and Marzocchi (see [4]).

Definition 1. Let $f : X \to \mathbb{R}$ be a continuous function defined on a metric space X. An element $u \in X$ is called a *critical point* of f if $|df|(u) = 0$, where by $|df|(u)$ we denote the supremum of the real numbers $\sigma \geq 0$ such that there exist $\delta > 0$ and a continuous map $\mathscr{H} : B_\delta(u) \times [0, \delta] \to X$ satisfying

$$d(\mathscr{H}(v,t), v) \leq t,$$
$$f(\mathscr{H}(v,t)) \leq f(v) - \sigma t, \tag{1}$$
$$\forall (v,t) \in B_\delta(u) \times [0,\delta].$$

Remark 2. (a) Note that $|df|(u) \in [0, +\infty]$ (take $\sigma = 0$, any $\delta > 0$, and $\mathscr{H}(v,t) \equiv v$ in (1)) and the map $X \to [0, +\infty]$, $u \mapsto |df|(u)$ is lower semicontinuous. The quantity $|df|(u)$ is usually called the weak slope of f at u.

(b) If $(X, \|\cdot\|)$ is a real Banach space, d is the distance induced by the norm $\|\cdot\|$, and $f \in C^1(X)$, then $|df|(u)$ coincides with the norm of the differential $f'(u)$. Thus, Definition 1 reduces to the usual notion of critical point in this context.

If $a \in \mathbb{R}$, we denote by K_a the set of critical points of f at level a; that is,

$$K_a := \{u \in X : |df|(u) = 0, f(u) = a\}. \tag{2}$$

The essential tool in our approach to critical point theory is the linking condition presented in the next definition. We say that (D, S) is a *compact pair* in X if S, D are compact subsets of X with $S \subset D$ and $D \neq \emptyset$.

Definition 3. A compact pair (D, S) in X and a closed subset A of X are *linking* if the following property holds:

$$S \cap A = \emptyset, \quad \gamma(D) \cap A \neq \emptyset, \quad \forall \gamma \in \Gamma, \tag{3}$$

where $\Gamma := \{\gamma \in C(D, X) : \gamma_{|S} = \mathrm{id}_S\}$.

Given a continuous function $f : X \to \mathbb{R}$, a compact pair (D, S) in X, and a nonempty, closed subset A of X, we set

$$a := \inf_A f, \qquad b := \sup_S f, \qquad c := \inf_{\gamma\in\Gamma} \sup_D (f \circ \gamma). \tag{4}$$

Clearly, $c \geq b$. Moreover, if (3) holds, then we have $c \geq a$. Note that $a, b, c \in \mathbb{R} \cup \{-\infty\}$. If $S \neq \emptyset$, then $b, c \in \mathbb{R}$.

The "classical" minimax principles are usually based on the assumption that $b < a$ (i.e., the "altitude" $a - b$ is positive; actually, refined results involve the weaker condition $b \leq a$), establishing in this situation the existence of a critical point for f. Our approach covers the situation where $b > a$ ("negative altitude"). We will distinguish the cases: $c = a$ and $c > a$. In the situation $c > a$, we will also distinguish the cases $c = b > a$ and $c > b$.

Now we formulate our main results. In all these results, unless otherwise stated, X is a metric space, $f : X \to \mathbb{R}$ is a continuous function, (D, S) is a compact pair in X and A is a closed subset of X satisfying (3).

The first statement is a preliminary result providing existence and location of critical points.

Proposition 4. *Assume that $a > -\infty$ (see (4)) and that there exist $\gamma \in \Gamma$ and a number $\rho > 0$ such that*

$$f(\gamma(u)) \leq a, \quad \forall u \in D \quad \text{with } \gamma(u) \in B_\rho(A). \tag{5}$$

Then $K_a \cap A \neq \emptyset$.

Hereafter, $B_\rho(A) = \{u \in X : d(u, A) < \rho\}$ (the open ρ-neighborhood of A). The proof of this proposition will be done in Section 2.

In the following results, the metric space X is supposed to be *complete*.

Here are our main results in the cases $c = a$ and $c > a$.

Theorem 5. *Assume that $c = a > -\infty$. Then*

$$\inf\{|df|(u) : c \leq f(u) < c + \delta, u \in B_\delta(A)\} = 0, \quad \forall \delta > 0. \tag{6}$$

In particular, there exists a sequence $(u_n) \subset X$ such that

$$|df|(u_n) \longrightarrow 0, \quad f(u_n) \downarrow c, \quad d(u_n, A) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{7}$$

Theorem 6. *Assume that $c > a > -\infty$. Then*

(a) *for all $\delta > 0$ and $\rho \geq 2d(S, A)$, one has*

$$\inf\{|df|(u) : a \leq f(u) < c + \delta, u \in B_\rho(A)\} \leq \frac{c - a}{d(S, A)}; \tag{8}$$

(b) *for all $\delta > 0$ and $\rho \geq 2d(S \cap [f > a], A)$, one has*

$$\inf\{|df|(u) : a \leq f(u) < c + \delta, u \in B_\rho(A)\} \leq \frac{c - a}{d(S \cap [f > a], A)}. \tag{9}$$

Hereafter, $[f > a] = \{u \in X : f(u) > a\}$. The notation $d(S, A)$ stands for $\inf\{d(u, v) : u \in S, v \in A\}$.

Remark 7. The estimate in part (b) of Theorem 6 is better than the one in part (a) when $\rho \in [2d(S \cap [f > a], A), +\infty)$. Nevertheless, Theorem 6(a) provides an estimate for all $\rho \in [2d(S, A), +\infty)$.

The next result distinguishes the situations $c > b$ and $c = b$ in (4). Since Theorem 5 treats the case $c = a$, the next result is meaningful when $c > a$.

Theorem 8. (a) *Let (D, S) be a compact pair in X. Assume that $c > b$. Then*

$$\inf\{|df|(u) : c \leq f(u) < c + \delta\} = 0, \quad \forall \delta > 0. \tag{10}$$

In particular, there exists a sequence $(u_n) \subset X$ such that

$$|df|(u_n) \longrightarrow 0, \qquad f(u_n) \downarrow c \text{ as } n \longrightarrow \infty. \tag{11}$$

(b) *Let (D, S) be a compact pair in X and let A be a closed subset of X satisfying* (3). *Assume that $c = b > a > -\infty$. Then*

$$\inf\{|df|(u) : a \leq f(u) < c + \delta\} \leq \frac{b - a}{d(S \cap [f > a], A)}, \quad \forall \delta > 0. \tag{12}$$

As a consequence of Theorems 5, 6(b), and 8, we have the next result, which studies the situations $b \leq a$ and $b > a$ in (4).

Corollary 9. (a) *Assume that $-\infty < b \leq a$. Then* (10) *holds. If $c = a$, then* (6) *holds.*

(b) *Assume that $b > a > -\infty$. Then* (12) *holds. If $c > b$, then* (10) *holds.*

Theorems 5 and 8(a) and Corollary 9(a) lead to the construction of a Palais-Smale sequence at level c (i.e., a sequence $(u_n) \subset X$ such that $f(u_n) \to c$ and $|df|(u_n) \to 0$ as $n \to \infty$). Moreover, in the case of Theorem 5 (referring to the *limiting case* $c = a$), the Palais-Smale sequence is located near A (i.e., $d(u_n, A) \to 0$ as $n \to \infty$). Recall from [3] the following Palais-Smale condition.

Definition 10. We say that a continuous function $f : X \to \mathbb{R}$ satisfies the Palais-Smale condition at level ℓ (condition $(\mathrm{PS})_\ell$ for short), with $\ell \in \mathbb{R}$, if every sequence $(u_n) \subset X$ such that $f(u_n) \to \ell$ and $|df|(u_n) \to 0$ as $n \to \infty$ has a convergent subsequence in X.

If we assume condition $(\mathrm{PS})_c$ (see (4)), the previous results guarantee the existence of critical points.

Corollary 11. *If conditions* (3) *and* $(\mathrm{PS})_c$ *hold, then one has the following.*

(a) *In the case* $c = a > -\infty$, *then* $K_c \cap A \neq \emptyset$.

(b) *In the case* $c > a$, *if one assumes that* $c > b$ *(for instance, if* $b \leq a$*), then* $K_c \neq \emptyset$.

Remark 12. (a) Corollary 11 actually holds under a weaker condition than $(\mathrm{PS})_c$, namely, $(\mathrm{PS})_{c^+}$: every sequence $(u_n) \subset X$ such that $f(u_n) \downarrow c$ and $|df|(u_n) \to 0$ as $n \to \infty$ has a convergent subsequence in X.

(b) Corollary 11(a) holds under a weaker condition than $(\mathrm{PS})_{c^+}$, namely, $(\mathrm{PS})_{c^+,A}$: every sequence $(u_n) \subset X$ such that $f(u_n) \downarrow c$, $d(u_n, A) \to 0$, and $|df|(u_n) \to 0$ as $n \to \infty$ has a convergent subsequence in X. The conclusion in Corollary 11(a) is also more precise since it establishes the location of the critical point in the set A.

Theorems 6 and 8(b) and Corollary 9(b) go beyond the "classical geometry" $b \leq a$ in the situation of linking, allowing the case $b > a$ and providing an estimate of the infimum of $|df|$ at the points u with $f(u) \in [a, c]$ in terms of $b - a$ and of $d(S, A)$ or of $d(S \cap [f > a], A)$.

We may still obtain a Palais-Smale sequence from Theorem 8(b) (the case of "nonclassical geometry") under appropriate hypotheses for a sequence of sets (in the spirit of [5]) which are linking.

Corollary 13. *Let* $f : X \to \mathbb{R}$ *be a continuous function on the metric space* X, *and let* (A_n), (D_n), *and* (S_n) *be sequences of subsets of* X *such that for all* $n \geq 1$, A_n *is closed,* (D_n, S_n) *is a compact pair in* X, (D_n, S_n) *and* A_n *satisfy* (3) *with* Γ *replaced by* $\Gamma_n := \{\gamma \in C(D_n, X) : \gamma_{|S_n} = \mathrm{id}_{S_n}\}$. *Assume that*

$$a_n := \inf_{A_n} f \in \mathbb{R}, \qquad b_n := \sup_{S_n} f \in \mathbb{R},$$
$$c_n := \inf_{\gamma \in \Gamma_n} \sup_{D_n} (f \circ \gamma) \in \mathbb{R}, \qquad \forall n \geq 1, \tag{13}$$
$$\lim_{n \to \infty} \frac{\max\{b_n - a_n, 0\}}{d(S_n \cap [f > a_n], A_n)} = 0.$$

Then, denoting $\bar{a} := \liminf_{n \to \infty} a_n$ *and* $\bar{c} := \limsup_{n \to \infty} c_n$, *there exists a sequence* $(u_n) \subset X$ *satisfying*

$$\bar{a} \leq \liminf_{n \to \infty} f(u_n) \leq \limsup_{n \to \infty} f(u_n) \leq \bar{c},$$
$$\lim_{n \to \infty} |df|(u_n) = 0. \tag{14}$$

In particular, if f *satisfies condition* $(PS)_\ell$ *for all* $\ell \in [\bar{a}, \bar{c}]$, *then there exists* $u \in X$ *such that* $f(u) \in [\bar{a}, \bar{c}]$ *and* $|df|(u) = 0$.

Remark 14 (assume that $b \leq a$). Considering particular situations of linking as in (3) in a Banach space $(X, \|\cdot\|)$, we obtain through Corollary 11 generalizations of classical minimax results in smooth critical point theory.

(a) If $D = \{te : t \in [0, 1]\}$, $S = \{0, e\}$, and $A = \{u \in X : \|u\| = \rho\}$ (with $\rho > 0$ and $e \in X$ such that $\|e\| > \rho$), then we get the mountain pass theorem (see [1]).

(b) If $X = X_1 \oplus X_2$, with $\dim X_1 < +\infty$, $D = \{u \in X_1 : \|u\| \leq 1\}$, $S = \{u \in X_1 : \|u\| = 1\}$, and $A = X_2$, then we get the saddle point theorem (see [2]).

(c) If $X = X_1 \oplus X_2$, with $\dim X_1 < +\infty$, $D = \{u \in X_1 : \|u\| \leq r\} + \{te : t \in [0, r]\}$, $S = \partial D$, and $A = \{u \in X_2 : \|u\| = \rho\}$ (with $r > \rho > 0$ and $e \in X_2$ such that $\|e\| = 1$), we get the generalized mountain pass theorem (see [2]).

It is worth noting that generalizations of the minimax results cited in (a)–(c), relative to critical point theories for certain classes of nondifferentiable functionals, can already be found in the literature and have numerous applications to partial differential equations, differential inclusions, variational inequalities, hemivariational inequalities, and variational-hemivariational inequalities. In this respect, motivated by the study of existence of solutions of the so-called variational-hemivariational inequalities, introduced by Motreanu-Panagiotopoulos [6], we mention the critical point theory developed by these authors in [6] for functionals $f = \varphi + \psi : X \to \mathbb{R} \cup \{+\infty\}$, with $\varphi : X \to \mathbb{R}$ locally Lipschitz and $\psi : X \to \mathbb{R} \cup \{+\infty\}$ lower semicontinuous, convex, and not identically $+\infty$ (see also Chang [7] for the case when $\psi = 0$, and see Szulkin [8] for the case when $\varphi : X \to \mathbb{R}$ is of class C^1 and $\psi : X \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous, convex, and not identically $+\infty$).

The rest of the paper is organized as follows. Section 2 contains the proof of Proposition 4. Section 3 deals with the proofs of Theorems 5, 6, and 8 and Corollaries 9, 11, and 13. The method of proofs is based on suitable deformation results given in Sections 2, 3.

2. Proof of Proposition 4

We start with stating the following deformation result.

Theorem 15. *Let* X *be a metric space endowed with the metric* d, *and let* $f : X \to \mathbb{R}$ *be a continuous function. For all* $\rho > 0$, *there exists a deformation* $\eta : X \times [0, 1] \to X$ *(i.e.,* η *is continuous and satisfies* $\eta(u, 0) = u$ *for all* $u \in X$*) such that for all* $u \in X$ *and* $t \in [0, 1]$, *one has*

(a) $t \in (0, 1] \Rightarrow d(\eta(u, t), u) < \rho t$;

(b) $|df|(u) = 0 \Rightarrow \eta(u, t) = u$;

(c) $|df|(u) > 0$, $t \in (0, 1] \Rightarrow f(\eta(u, t)) < f(u)$.

Proof. First, assume that $|df| \not\equiv +\infty$. We consider the continuous function $\sigma : X \to [0, +\infty)$ given by

$$\sigma(u) := \frac{1}{2} \inf\{d(u, v) + |df|(v) : v \in X\}. \tag{15}$$

We have $\sigma(u) = 0$ for all $u \in X$ with $|df|(u) = 0$. Using that $u \mapsto |df|(u)$ is lower semicontinuous (see Remark 2(a)), we see that

$$|df|(u) \neq 0 \Longrightarrow |df|(u) > \sigma(u) > 0. \tag{16}$$

Applying [3, Theorem (2.8)], we find a deformation $\hat{\eta} : X \times [0, 1] \to X$ and a continuous map $\hat{\tau} : X \to [0, +\infty)$ with the properties (for all $u \in X$ and $t \in [0,1]$): (i) $\widehat{\tau}(u) = 0 \Leftrightarrow |df|(u) = 0$; (ii) $d(\widehat{\eta}(u,t),u) \le t$; (iii) $0 \le t \le \widehat{\tau}(u) \Rightarrow f(\widehat{\eta}(u,t)) \le f(u) - \sigma(u)t$. Then the deformation $\eta : X \times [0,1] \to X$ defined by

$$\eta(u,t) := \widehat{\eta}\left(u, \frac{1}{2}\min\{\widehat{\tau}(u), \rho\} t\right), \quad \forall u \in X, \forall t \in [0,1], \tag{17}$$

has properties (a)–(c) of the statement. If $|df| \equiv +\infty$, we make the same reasoning taking an arbitrary positive constant in place of σ. □

Proof of Proposition 4. Let $\overline{\rho} := \min\{d(S,A), \rho\} > 0$, and let $\eta : X \times [0,1] \to X$ be a deformation satisfying properties (a)–(c) of Theorem 15 (with $\overline{\rho}$ in place of ρ). Note that $S \subset X \setminus B_{\overline{\rho}}(A)$. Let $\vartheta : X \to [0,1]$ be a continuous function such that

$$\vartheta(u) = 0 \Longleftrightarrow u \in X \setminus B_{\overline{\rho}}(A). \tag{18}$$

Defining $\widetilde{\gamma} : D \to X$ by

$$\widetilde{\gamma}(u) := \eta(\gamma(u), \vartheta(\gamma(u))), \quad \forall u \in D, \tag{19}$$

with γ as in the statement of Proposition 4, we have $\widetilde{\gamma} \in \Gamma$. Fix $u_0 \in D$ with $\widetilde{\gamma}(u_0) \in A$ (see (3)). Then, using property (a) in Theorem 15, we infer that

$$d(\gamma(u_0), A) \le d(\gamma(u_0), \widetilde{\gamma}(u_0)) < \overline{\rho} \le \rho. \tag{20}$$

Hence, using properties (b), (c) of Theorem 15, relation (5), and the assumption that $a \in \mathbb{R}$, we obtain that

$$a \le f(\widetilde{\gamma}(u_0)) = f(\eta(\gamma(u_0), \vartheta(\gamma(u_0)))) \le f(\gamma(u_0)) \le a \tag{21}$$

and $|df|(\gamma(u_0)) = 0$. This combined with property (b) of Theorem 15 yields $\widetilde{\gamma}(u_0) = \gamma(u_0) \in K_a \cap A$, which completes the proof. □

3. Proofs of Theorems 5, 6, and 8 and Corollaries 9, 11, and 13

The following deformation result was shown in [9].

Theorem 16. *Let X be a metric space endowed with the metric d, let $f : X \to \mathbb{R}$ be a continuous function, let C be a subset of X, and let $\ell \in \mathbb{R}, \sigma > 0$, and $\rho > 0$ be real numbers. Assume that the metric space $f^{-1}([\ell, s])$ (endowed with the induced metric) is complete for all $s \in (\ell, \ell + \sigma\rho)$ and that*

$$|df|(u) > \sigma, \quad \forall u \in B_\rho(C) \cap [\ell < f < \ell + \sigma\rho]. \tag{22}$$

Then there exist a continuous function $\tau : C \cap [f < \ell + \sigma\rho] \to [0,\rho)$ and a continuous map $\eta : (C \cap [f < \ell + \sigma\rho]) \times [0,1] \to [f < \ell + \sigma\rho]$ such that for all $u \in C \cap [f < \ell + \sigma\rho]$ and $t \in [0,1]$, one has

(a) $\tau(u) \le (1/\sigma)\max\{f(u) - \ell, 0\}$;
(b) $d(\eta(u,t), u) \le \tau(u)t$;
(c) $f(\eta(u,t)) \le f(u) - \sigma\tau(u)t$;
(d) $f(u) \ge \ell \Rightarrow f(\eta(u,1)) = \ell$.

Hereafter, we denote $[f < s] = \{u \in X : f(u) < s\}$ and $[\ell < f < s] = \{u \in X : \ell < f(u) < s\}$.

Proof of Theorem 5. For a fixed $\delta > 0$, it suffices to prove that

$$\begin{aligned} &\inf\{|df|(u) : u \in [c < f < c+\delta] \cap B_\delta(A)\} > 0 \\ &\quad \Longrightarrow K_a \cap A \ne \emptyset. \end{aligned} \tag{23}$$

To this end, let σ be a number such that

$$|df|(u) > \sigma, \quad \forall u \in [c < f < c+\delta] \cap B_\delta(A). \tag{24}$$

Let $\rho > 0$ such that

$$\sigma\rho \le \delta, \quad 4\rho \le \min\{d(S,A), \delta\}. \tag{25}$$

In particular, these choices combined with (24) guarantee that

$$|df|(u) > \sigma, \quad \forall u \in B_\rho(\overline{B}_{3\rho}(A)) \cap [c < f < c+\sigma\rho]. \tag{26}$$

Hereafter, we denote $\overline{B}_{3\rho}(A) = \{u \in X : d(u,A) \le 3\rho\}$. Thus, applying Theorem 16 with the numbers ρ and σ, $C := \overline{B}_{3\rho}(A)$, and $\ell := c$, we find a deformation $\widetilde{\eta} : (C \cap [f < c+\sigma\rho]) \times [0,1] \to [f < c+\sigma\rho]$ such that

$$\begin{aligned} &d(\widetilde{\eta}(u,t), u) < \rho, \qquad f(\widetilde{\eta}(u,1)) \le c = a, \\ &\forall u \in C \cap [f < c+\sigma\rho], \ \forall t \in [0,1]. \end{aligned} \tag{27}$$

Using that $3\rho \le d(S,A)$, we see that $S \subset (X \setminus B_{3\rho}(A)) \cap [f < c+\sigma\rho]$. Let $\vartheta : [f < c+\sigma\rho] \to [0,1]$ be a continuous function such that $\vartheta(u) = 0$ if $u \in X \setminus B_{3\rho}(A)$ and $\vartheta(u) = 1$ if $u \in \overline{B}_{2\rho}(A)$. Define the deformation $\eta : [f < c+\sigma\rho] \times [0,1] \to [f < c+\sigma\rho]$ by

$$\eta(u,t) := \begin{cases} \widetilde{\eta}(u, \vartheta(u)t) & \text{if } u \in B_{3\rho}(A), \\ u & \text{otherwise.} \end{cases} \tag{28}$$

To prove the continuity of η, we note that the restriction of η to the sets $([f < c+\sigma\rho] \setminus B_{3\rho}(A)) \times [0,1]$ and $([f < c+\sigma\rho] \cap \overline{B}_{3\rho}(A)) \times [0,1]$, which are closed in $[f < c+\sigma\rho]$, is continuous. Indeed, the restriction of η to $([f < c+\sigma\rho] \setminus B_{3\rho}(A)) \times [0,1]$ coincides with the projection $(u,t) \mapsto u$, whereas the restriction of η to $([f < c+\sigma\rho] \times \overline{B}_{3\rho}(A)) \times [0,1]$ coincides with the map $(u,t) \mapsto \widetilde{\eta}(u, \vartheta(u)t)$ (since $\vartheta \equiv 0$ on $\partial B_{3\rho}(A) := \overline{B}_{3\rho}(A) \setminus B_{3\rho}(A)$) which is continuous (by the continuity of ϑ).

Let $\gamma_0 \in \Gamma$ with $\max_D(f \circ \gamma_0) < c+\sigma\rho$. Then the map $\gamma : D \to X$ defined by

$$\gamma(u) := \eta(\gamma_0(u), 1) \tag{29}$$

is well defined and $\gamma \in \Gamma$. Finally, let $u \in D$ such that $\gamma(u) \in B_\rho(A)$. Since

$$d(\gamma(u), \gamma_0(u)) = d(\eta(\gamma_0(u), 1), \gamma_0(u)) < \rho \tag{30}$$

(by the first part of (27)), we have that $\gamma_0(u) \in B_{2\rho}(A)$, so $\vartheta(\gamma_0(u)) = 1$; thus

$$f(\gamma(u)) = f(\eta(\gamma_0(u), 1)) = f(\tilde{\eta}(\gamma_0(u), 1)) \leq a \tag{31}$$

(by the second part of (27)). Consequently, we can apply Proposition 4. Therefore, $K_a \cap A \neq \emptyset$. This concludes the proof. □

Proof of Theorem 6. (a) By (3), we see that $d(S, A) > 0$. For a fixed $\delta > 0$ and $\overline{\rho} \geq 2d(S, A)$, it suffices to prove that

$$\begin{aligned} &\inf\left\{|df|(u) : u \in [a < f < c+\delta] \cap B_{\overline{\rho}}(A)\right\} \\ &\quad > \frac{c-a}{d(S,A)} \Longrightarrow K_a \cap A \neq \emptyset. \end{aligned} \tag{32}$$

To this end, let σ be a number such that

$$|df|(u) > \sigma > \frac{c-a}{d(S,A)}, \quad \forall u \in [a < f < c+\delta] \cap B_{\overline{\rho}}(A). \tag{33}$$

Let ρ, ε be positive numbers such that

$$\frac{c-a}{\sigma} < \rho < \frac{c-a+\delta}{\sigma}, \qquad \rho + 2\varepsilon < d(S,A). \tag{34}$$

Setting $\theta := a + \sigma\rho - c \in (0, \delta)$, since $2(\rho + \varepsilon) < \overline{\rho}$, we have

$$|df|(u) > \sigma, \quad \forall u \in B_\rho\left(\overline{B}_{\rho+2\varepsilon}(A)\right) \cap [a < f < c+\theta]. \tag{35}$$

This allows us to apply Theorem 16 with the numbers ρ and σ, $C := \overline{B}_{\rho+2\varepsilon}(A)$, and $\ell := a$. Hence we find a deformation $\tilde{\eta} : (C \cap [f < c+\theta]) \times [0,1] \to [f < c+\theta]$ such that

$$\begin{aligned} &d(\tilde{\eta}(u,t), u) < \rho, \qquad f(\tilde{\eta}(u,1)) \leq a, \\ &\forall u \in C \cap [f < c+\theta], \ \forall t \in [0,1]. \end{aligned} \tag{36}$$

The fact that $\theta > 0$ in conjunction with the inequality $\rho + 2\varepsilon < d(S, A)$ yields $S \subset (X \setminus B_{\rho+2\varepsilon}(A)) \cap [f < c+\theta]$. Let $\vartheta : [f < c+\theta] \to [0,1]$ be a continuous function such that $\vartheta(u) = 0$ if $u \in X \setminus B_{\rho+2\varepsilon}(A)$ and $\vartheta(u) = 1$ if $u \in \overline{B}_{\rho+\varepsilon}(A)$. Define the deformation $\eta : [f < c+\theta] \times [0,1] \to [f < c+\theta]$ by

$$\eta(u,t) := \begin{cases} \tilde{\eta}(u, \vartheta(u)t) & \text{if } u \in B_{\rho+2\varepsilon}(A) \\ u & \text{otherwise.} \end{cases} \tag{37}$$

The continuity of η can be shown as in the proof of Theorem 5. Let $\gamma_0 \in \Gamma$ with $\max_D(f \circ \gamma_0) < c + \theta$. Then the map $\gamma : D \to X$ given by

$$\gamma(u) := \eta(\gamma_0(u), 1), \quad \forall u \in D, \tag{38}$$

is well defined, and we have $\gamma \in \Gamma$. Let $u \in D$ such that $\gamma(u) \in B_\varepsilon(A)$. Since

$$d(\gamma(u), \gamma_0(u)) = d(\eta(\gamma_0(u), 1), \gamma_0(u)) < \rho, \tag{39}$$

it follows that $\gamma_0(u) \in B_{\rho+\varepsilon}(A)$, which yields

$$f(\gamma(u)) = f(\eta(\gamma_0(u), 1)) = f(\tilde{\eta}(\gamma_0(u), 1)) \leq a. \tag{40}$$

Consequently, Proposition 4 (with ρ replaced by ε) can be applied, and we conclude that $K_a \cap A \neq \emptyset$.

(b) We see that $d(S \cap [f > a], A) \geq d(S, A) > 0$. For a fixed $\delta > 0$ and $\overline{\rho} \geq 2d(S \cap [f > a], A)$, it suffices to show that

$$\begin{aligned} &\inf\left\{|df|(u) : u \in [a < f < c+\delta] \cap B_{\overline{\rho}}(A)\right\} \\ &\quad > \frac{c-a}{d(S \cap [f > a], A)} \Longrightarrow K_a \cap A \neq \emptyset. \end{aligned} \tag{41}$$

To this end, let σ be a number such that

$$\begin{aligned} &|df|(u) > \sigma > \frac{c-a}{d(S \cap [f > a], A)}, \\ &\forall u \in [a < f < c+\delta] \cap B_{\overline{\rho}}(A). \end{aligned} \tag{42}$$

Let ρ, ε be positive numbers such that

$$\frac{c-a}{\sigma} < \rho < \frac{c-a+\delta}{\sigma}, \qquad \rho + 2\varepsilon < d(S \cap [f > a], A). \tag{43}$$

Setting $\theta := a + \sigma\rho - c \in (0, \delta)$, we have

$$|df|(u) > \sigma, \quad \forall u \in B_\rho\left(\overline{B}_{\rho+2\varepsilon}(A)\right) \cap [a < f < c+\theta]. \tag{44}$$

Thus, we can apply Theorem 16 with the numbers ρ and σ, $C := \overline{B}_{\rho+2\varepsilon}(A)$, and $\ell := a$. So we find a deformation $\tilde{\eta} : (C \cap [f < c+\theta]) \times [0,1] \to [f < c+\theta]$ such that

$$\begin{aligned} &\tilde{\eta}(u,t) = u, \quad \forall (u,t) \in (C \cap [f < c+\theta]) \times [0,1] \\ &\qquad\qquad\qquad\qquad \text{with } f(u) \leq a, \end{aligned} \tag{45}$$

$$\begin{aligned} &d(\tilde{\eta}(u,t), u) < \rho, \qquad f(\tilde{\eta}(u,1)) \leq a, \\ &\forall u \in C \cap [f < c+\theta], \ \forall t \in [0,1]. \end{aligned} \tag{46}$$

Let $\vartheta : [f < c+\theta] \to [0,1]$ be a continuous function such that $\vartheta(u) = 0$ if $u \in X \setminus B_{\rho+2\varepsilon}(A)$ and $\vartheta(u) = 1$ if $u \in \overline{B}_{\rho+\varepsilon}(A)$. Define the deformation $\eta : [f < c+\theta] \times [0,1] \to [f < c+\theta]$ by

$$\eta(u,t) := \begin{cases} \tilde{\eta}(u, \vartheta(u)t) & \text{if } u \in B_{\rho+2\varepsilon}(A), \\ u & \text{otherwise.} \end{cases} \tag{47}$$

The continuity of η can be shown as in the proof of Theorem 5.

Let $\gamma_0 \in \Gamma$ with $\max_D(f \circ \gamma_0) < c + \theta$. Define $\gamma : D \to X$ by

$$\gamma(u) := \begin{cases} \eta(\gamma_0(u), 1) & \text{if } f(\gamma_0(u)) > a, \\ \gamma_0(u) & \text{otherwise.} \end{cases} \tag{48}$$

We show that $\gamma \in \Gamma$. To prove that γ is continuous, it suffices to note that the restrictions of γ to the closed sets $\{u \in D :$

$f(\gamma_0(u)) \geq a\}$ and $\{u \in D : f(\gamma_0(u)) \leq a\}$ are continuous. This is immediate for the latter closed set. Concerning the former, this follows from the fact that the equality $\gamma(u) = \eta(\gamma_0(u), 1)$ holds whenever $f(\gamma_0(u)) = a$ (which is a consequence of the definition of η and relation (45)). Whence γ is continuous. Using that $S \cap [f > a] \subset X \setminus \overline{B}_{\rho+2\varepsilon}(A)$, we infer that $\gamma \in \Gamma$.

Let an arbitrary point $u \in D$ with $d(\gamma(u), A) < \varepsilon$. If $f(\gamma_0(u)) > a$, then

$$d(\gamma(u), \gamma_0(u)) = d(\eta(\gamma_0(u), 1), \gamma_0(u)) < \rho \tag{49}$$

(by the first part of (46)), so $\gamma_0(u) \in B_{\rho+\varepsilon}(A)$, and thus $f(\gamma(u)) = f(\eta(\gamma_0(u), 1)) = f(\tilde{\eta}(\gamma_0(u), 1)) \leq a$ (by the second part of (46)). If $f(\gamma_0(u)) \leq a$, then it is clear that $f(\gamma(u)) \leq a$. Consequently, in both cases, we have $f(\gamma(u)) \leq a$. This allows us to apply Proposition 4 (with ρ replaced by ε) and thus to obtain that $K_a \cap A \neq \emptyset$, which completes the proof. □

Proof of Theorem 8. (a) The hypotheses allow us to apply Theorem 5 with $A := \{u \in X : f(u) \geq c\}$. Indeed, since $c > b$, we have that $S \cap A = \emptyset$. Moreover, since D is compact, for each $\gamma \in \Gamma$, there exists $u_\gamma \in D$ with $\max_D(f \circ \gamma) = f(\gamma(u_\gamma)) \geq c$. It follows that condition (3) is satisfied and

$$c = \inf_{\gamma\in\Gamma} f(\gamma(u_\gamma)) \geq \inf_A f \geq c, \tag{50}$$

so $\inf_A f = c > -\infty$. The conclusion follows now from Theorem 5.

(b) This follows from Theorem 6(b). □

Proof of Corollary 9. (a) If $c > a$, then we have $c > b$. Thus, we can apply Theorem 8(a). In the case $c = a$, we apply Theorem 5.

(b) If $c > b$, then we obtain from Theorem 8(a) that (10) holds (so that a fortiori (12) holds since $b - a \geq 0$). If $c = b$, then we apply Theorem 8(b). □

Proof of Corollary 11. Parts (a) and (b) follow from Theorems 5 and 8(a), respectively, taking into account the lower semicontinuity of the map $u \mapsto |df|(u)$ (see Remark 2(a)). □

Proof of Corollary 13. Fix an arbitrary $n \geq 1$. If $b_n \leq a_n$, then applying Corollary 9(a), we have

$$\inf\left\{|df|(u) : c_n \leq f(u) < c_n + \frac{1}{n}\right\} = 0. \tag{51}$$

If $b_n > a_n$, then using Corollary 9(b), we have

$$\inf\left\{|df|(u) : a_n \leq f(u) < c_n + \frac{1}{n}\right\} \leq \frac{b_n - a_n}{d(S_n \cap [f > a_n], A_n)}. \tag{52}$$

Let

$$\sigma_n := \frac{\max\{b_n - a_n, 0\}}{d(S_n \cap [f > a_n], A_n)}. \tag{53}$$

Then we have $\sigma_n \to 0$ as $n \to \infty$ and

$$\inf\left\{|df|(u) : a_n \leq f(u) < c_n + \frac{1}{n}\right\} \leq \sigma_n, \quad \forall n \geq 1. \tag{54}$$

This completes the proof. □

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