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Research Article

Qualitative Analysis of Solutions of Nonlinear Delay Dynamic Equations

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We use the fixed point theory to investigate the qualitative analysis of a nonlinear delay dynamic equation on an arbitrary time scales. We illustrate our results by applying them to various kind of time scales.

1. Introduction

In this paper, we investigate the qualitative analysis of solutions of nonlinear delay dynamic equation of the form

$$x^{\Delta}(t) = -a(t)\, g(x(\delta(t)))\, \delta^{\Delta}(t), \quad t \in [t_0, \infty)_{\mathbb{T}} \tag{1}$$

on an arbitrary time scale $\mathbb{T}$ which is unbounded above, where the functions a and g are rd-continuous, the delay function $\delta : [t_0, \infty)_{\mathbb{T}} \to [\delta(t_0), \infty)_{\mathbb{T}}$ is strictly increasing, invertible, and delta differentiable such that $\delta(t) < t$, $|\delta^{\Delta}(t)| < \infty$ for $t \in \mathbb{T}$, and $\delta(t_0) \in \mathbb{T}$.

Although it is assumed that the reader is already familiar with the time scale calculus, for completeness, we will provide some essential information about time scale calculus in the Section 1.1. We should only mention here that this theory was introduced in order to unify continuous and discrete analysis; however it is not only unify the theories of differential equations and of difference equations, but also it is able to extend these classical cases to cases "in between," for example, to so-called q-difference equations. Also note that, when $\mathbb{T} = \mathbb{R}$, (1) is reduced to the nonlinear delay differential equation

$$x'(t) = -a(t)\, g(x(t - r)) \tag{2}$$

and when $\mathbb{T} = \mathbb{Z}$, it becomes a nonlinear delay difference equation

$$\Delta x(t) = -a(t)\, g(x(t - r)). \tag{3}$$

In the case of quantum calculus which defined as $\mathbb{T} = q^{\mathbb{N}} := \{q^m : m \in \mathbb{N}\}$, $q > 1$ is a real number, (1) leads to the nonlinear delay q-difference equation

$$\Delta_q x(t) = -a(t)\, g(x(\delta(t)))\, \Delta_q \delta(t), \tag{4}$$

where $\Delta_q f(t) = (f(qt) - f(t))/(q-1)t$.

Motivated by the papers [1, 2], in this paper we study the qualitative properties of solution of nonlinear delay dynamic equation (1) by means of fixed point theory. The results of this paper unify the results given by [1] for (2) and by [2] for (3). Moreover, we obtain new results for the q-difference equation (4) and explicitly provide an example in which we show how our conditions can be applied. Our technique in proving the results naturally has some common features with the ones employed in both [1] and [2] but it is actually quite different due to difficulties that are peculiar to the time scale calculus. Also, our results may be considered as generalization of the ones obtained in [3, 4] and [5] in which the authors studied the stability of the delay dynamic equation

$$x^{\Delta}(t) = -a(t)\, x(\delta(t))\, \delta^{\Delta}(t). \tag{5}$$

In [6], the authors establish some sufficient conditions for the uniform stability and the uniformly asymptotical stability of the first order delay dynamic equation

$$x^{\Delta}(t) = p(t)\, x(t - \tau(t)). \tag{6}$$

One can easily see that the results of (6) cannot be applied to q-difference equations. Moreover, it requires that $t-\tau(t)$ be in the time scale. For resent results regarding existence, uniqueness and continuous dependence of the solution for nonlinear delay dynamic equations, we refer to [7].

1.1. Preliminaries on Time Scales. In this subsection, we recall some of the notations, definitions, and theorems on time scale calculus that we use throughout the paper. An excellent comprehensive treatment of calculus on time scales can be found in [8, 9]. Most of the material in this subsection can be found in [8, Chapter 1]. We should start mentioning that this theory was introduced in order to unify continuous and discrete analysis; however, it does not only unify the theories of differential equations and difference equations, but also it enable us to extend these classical cases to cases "in between," for example, to so-called q-difference equations.

A time scale is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$ and is denoted by the symbol $\mathbb{T}$. The two most popular examples are $\mathbb{T}=\mathbb{R}$ and $\mathbb{T}=\mathbb{Z}$. Several other interesting time scales exist, and they give rise to plenty of applications such as the study of population dynamic models (see [8, pages 15 and 71]). Define the time scale interval $[a,b]_{\mathbb{T}}$ by $[a,b]_{\mathbb{T}} := [a,b]\cap\mathbb{T}$. Other time scale intervals are defined similarly. The forward jump operator $\sigma(t)$ at $t\in\mathbb{T}$ for $t<\sup\mathbb{T}$ is defined by $\sigma(t) := \inf\{s\in\mathbb{T} : s>t\}$ with $\sigma(\sup\mathbb{T})=\sup\mathbb{T}$. The graininess function $\mu:\mathbb{T}\to\mathbb{R}$ is defined by $\mu(t)=\sigma(t)-t$. The backward jump operator $\rho(t)$ at $t\in\mathbb{T}$ for $-\infty<\inf\mathbb{T}$ is defined by $\rho(t) := \sup\{s\in\mathbb{T} : s<t\}$ with $\rho(\inf\mathbb{T})=\inf\mathbb{T}$. The point $t\in\mathbb{T}$ is called right scattered if $\sigma(t)>t$, left scattered if $\rho(t)<t$, and dense if $\sigma(t)=t$.

Fix $t\in\mathbb{T}$ and let $f:\mathbb{T}\to\mathbb{R}$. Define $f^{\Delta}(t)$ (called the delta derivatives of f at t) to be the number (if exists) with the property that given $\epsilon>0$ there is a neighborhood U of t such that, for all $s\in U$,

$$\left|\left[f(\sigma(t))-f(s)\right]-f^{\Delta}(t)\left[\sigma(t)-s\right]\right|\le\epsilon\left|\sigma(t)-s\right|. \quad (7)$$

Some elementary facts concerning the delta derivative are as follows:

(1) if f is differentiable at t, then

$$f^{\sigma}(t)=f(\sigma(t))=f(t)+\mu(t)f^{\Delta}(t); \quad (8)$$

(2) if f and g are differentiable at t, then fg is differentiable at t with

$$\begin{aligned}(fg)^{\Delta}(t)&=f^{\sigma}(t)g^{\Delta}(t)+f^{\Delta}(t)g(t)\\&=f(t)g^{\Delta}(t)+f^{\Delta}(t)g^{\sigma}(t);\end{aligned} \quad (9)$$

(3) if f and g are differentiable at t and $g(t)g(\sigma(t))\neq0$, then f/g is differentiable at t with

$$\left(\frac{f}{g}\right)^{\Delta}(t)=\frac{f^{\Delta}(t)g(t)-f(t)g^{\Delta}(t)}{g(t)g^{\sigma}(t)}. \quad (10)$$

We say $f:\mathbb{T}\to\mathbb{R}$ is right-dense continuous ($f\in C_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$) provided f is continuous at right-dense points in $\mathbb{T}$ and its left-sided limit exists (finite) at left-dense points in $\mathbb{T}$. The importance of rd-continuous functions is that *every rd-continuous function possesses an antiderivative*. A function $F:\mathbb{T}^{\kappa}\to\mathbb{R}$ is called an antiderivative of $f:\mathbb{T}\to\mathbb{R}$ provided $F^{\Delta}(t)=f(t)$ holds for all $t\in\mathbb{T}^{\kappa}$.

Some elementary facts concerning the delta integral are as follows:

if $a,b,c\in\mathbb{T}$, $\alpha\in\mathbb{R}$, and $f,g\in C_{\mathrm{rd}}$, then

(a) $\int_a^b f(\sigma(t))g^{\Delta}(t)\Delta t=(fg)(b)-(fg)(a)-\int_a^b f^{\Delta}(t)\times g(t)\Delta t$;

(b) $\int_a^b f(t)g^{\Delta}(t)\Delta t=(fg)(b)-(fg)(a)-\int_a^b f^{\Delta}(t)\times g(\sigma(t))\Delta t$;

(c) If $|f(t)|\le g(t)$ on $[a,b)_{\mathbb{T}}$, then $|\int_a^b f(t)\Delta t|\le\int_a^b g(t)\Delta t$;

(d) If $f(t)\ge0$ for all $t\in[a,b)_{\mathbb{T}}$, then $\int_a^b f(t)\Delta t\ge0$.

Now, we present a chain rule: Assume that $\nu:\mathbb{T}\to\mathbb{R}$ is a strictly increasing and $\widetilde{\mathbb{T}} := \upsilon(\mathbb{T})$ is a time scale. Let $\omega:\widetilde{\mathbb{T}}\to\mathbb{R}$. If $\nu^{\Delta}(t)$ and $\omega^{\tilde{\Delta}}(\nu(t))$ exists for $t\in\mathbb{T}^{\kappa}$, then $(w\circ\nu)^{\Delta}=(\omega^{\tilde{\Delta}}\circ\nu)\nu^{\Delta}$.

A function $p:\mathbb{T}\to\mathbb{R}$ is called regressive if it is rd-continuous and satisfies

$$1+\mu(t)p(t)\neq0\quad\forall t\in\mathbb{T}. \quad (11)$$

The set of all regressive functions will be denoted by $\mathscr{R}$. Also $p\in\mathscr{R}^{+}$ (positively regressive) if and only if $p\in\mathscr{R}$ and $1+\mu(t)p(t)>0$, for all $t\in\mathbb{T}$.

For $h>0$, define the cylinder transformation $\xi_h(z)=\mathbb{C}_h\to\mathbb{Z}_h$ by

$$\xi_h(z)=\frac{1}{h}\operatorname{Log}(1+zh), \quad (12)$$

where Log is the principal logarithm function. For $h=0$, we define $\xi_h(z)=z$ for all $z\in\mathbb{C}$.

If $p\in\mathscr{R}$, then we define the generalized exponential function

$$e_p(t,t_0)=\exp\left(\int_{t_0}^{t}\xi_{\mu(\tau)}(p(\tau))\Delta\tau\right) \quad (13)$$

for all $t,t_0\in\mathbb{T}$. Note that one can also define the generalized exponential function $e_p(t,t_0)$ to be the unique solution of the initial value problem

$$x^{\Delta}(t)=p(t)x(t),\quad x(t_0)=1. \quad (14)$$

Also, it is well know that, if $p\in\mathscr{R}^{+}$, then $e_p(t,s)>0$ for all $t\in\mathbb{T}$. We will use many of the following properties of the generalized exponential function $e_p(t,t_0)$ in our calculations.

If $p, q \in \mathscr{R}$ and $t, s, r, c \in \mathbb{T}$, then the properties of generalized exponential function are:

(1) $e_0(t,s) \equiv 1$ and $e_p(t,t) \equiv 1$;
(2) $e_p(\sigma(t),s) = (1+\mu(t)p(t))e_p(t,s)$;
(3) $1/e_p(t,s) = e_{\ominus p}(t,s)$, where $\ominus p := -p/(1+\mu(t)p(t))$;
(4) $e_p(t,s) = 1/e_p(s,t) = e_{\ominus p}(s,t)$;
(5) $e_p(t,s)e_p(s,r) = e_p(t,r)$;
(6) $e_p(t,s)e_q(t,s) = e_{p\oplus q}(t,s)$, where $p \oplus q := p+q+\mu pq$;
(7) $e_p(t,s)/e_p(t,s) = e_{p\ominus q}(t,s)$;
(8) $[e_p(c,\cdot)]^\Delta = -p[e_p(c,\cdot)]^\sigma$.

Another useful tool is a variation of parameters formula for first order linear nonhomogeneous dynamic equations which now we state next. Suppose that $p \in \mathscr{R}$ and f is rd-continuous function. Let $t_0 \in \mathbb{T}$ and $x_0 \in \mathbb{R}$. Then the unique solution of

$$x^\Delta = p(t)x + f(t), \quad x(t_0) = x_0 \tag{15}$$

is given by

$$x(t) = e_p(t,t_0)x_0 + \int_{t_0}^{t} e_p(t,\sigma(\tau)) f(\tau)\Delta\tau. \tag{16}$$

1.2. Solution. For each $t_0 \in \mathbb{T}$ and for a given rd-continuous initial function $\psi := [\delta(t_0), t_0]_\mathbb{T} \to \mathbb{R}$, we say that $x(t) := x(t; t_0, \psi)$ is the solution of (1) if $x(t) = \psi(t)$ on $[\delta(t_0), t_0]_\mathbb{T}$ and satisfies (1) for all $t \geq t_0$. The zero solution of (1) is called stable if for any $\epsilon > 0$ and $t_0 \in \mathbb{T}$, there exists a $\eta(t_0,\epsilon) > 0$ such that $|\psi| < \eta$ implies $|x(t;t_0,\psi)| < \epsilon$ for all $t \geq t_0$.

We need the following lemmas in proving our main theorem.

Lemma 1 (see [10, Lemma 2]). *For nonnegative p with $-p \in \mathscr{R}^+$ one has the inequalities*

$$1 - \int_s^t p(u)\Delta u \leq e_{-p}(t,s) \leq \exp\left\{-\int_s^t p(u)\Delta u\right\} \quad \forall t \geq s. \tag{17}$$

Lemma 2 (see [3, Lemma 3], [4]). *Suppose that $\mathbb{T}$ is a time scale having a strictly increasing, and invertible delay function $\delta : [t_0,\infty)_\mathbb{T} \to [\delta(t_0),\infty)_\mathbb{T}$, $t_0 \in \mathbb{T}$ such that $\delta(t) < t$ and $|\delta^\Delta(t)| < \infty$. Then for a given rd-continuous function $f : \mathbb{T} \to \mathbb{R}$, one has*

$$\left(\int_{\delta(t)}^{t} f(s)\Delta s\right)^\Delta = f(t) - f(\delta(t))\delta^\Delta(t). \tag{18}$$

Lemma 3. *Assume $\mathbb{T}$ is a time scale having a strictly increasing and invertible delay function $\delta : [t_0,\infty)_\mathbb{T} \to [\delta(t_0),\infty)_\mathbb{T}$, $t_0 \in \mathbb{T}$ such that $\delta(t) < t$ and $|\delta^\Delta(t)| < \infty$. Then, the nonlinear delay equation* (1) *is equivalent to*

$$\begin{aligned} x^\Delta(t) = &-a\left(\delta^{-1}(t)\right)g(x(t)) \\ &+\left(\int_{\delta(t)}^{t} a\left(\delta^{-1}(s)\right)g(x(s))\Delta s\right)^\Delta. \end{aligned} \tag{19}$$

Proof. Assume x is a solution of (1). Then the proof immediately follows from Lemma 2 since

$$\begin{aligned} &\left(\int_{\delta(t)}^{t} a\left(\delta^{-1}(s)\right)g(x(s))\Delta s\right)^\Delta \\ &\quad = a\left(\delta^{-1}(t)\right)g(x(t)) - a(t)g(x(\delta(t)))\delta^\Delta(t). \end{aligned} \tag{20}$$

□

Lemma 4. *Suppose that $-a \in \mathscr{R}$. If x is a solution of* (1) *with initial function ψ, then*

$$\begin{aligned} x(t) = {}& e_{-a(\delta^{-1})}(t,t_0)\psi(t_0) \\ &+ \int_{\delta(t)}^{t} a\left(\delta^{-1}(s)\right)g(x(s))\Delta s - e_{-a(\delta^{-1})}(t,t_0) \\ &\times \int_{\delta(t_0)}^{t_0} a\left(\delta^{-1}(s)\right)g(\psi(s))\Delta s \\ &- \int_{t_0}^{t} a\left(\delta^{-1}(s)\right)e_{-a(\delta^{-1})}(t,\sigma(s)) \\ &\times \left(\int_{\delta(s)}^{s} a\left(\delta^{-1}(u)\right)g(x(u))\Delta u\right)\Delta s \\ &+ \int_{t_0}^{t} a\left(\delta^{-1}(s)\right)e_{-a(\delta^{-1})}(t,\sigma(s))\left[x(s)-g(x(s))\right]\Delta s. \end{aligned} \tag{21}$$

Proof. We know from Lemma 3 that (1) is equivalent to (19). To create a linear term, we add and subtract $a(\delta^{-1}(t))x(t)$ in (19) to obtain

$$\begin{aligned} x^\Delta(t) = &-a\left(\delta^{-1}(t)\right)x(t) + a\left(\delta^{-1}(t)\right)\left[x(t)-g(x(t))\right] \\ &+\left(\int_{\delta(t)}^{t} a\left(\delta^{-1}(s)\right)g(x(s))\Delta s\right)^\Delta. \end{aligned} \tag{22}$$

Using the variation of constants formula page 77 [8] for (22) yields

$$\begin{aligned} x(t) = {}& e_{-a(\delta^{-1})}(t,t_0)x(t_0) \\ &+ \int_{t_0}^{t} e_{-a(\delta^{-1})}(t,\sigma(s))a\left(\delta^{-1}(s)\right)\left[x(s)-g(x(s))\right]\Delta s \\ &+ \int_{t_0}^{t} e_{-a(\delta^{-1})}(t,\sigma(s)) \\ &\times \left(\int_{\delta(s)}^{s} a\left(\delta^{-1}(u)\right)g(x(u))\Delta u\right)^{\Delta_s}\Delta s, \end{aligned} \tag{23}$$

where Δ_s denotes the delta derivative with respect to s. The proof follows from using integration by parts formula on the last term of the right hand side of (23). □

2. Main Results

Let $\mathbb{T}$ be an arbitrary time scale which is unbounded above and consider the nonlinear delay dynamic equation

$$x^\Delta(t) = -a(t)g(x(\delta(t)))\delta^\Delta(t), \quad t \in [t_0,\infty)_\mathbb{T}, \tag{24}$$

with rd-continuous initial function $\psi := [\delta(t_0), t_0]_\mathbb{T} \to \mathbb{R}$.

In the sequel we assume the following:

(a) $\sup \mathbb{T} = \infty$;

(b) $a : [t_0, \infty)_{\mathbb{T}} \to \mathbb{R}$ rd-continuous and $-a \in \mathscr{R}^+$;

(c) the delay function $\delta : [t_0, \infty)_{\mathbb{T}} \to [\delta(t_0), \infty)_{\mathbb{T}}$ is strictly increasing, invertible, and delta differentiable such that $\delta(t) < t$, $|\delta^{\Delta}(t)| < \infty$ for $t \in \mathbb{T}$, and $\delta(t_0) \in \mathbb{T}$;

(d) $g : \mathbb{R} \to \mathbb{R}$ is continuous, locally Lipschitz and odd while $g(x)$ is rd-continuous, $x - g(x)$ is nondecreasing and $g(x)$ is increasing on an interval $[0, L]$ with $L > 0$, where $x : \mathbb{T} \to \mathbb{R}$ is rd-continuous function.

We should remark here that condition (d) in our hypotheses ensures that the function $g(x)$ and $x - g(x)$ are locally Lipschitz with the same Lipschitz constant $K > 0$. Also it is clear that if $0 < L_1 < L$, then the condition on g given by (d) hold on $[-L_1, L_1]$. Moreover, for any given rd-continuous function $\phi : [t_0, \infty)_{\mathbb{T}} \to \mathbb{R}$ with $\phi_0 = \psi$ is bounded with $|\phi(t)| \le L$, then for $t \ge t_0$ we have

$$|\phi(t) - g(\phi(t))| \le L - g(L), \tag{25}$$

since $x - g(x)$ is odd and nondecreasing on $(0, L)$.

We need to construct a mapping which is suitable for fixed point theory. Instead of using the supremum norm, we will use nonconventional metric to define a new norm in order to overcome the difficulties that arise from the contraction constant which, in turn rely on the Lipschitz constant.

Lemma 5. *Let (a)–(d) of our hypotheses hold. Let $L > 0$ and $\psi(t_0)$ be a fixed number for $t_0 \in \mathbb{T}$,*

$$\begin{aligned} S = \{&\phi : [t_0, \infty)_{\mathbb{T}} \longrightarrow \mathbb{R} \mid \phi \in C_{rd},\ \phi(t_0) = \psi(t_0), \\ &|\phi(t)| \le L\}, \end{aligned} \tag{26}$$

and $f : [-L, L] \to \mathbb{R}$ satisfy a Lipschitz condition with constant $K > 0$. Suppose that $h : [t_0, \infty)_{\mathbb{T}} \to \mathbb{R}$ is rd-continuous and for $\phi \in S$ define

$$(P\phi)(t) = h(t) + \int_{t_0}^{t} a(\delta^{-1}(s)) e_{-a(\delta^{-1})}(t, \sigma(s)) f(\phi(s)) \Delta s. \tag{27}$$

If $P : S \to S$, then for each $d > 1$ there is a metric ν on S such that P is a contraction with constant $1/d$ and (S, ν) is a complete metric space.

Proof. Let $(\mathscr{B}, |\cdot|_K)$ be the Banach space of rd-continuous function $\phi : [t_0, \infty)_{\mathbb{T}} \to \mathbb{R}$ for which

$$|\phi|_K := \sup_{t \ge t_0} e_{-(dK+1)|a(\delta^{-1})|}(t, t_0) |\phi(t)| \tag{28}$$

exists. If $\phi, \varphi \in M$, then

$$\begin{aligned} &|P\phi - P\varphi|_K \\ &\quad \le \sup_{t \ge t_0} e_{-(dK+1)|a(\delta^{-1})|}(t, t_0) \\ &\qquad \times \int_{t_0}^{t} |a(\delta^{-1}(s))| e_{-a(\delta^{-1})}(t, \sigma(s)) \\ &\qquad\qquad \times |f(\phi(s)) - f(\varphi(s))| \Delta s \\ &\quad \le \sup_{t \ge t_0} e_{-(dK+1)|a(\delta^{-1})|}(t, t_0) \\ &\qquad \times \int_{t_0}^{t} |a(\delta^{-1}(s))| e_{-a(\delta^{-1})}(t, \sigma(s)) K |\phi(s) - \varphi(s)| \Delta s \\ &\quad = \sup_{t \ge t_0} \int_{t_0}^{t} |a(\delta^{-1}(s))| e_{-a(\delta^{-1})}(t, \sigma(s)) \\ &\qquad\qquad \times K |\phi(s) - \varphi(s)| e_{-(dK+1)|a(\delta^{-1})|}(t, t_0) \Delta s. \end{aligned} \tag{29}$$

Since

$$\begin{aligned} &e_{-(dK+1)|a(\delta^{-1})|}(t, t_0) \\ &\quad = e_{-(dK+1)|a(\delta^{-1})|}(t, s) e_{-(dK+1)|a(\delta^{-1})|}(s, t_0), \\ &|\phi - \varphi|_K = \sup_{s \ge t_0} e_{-(dK+1)|a(\delta^{-1})|}(s, t_0) |\phi(s) - \varphi(s)|, \end{aligned} \tag{30}$$

we obtain

$$\begin{aligned} &|P\phi - P\varphi|_K \\ &\quad \le K|\phi - \varphi|_K \\ &\qquad \times \sup_{t \ge t_0} \int_{t_0}^{t} |a(\delta^{-1}(s))| e_{-a(\delta^{-1})}(t, \sigma(s)) \\ &\qquad\qquad \times e_{-(dK+1)|a(\delta^{-1})|}(t, s) \Delta s. \end{aligned} \tag{31}$$

On the one hand, using Lemma 1 in the integral on the right hand side of (31), we have

$$\begin{aligned} &\int_{t_0}^{t} |a(\delta^{-1}(s))| e_{-a(\delta^{-1})}(t, \sigma(s)) e_{-(dK+1)|a(\delta^{-1})|}(t, s) \Delta s \\ &\quad \le \int_{t_0}^{t} |a(\delta^{-1}(s))| e_{-a(\delta^{-1})}(t, \sigma(s)) \\ &\qquad \times \left(e^{-(dK+1)\int_s^t |a(\delta^{-1}(u))| \Delta u}\right) \Delta s. \end{aligned} \tag{32}$$

In addition, by setting

$$\begin{aligned} f(s) &= e_{-|a(\delta^{-1})|}(t, s), \\ g(s) &= e^{-(dK+1)\int_s^t |a(\delta^{-1}(u))| \Delta u}, \end{aligned} \tag{33}$$

and using the fact that

$$g^{\Delta_s}(s) = (dK+1) |a(\delta^{-1}(s))| e^{-(dK+1)\int_s^t |a(\delta^{-1}(u))| \Delta u} \tag{34}$$

yield

$$\int_{t_0}^{t} \left|a\left(\delta^{-1}(s)\right)\right| e_{-a(\delta^{-1})}(t,\sigma(s)) \left(e^{-(dK+1)\int_s^t |a(\delta^{-1}(u))|\Delta u}\right) \Delta s$$
$$= \frac{1}{(dK+1)} \int_{t_0}^{t} f(\sigma(s)) g^{\Delta_s}(s)\Delta s, \tag{35}$$

where Δ_s denotes the delta derivative with respect to variable s. And hence, by fixing t and using the integration by part formula by taking into account that

$$f^{\Delta_s}(s) = \left(e_{-|a(\delta^{-1})|}(t,s)\right)^{\Delta_s} = \left|a\left(\delta^{-1}\right)\right| e_{-|a(\delta^{-1})|}(t,\sigma(s)), \tag{36}$$

we obtain

$$\int_{t_0}^{t} \left|a\left(\delta^{-1}(s)\right)\right| e_{-a(\delta^{-1})}(t,\sigma(s)) \left(e^{-(dK+1)\int_s^t |a(\delta^{-1}(u))|\Delta u}\right) \Delta s$$
$$= \frac{1}{(dK+1)} - \frac{1}{(dK+1)} e_{-|a(\delta^{-1})|} \times (t,t_0) e^{-(dK+1)\int_{t_0}^t |a(\delta^{-1}(u))|\Delta u}$$
$$- \frac{1}{(dK+1)} \int_{t_0}^{t} \left|a\left(\delta^{-1}(s)\right)\right| e_{-a(\delta^{-1})}(t,\sigma(s)) \times \left(e^{-(dK+1)\int_s^t |a(\delta^{-1}(u))|\Delta u}\right) \Delta s. \tag{37}$$

Thus, we get

$$\int_{t_0}^{t} \left|a\left(\delta^{-1}(s)\right)\right| e_{-a(\delta^{-1})}(t,\sigma(s)) \left(e^{-(dK+1)\int_s^t |a(\delta^{-1}(u))|\Delta u}\right) \Delta s$$
$$= \frac{1}{(dK+2)} \left(1 - e_{-|a(\delta^{-1})|}(t,t_0) e^{-(dK+1)\int_{t_0}^t |a(\delta^{-1}(u))|\Delta u}\right)$$
$$\le \frac{1}{(dK+2)}. \tag{38}$$

Finally, substituting (38) into (31) we get

$$\left|P\phi - P\varphi\right|_K \le \frac{K}{dK+2}\left|\phi-\varphi\right|_K \le \frac{1}{d}\left|\phi-\varphi\right|_K. \tag{39}$$

Since S is a subset of the Banach space $\mathscr{B}$ and S is closed, hence S is complete. Thus, $P : S \longrightarrow S$ has a unique fixed point. □

For any given rd-continuous initial function ψ defined on $[\delta(t_0), t_0]_{\mathbb{T}}$ with $|\psi(t)| < L$, we let

$$S = \left\{\phi : [\delta(t_0), \infty)_{\mathbb{T}} \longrightarrow \mathbb{R} \mid \phi \in C_{\mathrm{rd}},\ \phi_0 = \psi,\ \left|\phi(t)\right| \le L\right\}, \tag{40}$$

where ϕ_0 denotes the segment of ϕ on $[\delta(t_0), t_0]$.

Theorem 6. *Let (a)–(d) of our hypotheses hold, and suppose that for each $L_1 \in (0, L]$ the inequality*

$$\left|L_1 - g(L_1)\right| \sup_{t\ge t_0} \int_{t_0}^{t} \left|a\left(\delta^{-1}(s)\right)\right| e_{-a(\delta^{-1})}(t,\sigma(s))\Delta s$$
$$+ g(L_1) \sup_{t\ge t_0} \int_{\delta(t)}^{t} \left|a\left(\delta^{-1}(s)\right)\right| \Delta s$$
$$+ g(L_1) \sup_{t\ge t_0} \int_{t_0}^{t} \left|a\left(\delta^{-1}(s)\right)\right| e_{-a(\delta^{-1})}(t,\sigma(s)) \times \left(\int_{\delta(s)}^{s} \left|a\left(\delta^{-1}(u)\right)\right| \Delta u\right) \Delta s < L_1 \tag{41}$$

and for $J > 0$ the inequality

$$e_{-a(\delta^{-1})}(t,t_0) \le J, \quad t \ge t_0 \tag{42}$$

hold. Then every solution of (24) is bounded. In addition if $g(0) = 0$, then the zero solution of (24) is stable.

Proof. Define a mapping P on S using (21) in such a way that for $\phi \in S$ we have

$$(P\phi)(t) = \psi(t), \quad \delta(t_0) \le t \le t_0,$$
$$(P\phi)(t) = e_{-a(\delta^{-1})}(t,t_0)\psi(t_0) + \int_{\delta(t)}^{t} a\left(\delta^{-1}(s)\right) g(\phi(s))\Delta s$$
$$- e_{-a(\delta^{-1})}(t,t_0) \int_{\delta(t_0)}^{t_0} a\left(\delta^{-1}(s)\right) g(\psi(s))\Delta s$$
$$- \int_{t_0}^{t} a\left(\delta^{-1}(s)\right) e_{-a(\delta^{-1})}(t,\sigma(s)) \times \left(\int_{\delta(s)}^{s} a\left(\delta^{-1}(u)\right) g(\phi(u))\Delta u\right)\Delta s$$
$$+ \int_{t_0}^{t} a\left(\delta^{-1}(s)\right) e_{-a(\delta^{-1})}(t,\sigma(s)) \times \left[\phi(s) - g(\phi(s))\right]\Delta s, \quad t \ge t_0. \tag{43}$$

By (41) there exists an $\alpha \in (0,1)$ such that if $\phi \in S$ and for $t \ge t_0$ we have

$$\left|(P\phi)(t)\right| \le e_{-a(\delta^{-1})}(t,t_0)\left\|\psi\right\| + g(L) \sup_{t\ge t_0} \int_{\delta(t)}^{t} \left|a\left(\delta^{-1}(s)\right)\right| \Delta s$$
$$+ e_{-a(\delta^{-1})}(t,t_0)\left\|g(\psi)\right\| \int_{\delta(t_0)}^{t_0} \left|a\left(\delta^{-1}(s)\right)\right| \Delta s$$
$$+ g(L) \sup_{t\ge t_0} \int_{t_0}^{t} \left|a\left(\delta^{-1}(s)\right)\right| e_{-a(\delta^{-1})}(t,\sigma(s)) \times \left(\int_{\delta(s)}^{s} \left|a\left(\delta^{-1}(u)\right)\right| \Delta u\right)\Delta s$$
$$+ \left|L - g(L)\right| \sup_{t\ge t_0} \int_{t_0}^{t} a\left(\delta^{-1}(s)\right) e_{-a(\delta^{-1})} \times (t,\sigma(s))\Delta s. \tag{44}$$

Since $e_{-a(\delta^{-1})}(t,t_0) \le J$ we obtain

$$\left|(P\phi)(t)\right| \le J\left(\|\psi\| + \|g(\psi)\| \int_{\delta(t_0)}^{t_0} \left|a\left(\delta^{-1}(s)\right)\right| \Delta s\right) + \alpha L. \tag{45}$$

By choosing the initial function ψ small enough we have

$$J\left(\|\psi\| + K\|\psi\| \int_{\delta(t_0)}^{t_0} \left|a\left(\delta^{-1}(s)\right)\right| \Delta s\right) \le (1-\alpha)L, \tag{46}$$

where K is the Lipschitz constant of g on $[0,L]$. Hence, we obtain

$$\left|(P\phi)(t)\right| \le (1-\alpha)L + \alpha L = L. \tag{47}$$

Thus, we have showed that $P : S \to S$ and any solution of (24) that is in S is bounded.

Next we need to show that P is a contraction. To do this, we proceed as in the proof of Lemma 5. First note that for ϕ, $\varphi \in S$ we have

$$\begin{aligned}
&\left|(P\phi)(t) - (P\varphi)(t)\right| \\
&\quad \le \int_{\delta(t)}^{t} \left|a\left(\delta^{-1}(s)\right)\right| \left|g(\phi(s)) - g(\varphi(s))\right| \Delta s \\
&\qquad + \int_{t_0}^{t} \left|a\left(\delta^{-1}(s)\right)\right| e_{-a(\delta^{-1})}(t,\sigma(s)) \\
&\qquad\quad \times \left(\int_{\delta(s)}^{s} \left|a\left(\delta^{-1}(u)\right)\right| \left|g(\phi(u)) - g(\varphi(u))\right| \Delta u\right) \Delta s \\
&\qquad + \int_{t_0}^{t} \left|a\left(\delta^{-1}(s)\right)\right| e_{-a(\delta^{-1})}(t,\sigma(s)) \\
&\qquad\quad \times \left|\phi(s) - g(\phi(s)) - \varphi(s) + g(\varphi(s))\right| \Delta s,
\end{aligned} \tag{48}$$

and hence we take the metric on S which is induced by the norm

$$|\phi|_K := \sup_{t\ge t_0} e_{-(dK+1)|a(\delta^{-1})|}(t,t_0)\left|\phi(t)\right|. \tag{49}$$

Our aim is to simplify (48). It follows from Lemma 5 that the last term on the right hand side of (48) has a contraction constant $1/d$ since $g(x)$ and $x - g(x)$ both satisfy a Lipschitz condition with the same constant $K > 0$ that is;

$$\begin{aligned}
&\sup_{t\ge t_0} e_{-(dK+1)|a(\delta^{-1})|}(t,t_0) \\
&\quad \times \int_{t_0}^{t} \left|a\left(\delta^{-1}(s)\right)\right| e_{-a(\delta^{-1})}(t,\sigma(s)) \\
&\qquad \times \left|\phi(s) - g(\phi(s)) - \varphi(s) + g(\varphi(s))\right| \Delta s \\
&\le \frac{1}{d}|\phi - \varphi|_K.
\end{aligned} \tag{50}$$

The first term satisfies

$$\begin{aligned}
&\sup_{t\ge t_0} e_{-(dK+1)|a(\delta^{-1})|}(t,t_0) \\
&\quad \times \int_{\delta(t)}^{t} \left|a\left(\delta^{-1}(s)\right)\right| \left|g(\phi(s)) - g(\varphi(s))\right| \Delta s \\
&\quad \le \sup_{t\ge t_0} \int_{\delta(t)}^{t} \left|a\left(\delta^{-1}(s)\right)\right| K\left|\phi(s) - \varphi(s)\right| \\
&\qquad\qquad \times e_{-(dK+1)|a(\delta^{-1})|}(t,t_0)\Delta s \\
&\quad \le \sup_{t\ge t_0} \int_{\delta(t)}^{t} \left|a\left(\delta^{-1}(s)\right)\right| K\left|\phi(s) - \varphi(s)\right| \\
&\qquad\qquad \times e_{-(dK+1)|a(\delta^{-1})|}(t,s)\, e_{-(dK+1)|a(\delta^{-1})|}(s,t_0)\Delta s \\
&\quad \le K|\phi - \varphi|_K \sup_{t\ge t_0} \int_{\delta(t)}^{t} \left|a\left(\delta^{-1}(s)\right)\right| e_{-(dK+1)|a(\delta^{-1})|}(t,s)\Delta s \\
&\quad \le K|\phi - \varphi|_K \sup_{t\ge t_0} \int_{\delta(t)}^{t} \left|a\left(\delta^{-1}(s)\right)\right| e^{-(dK+1)\int_s^t |a(\delta^{-1}(u))|\Delta u}\Delta s.
\end{aligned} \tag{51}$$

Setting

$$f(s) = e^{-(dK+1)\int_s^t |a(\delta^{-1}(u))|\Delta u} \tag{52}$$

and using the fact that

$$f^{\Delta_s}(s) = (dK+1)\left|a\left(\delta^{-1}(s)\right)\right| e^{-(dK+1)\int_s^t |a(\delta^{-1}(u))|\Delta u} \tag{53}$$

we obtain

$$\begin{aligned}
&\sup_{t\ge t_0} \int_{\delta(t)}^{t} \left|a\left(\delta^{-1}(s)\right)\right| e^{-(dK+1)\int_s^t |a(\delta^{-1}(u))|\Delta u}\Delta s \\
&\quad = \sup_{t\ge t_0} \frac{1}{dK+1}\left(1 - e^{-(dK+1)\int_{\delta(t)}^t |a(\delta^{-1}(u))|\Delta u}\right) \\
&\quad \le \frac{1}{dK+1}.
\end{aligned} \tag{54}$$

Thus,

$$\begin{aligned}
&\sup_{t\ge t_0} e_{-(dK+1)|a(\delta^{-1})|}(t,t_0) \\
&\quad \times \int_{\delta(t)}^{t} \left|a\left(\delta^{-1}(s)\right)\right| \left|g(\phi(s)) - g(\varphi(s))\right| \Delta s \\
&\quad \le \frac{K}{dK+1}|\phi - \varphi|_K \le \frac{1}{d}|\phi - \varphi|_K.
\end{aligned} \tag{55}$$

Now, we turn our attention to the second term of (48). Multiply this term by

$$e_{-(dK+1)|a(\delta^{-1})|}(t,t_0) \tag{56}$$

we get

$$\begin{aligned}
&\sup_{t\ge t_0} e_{-(dK+1)|a(\delta^{-1})|}(t,t_0)\\
&\quad\times\int_{t_0}^{t}\left|a\left(\delta^{-1}(s)\right)\right| e_{-a(\delta^{-1})}(t,\sigma(s))\\
&\qquad\times\left(\int_{\delta(s)}^{s}\left|a\left(\delta^{-1}(u)\right)\right|\left|g(\phi(u))-g(\varphi(u))\right|\Delta u\right)\Delta s\\
&\le\sup_{t\ge t_0}\int_{t_0}^{t}\left|a\left(\delta^{-1}(s)\right)\right| e_{-a(\delta^{-1})}(t,\sigma(s))\\
&\qquad\times\left(\int_{\delta(s)}^{s}\left|a\left(\delta^{-1}(u)\right)\right| K\left|\phi(u)-\varphi(u)\right|\right.\\
&\qquad\left.\times e_{-(dK+1)|a(\delta^{-1})|}(t,t_0)\Delta u\right)\Delta s\\
&\le K|\phi-\varphi|_K\int_{t_0}^{t}\left|a\left(\delta^{-1}(s)\right)\right| e_{-a(\delta^{-1})}(t,\sigma(s))\\
&\qquad\times\left(\int_{\delta(s)}^{s}\left|a\left(\delta^{-1}(u)\right)\right|\right.\\
&\qquad\left.\times e_{-(dK+1)|a(\delta^{-1})|}(t,u)\Delta u\right)\Delta s\\
&\le K|\phi-\varphi|_K\int_{t_0}^{t}\left|a\left(\delta^{-1}(s)\right)\right| e_{-a(\delta^{-1})}\\
&\qquad\times(t,\sigma(s))\, e_{-(dK+1)|a(\delta^{-1})|}(t,s)\\
&\qquad\times\left(\int_{\delta(s)}^{s} a\left(\delta^{-1}(u)\right)\right.\\
&\qquad\left.\times e^{-(dK+1)\int_{u}^{s}|a(\delta^{-1}(\tau))|\Delta\tau}\Delta u\right)\Delta s\\
&\le\frac{K}{dK+1}|\phi-\varphi|_K\\
&\quad\times\int_{t_0}^{t}\left|a\left(\delta^{-1}(s)\right)\right| e_{-a(\delta^{-1})}(t,\sigma(s))\, e_{-(dK+1)|a(\delta^{-1})|}\\
&\quad\times(t,s)\Delta s\\
&\le\frac{K}{dK+1}|\phi-\varphi|_K\\
&\quad\times\int_{t_0}^{t}\left|a\left(\delta^{-1}(s)\right)\right| e_{-a(\delta^{-1})}(t,\sigma(s))\\
&\quad\times e^{-(dK+1)\int_{s}^{t}|a(\delta^{-1}(\tau))|\Delta\tau}\Delta s\\
&\le\frac{K}{dK+1}|\phi-\varphi|_K\frac{1}{(dK+2)}\le\frac{1}{d}|\phi-\varphi|_K.
\end{aligned}\tag{57}$$

A substitution of (50), (55), and (57) into (48) gives

$$\left|(P\phi)(t)-(P\varphi)(t)\right|_K\le\left(\frac{1}{d}+\frac{1}{d}+\frac{1}{d}\right)|\phi-\varphi|_K.\tag{58}$$

Hence, we have showed that P is a contraction for $d>3$. Thus, taking into account Lemma 5, $P:S\to S$ has a unique fixed point. This completes the proof. □

When $\mathbb{T}=\mathbb{R}$ and $\mathbb{T}=\mathbb{Z}$, we have the following two corollaries which are immediate consequences of Theorem 6 and thus the proofs are omitted.

Corollary 7. *Let (a)–(d) of our hypotheses hold in case of $\mathbb{T}=\mathbb{R}$ and suppose that for each $L_1\in(0,L]$ the inequality*

$$\begin{aligned}
&\left|L_1-g(L_1)\right|\sup_{t\ge t_0}\int_{t_0}^{t}|a(s+r)|\,e^{-\int_s^t a(u+r)du}ds\\
&\quad+g(L_1)\sup_{t\ge t_0}\int_{t-r}^{t}|a(s+r)|\,ds\\
&\quad+g(L_1)\sup_{t\ge t_0}\int_{t_0}^{t}|a(s+r)|\,e^{-\int_s^t a(u+r)du}\\
&\qquad\times\left(\int_{s-r}^{s}|a(u+r)|\,du\right)ds<L_1
\end{aligned}\tag{59}$$

and for $J>0$ the inequality

$$e^{-\int_{t_0}^{t}a(s+r)ds}\le J,\quad t\ge t_0\tag{60}$$

hold. Then every solution of (2) is bounded. In addition, if $g(0)=0$, then the zero solution of (2) is stable.

Corollary 8. *Let (a)–(d) of our hypotheses hold in case of $\mathbb{T}=\mathbb{Z}$, and suppose that for each $L_1\in(0,L]$ the inequality*

$$\begin{aligned}
&\left|L_1-g(L_1)\right|\max_{t\ge 0}\sum_{s=0}^{t-1}|a(s+r)|\prod_{u=s}^{t-1}|(1-a(u+r))|\\
&\quad+g(L_1)\max_{t\ge 0}\sum_{s=t-r}^{t-1}|a(s+r)|\\
&\quad+g(L_1)\max_{t\ge 0}\sum_{s=0}^{t-1}|a(s+r)|\prod_{u=s}^{t-1}|(1-a(u+r))|\\
&\quad\times\left(\sum_{k=s-r}^{s-1}|a(k+r)|\right)<L_1
\end{aligned}\tag{61}$$

and for $J>0$ the inequality

$$\prod_{s=0}^{t-1}|(1-a(s+r))|\le J,\quad t\ge 0\tag{62}$$

hold. Then every solution of (3) is bounded. In addition if $g(0)=0$, then the zero solution of (3) is stable.

Remark 9. We may deduce [11, Theorem 4.1] and [2, Theorem 2.2] as Corollaries 7 and 8, respectively. Thus, we have unified these results and moreover, we have extended them the general time scales. In particular, the next results concerning the q-difference equation (4) are new.

Example 10. For any $q > 1$ and fixed positive integer m, define

$$\mathbb{T} := \left\{q^{-m}, q^{-m+1}, \ldots, q^{-1}, 1, q, q^2, \ldots\right\}. \tag{63}$$

For any initial function $\psi(t)$, $t \in [q^{-m}, 1]_{\mathbb{T}}$, Consider the following nonlinear delay initial value problem

$$\begin{gathered} D_q x(t) = -\frac{c}{t} x^3\left(q^{-m} t\right) q^{-m}, \quad t \in [1, \infty)_{\mathbb{T}} \\ x(t) = \psi(t), \quad t \in [q^{-m}, 1]_{\mathbb{T}}, \end{gathered} \tag{64}$$

where $0 < c < q^m/2m(q-1)$. We show that the solution of (64) is bounded and zero solution is stable.

A simple comparison of (64) to (24) yield that $a(t) = c/t$, $\delta(t) = q^{-m}t$, $\delta^{-1}(t) = q^m t$, $D_q\delta(t) = q^{-m}$, and $a(\delta^{-1}(t)) = c/q^m t = q^{-m}(c/t)$.

Notice that $g(x(t)) = x^3(t)$ is increasing, continuous and odd and that $x - g(x)$ and $g(x)$ locally Lipschitz on $[0, 1/\sqrt{3}q]$. Thus, every condition of our hypotheses is satisfied.

In order to make use of Theorem 6, we must perform some calculations so that conditions (41) and (42) are satisfied.

For $t_0 = 1$, $t = q^n$ we first calculate $e_{-a(\delta^{-1})}(t, t_0)$:

$$\begin{aligned} e_{-a(\delta^{-1})}(t, t_0) &= e_{-a(\delta^{-1})}(t, 1) \\ &= e^{(1/(q-1)) \int_1^{q^n} (1/s) \log[1-(q-1)s(c/(q^m s))] d_q s} \\ &= e^{(1/(q-1)) \log[1-(q-1)q^{-m}c] \int_1^{q^n} (1/s) d_q s}, \end{aligned} \tag{65}$$

and since

$$\begin{aligned} \int_1^{q^n} \frac{1}{s} d_q s &= \int_{q^0}^{q} \frac{1}{s} d_q s + \int_q^{q2} \frac{1}{s} d_q s + \int_{q^2}^{q^3} \frac{1}{s} d_q s \\ &\quad + \cdots + \int_{q^{n-1}}^{q^n} \frac{1}{s} d_q s \\ &= \sum_{k=0}^{n-1} \int_{q^k}^{\sigma(q^k)} \frac{1}{s} d_q s = \sum_{k=0}^{n-1} \mu\left(q^k\right) \frac{1}{q^k} \\ &= \sum_{k=0}^{n-1} (q-1) q^k \frac{1}{q^k} = n(q-1), \end{aligned} \tag{66}$$

we obtain

$$\begin{aligned} e_{-a(\delta^{-1})}(t, 1) &= e^{(1/(q-1)) \log[1-(q-1)q^{-m}c] \int_1^{q^n} (1/s) d_q s} \\ &= e^{(1/(q-1)) \log[1-(q-1)q^{-m}c] n(q-1)} \\ &= e^{n \log[1-(q-1)q^{-m}c]} \\ &= \left[1 - (q-1) q^{-m} c\right]^n. \end{aligned} \tag{67}$$

Next, we calculate $\int_1^t |a(\delta^{-1}(s))| e_{-a(\delta^{-1})}(t, \sigma(s)) d_q s$. For $1 \le s \le t = q^n$, we have $s = q^{k-1}$ where $1 \le k - 1 \le n \Rightarrow \sigma(s) = q^k$. This implies that

$$\begin{aligned} e_{-a(\delta^{-1})}(t, \sigma(s)) &= e^{\int_{\sigma(s)}^{t} (1/(q-1)u) \log[1-(q-1)u a(\delta^{-1}(u))] d_q u} \\ &= e^{\int_{q^k}^{q^n} (1/(q-1)u) \log[1-(q-1)u a(\delta^{-1}(u))] d_q u} \\ &= e^{(1/(q-1)) \log[1-(q-1)q^{-m}c] \int_{q^k}^{q^n} (1/u) d_q u}. \end{aligned} \tag{68}$$

Since

$$\begin{aligned} \int_{q^k}^{q^n} \frac{1}{u} d_q u &= \int_{q^k}^{q^{k+1}} \frac{1}{u} d_q u + \int_{q^{k+1}}^{q^{k+2}} \frac{1}{u} d_q u + \int_{q^{k+2}}^{q^{k+3}} \frac{1}{u} d_q u \\ &\quad + \cdots + \int_{q^{n-1}}^{q^n} \frac{1}{u} d_q u \\ &= \int_{q^k}^{\sigma(q^k)} \frac{1}{u} d_q u + \int_{q^{k+1}}^{\sigma(q^{k+1})} \frac{1}{u} d_q u + \int_{q^{k+2}}^{\sigma(q^{k+2})} \frac{1}{u} d_q u \\ &\quad + \cdots + \int_{q^{n-1}}^{\sigma(q^{n-1})} \frac{1}{u} d_q u \\ &= \mu\left(q^k\right) \frac{1}{q^k} + \mu\left(q^{k+1}\right) \frac{1}{q^{k+1}} + \mu\left(q^{k+2}\right) \frac{1}{q^{k+2}} \\ &\quad + \cdots + \mu\left(q^{n-1}\right) \frac{1}{q^{n-1}} \\ &= \sum_{i=k}^{n-1} \mu\left(q^i\right) \frac{1}{q^i} = \sum_{i=k}^{n-1} (q-1) q^i \frac{1}{q^i} \\ &= (q-1)(n-k), \end{aligned} \tag{69}$$

we have

$$\begin{aligned} e_{-a(\delta^{-1})}(t, \sigma(s)) &= e^{(1/(q-1)) \log[1-(q-1)q^{-m}c] \int_{q^k}^{q^n} (1/u) d_q u} \\ &= e^{(1/(q-1)) \log[1-(q-1)q^{-m}c](q-1)(n-k)} \\ &= e^{(n-k) \log[1-(q-1)q^{-m}c]} \\ &= \left[1 - (q-1) q^{-m} c\right]^{n-k}. \end{aligned} \tag{70}$$

Since $|a(\delta^{-1}(s))| = c/q^m s$, we have that

$$\begin{aligned} &\int_1^t \left|a\left(\delta^{-1}(s)\right)\right| e_{-a(\delta^{-1})}(t, \sigma(s)) d_q s \\ &\quad = \sum_{k=0}^{n-1} \int_{q^k}^{\sigma(q^k)} \left|a\left(\delta^{-1}(s)\right)\right| e_{-a(\delta^{-1})}(t, \sigma(s)) d_q s \\ &\quad = \left(\int_{q^0}^{\sigma(q^0)} + \int_{q^1}^{\sigma(q^1)} + \cdots + \int_{q^n}^{\sigma(q^n)}\right) \\ &\quad\quad \times \left(\left|a\left(\delta^{-1}(s)\right)\right| e_{-a(\delta^{-1})}(t, \sigma(s)) d_q s\right) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{n-1} \mu\left(q^k\right) a\left(\delta^{-1}\left(q^k\right)\right) e_{-a(\delta^{-1})}\left(t, \sigma\left(q^k\right)\right) \\
&= \sum_{k=0}^{n-1} (q-1) q^k \frac{c}{q^{m+k}} \left[1-(q-1) q^{-m} c\right]^{n-(k+1)} \\
&= \sum_{k=0}^{n-1} (q-1) \frac{c}{q^m} \left[1-(q-1) q^{-m} c\right]^{n-(k+1)} \\
&= (q-1) \frac{c}{q^m} \left[1-(q-1) q^{-m} c\right]^{n-1} \\
&\quad \times \sum_{k=0}^{n-1} \left[1-(q-1) q^{-m} c\right]^{-k},
\end{aligned} \tag{71}
$$

where we have used the fact that

$$
e_{-a(\delta^{-1})}\left(t, \sigma\left(q^k\right)\right) = \left[1-(q-1) q^{-m} c\right]^{n-(k+1)}. \tag{72}
$$

We know that $\sum_{i=0}^{n-1} a^i = (1-a^n)/(1-a)$, and hence, by taking $a = 1/(1-(q-1)q^{-m}c)$, we obtain

$$
\begin{aligned}
\frac{1-a^n}{1-a} &= \frac{1-\left(1/(1-(q-1) q^{-m} c)\right)^n}{1-\left(1/\left(1-(q-1) q^{-m} c\right)\right)} \\
&= \frac{1-\left(1/\left[1-(q-1) q^{-m} c\right]^n\right)}{1-\left(1/\left[1-(q-1) q^{-m} c\right]\right)} \\
&= \frac{\left(\left[1-(q-1) q^{-m} c\right]^n-1\right)/\left[1-(q-1) q^{-m} c\right]^n}{\left(\left[1-(q-1) q^{-m} c\right]-1\right)/\left[1-(q-1) q^{-m} c\right]} \\
&= \frac{\left(\left[1-(q-1) q^{-m} c\right]^n-1\right)}{\left[1-(q-1) q^{-m} c\right]^n} \frac{\left[1-(q-1) q^{-m} c\right]}{\left[-(q-1) q^{-m} c\right]} \\
&= \frac{\left[1-(q-1) q^{-m} c\right]}{(q-1) q^{-m} c} \frac{\left(1-\left[1-(q-1) q^{-m} c\right]^n\right)}{\left[1-(q-1) q^{-m} c\right]^n}.
\end{aligned} \tag{73}
$$

Thus

$$
\begin{aligned}
&(q-1) \frac{c}{q^m} \left[1-(q-1) q^{-m} c\right]^{n-1} \sum_{k=0}^{n-1} \left[1-(q-1) q^{-m} c\right]^{-k} \\
&= (q-1) \frac{c}{q^m} \left[1-(q-1) q^{-m} c\right]^{n-1} \\
&\quad \times \frac{\left[1-(q-1) q^{-m} c\right]}{(q-1) q^{-m} c} \frac{\left(1-\left[1-(q-1) q^{-m} c\right]^n\right)}{\left[1-(q-1) q^{-m} c\right]^n} \\
&= \left[1-(q-1) q^{-m} c\right]^{n-1} \left[1-(q-1) q^{-m} c\right] \\
&\quad \times \frac{\left(1-\left[1-(q-1) q^{-m} c\right]^n\right)}{\left[1-(q-1) q^{-m} c\right]^n} \\
&= \left(1-\left[1-(q-1) q^{-m} c\right]^n\right).
\end{aligned} \tag{74}
$$

As a consequence, we obtain

$$
\begin{aligned}
&\int_1^t \left|a\left(\delta^{-1}(s)\right)\right| e_{-a(\delta^{-1})}(t, \sigma(s))\, d_q s \\
&\quad = 1-\left(1-(q-1) q^{-m} c\right)^n.
\end{aligned} \tag{75}
$$

Left to calculate $\int_{\delta(t)}^t |a(\delta^{-1}(s))| d_q s$. Consider

$$
\begin{aligned}
&\int_{\delta(t)}^t \left|a\left(\delta^{-1}(s)\right)\right| d_q s \\
&= \int_{q^{-m}t}^t \frac{c}{q^m s} d_q s = \int_{q^{-m}t}^t \frac{c}{q^m s} d_q s \\
&= \frac{c}{q^m} \left\{\int_{q^{-m}t}^{q^{-m+1}t} \frac{1}{s} d_q s + \int_{q^{-m+1}t}^{q^{-m+2}t} \frac{1}{s} d_q s + \cdots + \int_{q^{-1}t}^{t} \frac{1}{s} d_q s\right\} \\
&= \frac{c}{q^m} \sum_{k=1}^{m} \int_{q^{-k}t}^{\sigma(q^{-k}t)} \frac{1}{s} d_q s = \frac{c}{q^m} \sum_{k=1}^{m} \mu\left(q^{-k} t\right) \frac{1}{q^{-k} t} \\
&= \frac{c}{q^m} \sum_{k=1}^{m} (q-1) q^{-k} t \frac{1}{q^{-k} t} = \frac{c}{q^m} (q-1)(m).
\end{aligned} \tag{76}
$$

Thus, the above integral is independent of t.

Left to compute

$$
\int_1^t \left|a\left(\delta^{-1}(s)\right)\right| e_{-a(\delta^{-1})}(t, \sigma(s)) \left(\int_{\delta(s)}^s \left|a\left(\delta^{-1}(u)\right)\right| d_q u\right) d_q s. \tag{77}
$$

We have already shown $\int_{\delta(s)}^s |a(\delta^{-1}(u))| d_q u = (c/q^m)(q-1)(m)$ which is independent of integration variable. This leads us to

$$
\begin{aligned}
&\int_1^t \left|a\left(\delta^{-1}(s)\right)\right| e_{-a(\delta^{-1})}(t, \sigma(s)) \left(\int_{\delta(s)}^s \left|a\left(\delta^{-1}(u)\right)\right| d_q u\right) d_q s \\
&= \frac{mc}{q^m} (q-1) \int_1^t \left|a\left(\delta^{-1}(s)\right)\right| e_{-a(\delta^{-1})}(t, \sigma(s))\, d_q s \\
&= \frac{mc}{q^m} (q-1) \left[1-\left(1-(q-1) q^{-m} c\right)^n\right].
\end{aligned} \tag{78}
$$

As consequence of all of the above calculations, we arrive at, for $L_1 \in [0, 1/\sqrt{3}q]$ that

$$
\begin{aligned}
&\left|L_1 - g\left(L_1\right)\right| \sup_{t \geq t_0} \int_{t_0}^t \left|a\left(\delta^{-1}(s)\right)\right| e_{-a(\delta^{-1})}(t, \sigma(s))\, d_q s \\
&\quad + g\left(L_1\right) \sup_{t \geq t_0} \int_{\delta(t_0)}^t \left|a\left(\delta^{-1}(s)\right)\right| d_q s \\
&\quad + g\left(L_1\right) \sup_{t \geq t_0} \int_1^t \left|a\left(\delta^{-1}(s)\right)\right| e_{-a(\delta^{-1})}(t, \sigma(s)) \\
&\qquad \times \left(\int_{\delta(s)}^s \left|a\left(\delta^{-1}(u)\right)\right| d_q u\right) d_q s
\end{aligned}
$$

$$\begin{aligned}&\leq \left|L_1 - g(L_1)\right| \sup_{t\geq 1}\left[1-\left(1-(q-1)q^{-m}c\right)^n\right]\\&\quad + g(L_1)\sup_{t\geq 1}\frac{mc}{q^m}(q-1)\\&\quad + g(L_1)\sup_{t\geq 1}\frac{mc}{q^m}(q-1)\left[1-\left(1-(q-1)q^{-m}c\right)^n\right]\\&\leq L_1.\end{aligned} \tag{79}$$

Since $c < q^m/2m(q-1)$, we obtain

$$\begin{gathered}\sup_{t\geq 1}\left[1-\left(1-(q-1)q^{-m}c\right)^n\right] = 1,\\ \sup_{t\geq 1}\frac{mc}{q^m}(q-1) = \frac{1}{2},\\ \sup_{t\geq 1}\frac{mc}{q^m}(q-1)\left[1-\left(1-(q-1)q^{-m}c\right)^n\right] = \frac{1}{2}.\end{gathered} \tag{80}$$

Also since

$$e_{-a(\delta^{-1})}(t,1) = \left[1-(q-1)q^{-m}c\right]^n, \tag{81}$$

we can find sufficiently large N such that for $n \geq N$

$$e_{-a(\delta^{-1})}(t,1) = \left[1-(q-1)q^{-m}c\right]^n \leq \left(1-\frac{1}{2m}\right)^N = J \tag{82}$$

since $cq^{-m}(q-1) < 1/2m < 1$. Hence all the conditions of Theorem 6 are satisfied and a result solution of (64) are bounded and its zero solution is stable.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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