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Research Article

Existence and Modification of Halpern-Mann Iterations for Fixed Point and Generalized Mixed Equilibrium Problems with a Bifunction Defined on the Dual Space

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We study and establish the existence of a solution for a generalized mixed equilibrium problem with a bifunction defined on the dual space of a Banach space. Furthermore, we also modify Halpern-Mann iterations for finding a common solution of a generalized mixed equilibrium problem and a fixed point problem. Under suitable conditions of the purposed iterative sequences, the strong convergence theorems are established by using sunny generalized nonexpansive retraction in Banach spaces. Our results extend and improve various results existing in the current literature.

1. Introduction

In the past years, variational inequalities is among the most important and interesting mathematical problems, since they have wide applications in the optimization and control, economics and equilibrium, engineering science and physical sciences.

Equilibrium problem represents an important area of mathematical sciences such as optimization, operations research, game theory, financial mathematics, and mechanics. Equilibrium problems include variational inequalities, optimization problems, Nash equilibria problems, saddle point problems, and fixed point problems as special cases.

Throughout this paper, we denote the strong convergence and weak convergence $\{x_n\}$ by $x_n \to x$, $x_n \rightharpoonup x$, respectively.

Let C be a closed and convex subset of a real Banach space E with the dual space E^*. Let C^* be a closed and convex subset of E^*. We recall the following definitions.

(1) A mapping $A : C \to E^*$ is said to be *monotone* if for each $x, y \in C$ such that

$$\langle x - y, Ax - Ay\rangle \geq 0. \tag{1}$$

(2) A mapping $A : C \to E^*$ is said to be δ-*strongly monotone*, if there exists a constant $\delta > 0$ such that

$$\langle x - y, Ax - Ay\rangle \geq \delta\|x - y\|^2, \quad \forall x, y \in C. \tag{2}$$

(3) A mapping $A : C \to E^*$ is said to be δ-*inverse strongly monotone*, if there exists a constant $\delta > 0$ such that

$$\langle x - y, Ax - Ay\rangle \geq \delta\|Ax - Ay\|^2, \quad \forall x, y \in C. \tag{3}$$

(4) A mapping $A : C^* \to E$ is said to be *skew monotone* if for each $x^*, y^* \in C^*$ such that

$$\langle Ax^* - Ay^*, x^* - y^*\rangle \geq 0. \tag{4}$$

(5) A mapping $A : D(A) \subset E^* \to E$ is said to be α-*inverse strongly skew monotone* if there exists a constant $\alpha > 0$ such that

$$\langle Ax^* - Ay^*, x^* - y^*\rangle \geq \alpha\|Ax^* - Ay^*\|^2, \quad \forall x^*, y^* \in D(A). \tag{5}$$

Definition 1. Let E be a Banach space. Then,

(1) E is said to be *strictly convex* if $(\|x+y\|/2) < 1$ for all $x, y \in U_E = \{z \in E : \|z\| = 1\}$ with $x \neq y$;

(2) E is said to be *uniformly convex* if for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that $(\|x+y\|/2) \leq 1-\delta$, for all $x, y \in U_E$ with $\|x+y\| > \epsilon$;

(3) E is said to be *smooth* if the limit

$$\lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{6}$$

exists, for each $x, y \in U_E$;

(4) E is said to be *uniformly smooth* if the limit (6) is attained uniformly, for all $x, y \in U_E$;

(5) E is said to have *uniformly Gâteaux differentiable norm* if for all $y \in U(E)$, the the limit (6) converges uniformly, for $x \in U_E$.

Definition 2. Let E be a Banach space. Then, a function $\rho_E : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be the *modulus of smoothness* of E if

$$\rho_E(t) = \sup\left\{\frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| = 1,\ \|y\| = t\right\}. \tag{7}$$

(1) E is said to be *smooth* if $\rho_E(t) > 0$, $\forall t > 0$.

(2) E is said to be *uniformly smooth* if and only if $\lim_{t \to 0^+}(\rho_E(t)/t) = 0$.

Definition 3. Let E be a Banach space. Then, the *modulus of convexity* of E is the function $\delta_E : [0, 2] \to [0, 1]$ defined by

$$\delta_E(\epsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| \leq 1,\ \|y\| \leq 1;\ \|x-y\| \geq \epsilon\right\}. \tag{8}$$

(1) E is said to be *uniformly convex* if and only if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0, 2]$.

(2) Let p be a fixed real number $p > 1$. Then, E is said to be *p-uniformly convex* if there exists a constant $c > 0$ such that $\delta_E(\epsilon) \geq c\epsilon^p$ for all $\epsilon \in [0, 2]$.

Observe that every p-uniformly convex is uniformly convex. One should note that no a Banach space is p-uniformly convex, for $1 < p < 2$. It is well known that a Hilbert space is 2-uniformly convex and uniformly smooth.

For any $p > 1$, the *generalized duality mapping* $J_p : E \to 2^{E^*}$ is defined by

$$J_p x = \left\{f^* \in E^* : \langle x, f^* \rangle = \|x\|^p, \|f^*\| = \|x\|^{p-1}\right\}, \quad \forall x \in E. \tag{9}$$

In particular, $J = J_2$ is called the *normalized duality mapping*. If E is a Hilbert space, then $J = I$, where I is the identity mapping. That is,

$$\begin{aligned} J_2 x &= Jx \\ &= \left\{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\right\}, \quad \forall x \in E. \end{aligned} \tag{10}$$

Remark 4. The basic properties below hold (see [1–3]).

(1) If E is uniformly smooth real Banach space, then J is uniformly continuous on each bounded subset of E.

(2) If E is uniformly smooth real Banach space, then $J^* : E^* \to 2^E$ is a normalized duality mapping on E^*, and then $J^{-1} = J^*$, $(J^*)J = I_E$, and $J(J^*) = I_{E^*}$, where on I_E and I_{E^*} are the identity mappings on E and E^*, respectively.

(3) Let E be a smooth, strictly convex reflexive Banach space, and let J be the duality mapping from E into E^*. Then, J^{-1} is also single-valued, one-to-one, and onto, and it is also the duality mapping from E^* into E.

(4) If E is a reflexive, strictly convex Banach space, then J^{-1} is hemicontinuous; that is, J^{-1} is norm-to-weak*-continuous.

(5) If E is a reflexive, smooth, and strictly convex Banach space, then J is single-valued, one-to-one, and onto.

(6) A Banach space E is uniformly smooth if and only if E^* is uniformly convex.

(7) Each uniformly convex Banach space E has the *Kadec-Klee property*; that is, for any sequence $\{x_n\} \subset E$, if $x_n \rightharpoonup x \in E$, and $\|x_n\| \to \|x\|$, then $x_n \to x$.

(8) A Banach space E is strictly convex if and only if J is strictly monotone; that is,

$$\begin{aligned} \langle x-y, x^* - y^* \rangle > 0, &\quad \text{whenever } x, y \in E, \\ x \neq y, &\quad x^* \in Jx, \quad y^* \in Jy. \end{aligned} \tag{11}$$

(9) Both uniformly smooth Banach space and uniformly convex Banach space are reflexive.

(10) If E^* is uniformly convex, then J is uniformly norm-to-norm continuous on each bounded subset of E.

(11) If E^* is strictly convex Banach space, then J is one-to-one; that is, $x \neq y$ implies $Jx \cap Jy \neq \emptyset$.

Let J be the *normalized duality mapping*; then J is said to be *weakly sequentially continuous* if the strong convergence of a sequence $\{x_n\}$ to $x \in E$ implies the weak* convergence of a sequence $\{Jx_n\}$ to Jx in E^*.

Let E be a smooth and strictly convex reflexive Banach space, and let C be a nonempty, closed, and convex subset of E. We assume that the *Lyapunov functional* $\phi : E \times E \to \mathbb{R}^+$ is defined by [3, 4]

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \forall x, y \in E. \tag{12}$$

Let C is nonempty, closed, and convex subset of a Banach space E. The *generalized projection* [3] $\Pi_C : E \to C$ is defined by for each $x \in E$,

$$\Pi_C(x) = \arg\min_{y \in C} \phi(x, y). \tag{13}$$

Remark 5. From the definition of ϕ. It is easy to see that

(1) $(\|x\| - \|y\|)^2 \le \phi(x, y) \le (\|x\| + \|y\|)^2$, for all $x, y \in E$;

(2) $\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy\rangle$, for all $x, y, z \in E$;

(3) $\phi(x, y) = \langle x, Jx - Jy\rangle + \langle y - x, Jy\rangle \le \|x\|\|Jx - Jy\| + \|y - x\|\|y\|$, for all $x, y, z \in E$;

(4) if E is a real Hilbert space H, then $\phi(x, y) = \|x - y\|^2$, and $\Pi_C = P_C$ (the metric projection of H onto C).

Lemma 6 (see [3, 4]). *If C is a nonempty, closed, and convex subset of a smooth and strictly convex reflexive real Banach space E, then*

(1) *for $x \in E$ and $u \in C$, one has*

$$u = \Pi_C(x) \Longleftrightarrow \langle u - y, Jx - Ju\rangle \ge 0, \quad \forall y \in C; \tag{14}$$

(2) $\phi(x, \Pi_C(y)) + \phi(\Pi_C(y), y) \le \phi(x, y), \forall x \in C$, *and* $y \in E$;

(3) $\phi(x, y) = 0$ *if and only if* $x = y, \forall x, y \in E$.

Let C be a nonempty, closed subset of a smooth, strictly convex, and reflexive Banach space E such that $J(C)$ is closed and convex. For solving the equilibrium problem, let us assume that a bifunction $F : J(C) \times J(C) \to \mathbb{R}$ satisfies the following conditions:

(DA1) $F(x^*, x^*) = 0$, for all $x^* \in J(C)$;

(DA2) F is monotone, that is, $F(x^*, y^*) + F(y^*, x^*) \le 0$, for all $x^*, y^* \in J(C)$

(DA3) for all $x^*, y^*, z^* \in J(C)$,

$$\limsup_{t \downarrow 0} F(tz^* + (1 - t)x^*, y^*) \le F(x^*, y^*); \tag{15}$$

(DA4) for all $x^* \in J(C)$, $F(x^*, \cdot)$ is convex and lower semicontinuous.

The following result is in Blum and Oettli [5], and see proof in [6].

Let $\mathbb{R}$ be the set of real numbers, let E be a real Banach space with the norm $\|\cdot\|$, and let $\langle\cdot,\cdot\rangle$ that is the dual pair between E and E^* by E^* be the dual space of E. Let C be a nonempty, closed, and convex subset of E, let J be the duality mapping from E into E^* such that $J(C)$ is closed and convex of E^*, let us assume that a bifunction $F : J(C) \times J(C) \to \mathbb{R}$ satisfies suitable conditions, let $A : C^* \to E$ be a skew monotone operator from $J(C)$ into E, and let $\varphi : J(C) \to \mathbb{R}$ be a real-valued function.

The *generalized mixed equilibrium problem* is to find $\hat{z} \in C$ such that

$$F(J\hat{z}, Jy) + \langle AJ\hat{z}, Jy - J\hat{z}\rangle + \varphi(Jy) - \varphi(J\hat{z}) \ge 0, \quad \forall y \in C. \tag{16}$$

The set of solutions of (16) is denoted by GMEP(F, A, φ); that is,

$$\begin{aligned}\mathrm{GMEP}(F, A, \varphi) = \{\hat{z} \in C : F(J\hat{z}, Jy) \\ + \langle AJ\hat{z}, Jy - J\hat{z}\rangle + \varphi(Jy) \\ -\varphi(J\hat{z}) \ge 0, \ \forall y \in C\}.\end{aligned} \tag{17}$$

If $A \equiv 0$, then the problem (16) reduces to the *mixed equilibrium problem* which is to find $\hat{z} \in C$ such that

$$F(J\hat{z}, Jy) + \varphi(Jy) - \varphi(J\hat{z}) \ge 0, \quad \forall y \in C. \tag{18}$$

The set of solution of problem (18) is denoted by MEP(F, φ); that is,

$$\begin{aligned}\mathrm{MEP}(F, \varphi) = \{\hat{z} \in C : F(J\hat{z}, Jy) \\ +\varphi(Jy) - \varphi(J\hat{z}) \ge 0, \ \forall y \in C\}.\end{aligned} \tag{19}$$

If $A \equiv 0$ and $\varphi \equiv 0$, then the problem (16) reduces to the *equilibrium problem* which is to find $\hat{z} \in C$ such that

$$F(J\hat{z}, Jy) \ge 0, \quad \forall y \in C. \tag{20}$$

The set of solution of problem (20) is denoted by EP(F); that is,

$$\mathrm{EP}(F) = \{\hat{z} \in C : F(J\hat{z}, Jy) \ge 0, \ \forall y \in C\}. \tag{21}$$

The above formulation (20) was considered in Takahashi and Zembayashi [7], and they proved a strong convergence theorem for finding a solution of the equilibrium problem (20) in Banach spaces.

If $F \equiv 0$ and $A \equiv 0$, then the problem (16) reduces to *variational inequality*, which is to find $\hat{z} \in C$ such that

$$\langle AJ\hat{z}, Jy - J\hat{z}\rangle \ge 0, \ \forall y \in C. \tag{22}$$

The set of solution of problem (22) is denoted by VI$(J(C), A)$; that is,

$$\mathrm{VI}(J(C), A) = \{\hat{z} \in C : \langle AJ\hat{z}, Jy - J\hat{z}\rangle \ge 0, \ \forall y \in C\}. \tag{23}$$

In the sequel, let $T : C \to C$ be a mapping, we denote by Fix(T) the set of fixed points of T; that is,

$$\mathrm{Fix}(T) = \{x \in C : Tx = x\}. \tag{24}$$

In 1953, Mann [8] introduced an iterative algorithm which is defined by the initial point x_0 is taken in C arbitrarily and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \quad n \ge 0 \tag{25}$$

where the sequence $\alpha_n \in [0, 1]$. Mann's iteration can yield only weak convergence.

In 1967, Halpern [9] introduced another iterative algorithm which is defined by the initial point x_0 is taken in C arbitrarily and

$$\begin{aligned} x_0 &= u \in C \text{ chosen arbitrary}, \\ x_{n+1} &= \alpha_n u + (1-\alpha_n) T x_n, \quad n \geq 0, \end{aligned} \tag{26}$$

which satisfied the conditions $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$. Then, the sequence $\{x_n\}$ is converges strongly to a fixed point of T.

In 2007, Takahashi and Zembayashi [7] introduced an iterative algorithm for finding a solution of an equilibrium problem with a bifunction defined on the dual space of a Banach space by using the shrinking projection method, and they established the strong convergence of the following results.

Theorem TZ. *Let E be a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable, and let C be a nonempty, closed and convex subset of E such that $J(C)$ is closed, and convex of E^*. Assume that a mapping $F : J(C) \times J(C) \to \mathbb{R}$ satisfied the conditions (DA1)–(DA4) such that $\mathrm{EP}(F) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$\begin{gathered} x_0 \in C, \qquad C_0 = C \text{ chosen arbitrary}, \\ u_n \in C \text{ such that } F(Ju_n, Jy) + \frac{1}{r_n}\langle u_n - x_n, Jy - Ju_n\rangle \geq 0, \\ \forall y \in C, \\ y_n = \alpha_n x_n + (1-\alpha_n) u_n, \\ C_{n+1} = \{z \in C_n : \phi(y_n, z) \leq \phi(x_n, z)\}, \\ x_{n+1} = R_{C_{n+1}}(x_0), \quad \forall n \in \mathbb{N}\cup\{0\}, \end{gathered} \tag{27}$$

where J is the duality mapping on E, the sequence $\{\alpha_n\} \subset [0,1]$ such that $\limsup_{n\to\infty}\alpha_n < 1$, $r_n \subset [a,\infty)$ for some $a > 0$, and $R_{C_{n+1}}$ is the sunny generalized nonexpansive retraction from E onto C_{n+1}. Then, the sequence $\{x_n\}$ converges strongly to some point $p = R_{\mathrm{EF}(F)}(x_0)$, where $R_{\mathrm{EP}(F)}$ is the sunny generalized nonexpansive retraction from E to $\mathrm{EP}(F)$.

In 2010, Plubtieng and Sriprad [10] proved the existence theorem of the variational inequality problem for skew monotone operator defined on the dual space of a smooth Banach space, and they established weak convergence theorem for finding a solution of the variational inequality problem using projection algorithm method with a new projection which was introduced by Ibaraki and Takahashi [11] and Iiduka and Takahashi [12] in Banach spaces.

Let E be a smooth, strictly convex, and reflexive Banach space, let E^* be the dual space of E, and let C be a nonempty, closed, and convex subset of E such that $J(C)$ is closed and convex of E^*, where J is the duality mapping on E. Let A be a skew monotone operator from $J(C)$ into E. Then, the *variational inequality problem* is to find $z \in C$ such that

$$\langle AJz, Jy - Jz\rangle \geq 0, \quad \forall y \in C. \tag{28}$$

The set of solution of problem (28) is denoted by $\mathrm{VI}(J(C), A)$; that is,

$$\mathrm{VI}(J(C), A) = \{z \in C : \langle AJz, Jy - Jz\rangle \geq 0, \ \forall y \in C\}. \tag{29}$$

Theorem PS. *Let E be a uniformly convex and 2-uniformly smooth Banach space whose duality mapping J is weakly sequentially continuous. Let C be a nonempty, closed and convex subset of E such that $J(C)$ is closed, and convex, and let A be an α-inverse-strongly-skew-monotone operator of JC into E such that $\mathrm{VI}(J(C), A) \neq \emptyset$, and $\|AJz\| \leq \|AJy - AJz\|$, for all $y \in C$ and $z \in \mathrm{VI}(J(C), A)$. Let $\{x_n\}$ be a sequence defined by $x_1 = x \in C$ and*

$$x_{n+1} = R_C(x_n - \lambda_n AJx_n), \tag{30}$$

for every $n = 1, 2, 3, \ldots$, where R_C is the sunny generalized nonexpansive retraction of E into C, $\{\alpha_n\} \subset [a, b]$ and for some a, b with $0 < a < b < (\alpha/c)$, where $c > 0$ is a constant that satisfies $\|Jx - Jy\| \leq c\|x - y\|$, for all $x, y \in C$. Then, the sequence $\{x_n\}$ converges weakly to some element $z \in \mathrm{VI}(J(C), A)$. Further $z = \lim_{n\to\infty} R_{\mathrm{VI}(J(C),A)} x_n$.

In 2011, Chen et al. [13] introduced a new iterative method for finding a solution of equilibrium with a bifunction defined on the dual space of a Banach space. They established the strong convergence theorem by using the sunny generalized nonexpansive retraction in Banach spaces.

Theorem CCW. *Let C be a nonempty, closed and convex subset of a uniformly convex and uniformly smooth Banach space such that $J(C)$ is closed and convex. Assume that a bifunction $F : J(C) \times J(C) \to \mathbb{R}$ satisfies the conditions (DA1)–(DA4). Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{gathered} x_0 \in C \text{ chosen arbitrary}, \\ u_n \in C \text{ such that } F(Ju_n, Jy) + \frac{1}{r_n}\langle u_n - x_n, Jy - Ju_n\rangle \geq 0, \\ \forall y \in C, \\ x_{n+1} = \alpha_n x_0 + (1-\alpha_n)(\beta_n x_n + (1-\beta_n) u_n), \quad \forall n \in \mathbb{N}, \end{gathered} \tag{31}$$

where J is the duality mapping on E, the sequences $\{\alpha_n\}$, $\{\beta_n\} \subset [0,1]$, and $\{r_n\} \subset [a,\infty)$, for some $a > 0$ such that

$$\sum_{n=0}^{\infty}\alpha_n < \infty, \qquad \liminf_{n\to\infty}\beta_n(1-\beta_n) > 0, \qquad \liminf_{n\to\infty} r_n > 0. \tag{32}$$

Then, the sequence $R_{\mathrm{EP}(F)}\{x_n\}$ converges strongly to some point $p \in \mathrm{EF}(F)$, where $R_{\mathrm{EP}(F)}$ is the sunny generalized nonexpansive retraction from E to $\mathrm{EP}(F)$.

In 2012, Saewan et al. [14] introduced a new iterative scheme for finding a common element of the set of solutions of the mixed equilibrium problems and the set of fixed points for a -ϕ-nonexpansive mapping in Banach spaces by using sunny generalized nonexpansive retraction in Banach spaces.

Theorem SCK. *Let E be a uniformly smooth and uniformly convex Banach space, and let C be a nonempty, closed, and convex subset of E such that $J(C)$ is closed and convex of E^*. Let $F : J(C) \times J(C) \to \mathbb{R}$ be a bifunction that satisfies the conditions (DA1)–(DA4), and let $T : C \to C$ be a closed and -ϕ-nonexpansive mapping. Assume that $\mathfrak{F} := \mathrm{Fix}(T) \cap \mathrm{MEP}(F, \varphi) \neq \emptyset$. For an initial point $x_0 \in C$ and define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{gathered} x_0 \in C \text{ chosen arbitrary}, \\ u_n \in C \text{ such that} \\ F(Ju_n, Jy) + \varphi(Jy) - \varphi(Ju_n) + \frac{1}{r_n}\langle u_n - x_n, Jy - Ju_n\rangle \geq 0, \\ \forall y \in C, \\ x_{n+1} = \alpha_n x_0 + (1-\alpha_n)(\beta_n x_n + (1-\beta_n)Tu_n), \\ \forall n \in \mathbb{N}, \end{gathered} \tag{33}$$

where J is the duality mapping on E, the sequence $\{\alpha_n\}, \{\beta_n\} \subset [a, b]$, and $\{r_n\} \subset [c, \infty)$, for some $a, b \in (0, 1)$ and $c > 0$. If the following conditions are satisfied:

$$\sum_{n=0}^{\infty}\alpha_n < \infty, \qquad \sum_{n=0}^{\infty}\beta_n < \infty, \qquad \liminf_{n\to\infty} r_n > 0. \tag{34}$$

Then, the sequence $R_{\mathfrak{F}}\{x_n\}$ converges strongly to some point $p \in \mathfrak{F}$, where $R_{\mathfrak{F}}$ is the sunny generalized nonexpansive retraction from E onto $\mathfrak{F}$.

In this paper, Motivated and inspired by the previously mentioned above results, we study and investigate the existence of theorem for a generalized mixed equilibrium problem with a bifunction defined on the dual space of a Banach space, and we construct an iterative procedure generated by the conditions for solving the common solution of a generalized mixed equilibrium problem and a fixed point problem by using the sunny generalized nonexpansive retraction. Under some suitable assumptions, the strong convergence theorem are established in Banach spaces. The results obtained in this paper extend and improve several recent results in this area.

2. Preliminaries

Definition 7. Let C be a nonempty, closed subset of a smooth Banach space.

(1) A mapping $T : C \to C$ is said to be *closed* if for each $\{x_n\} \subset C$, $x_n \to x$ and $Tx_n \to y$ imply $Tx = y$.

(2) A mapping $T : C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{35}$$

(3) A mapping $T : C \to C$ is said to be -ϕ-*nonexpansive* if $\mathrm{Fix}(T) \neq \emptyset$, and

$$\phi(Tx, Ty) \leq \phi(x, y), \quad \forall x, y \in C. \tag{36}$$

(4) A mapping $T : C \to C$ is said to be *generalized nonexpansive* [15] if $\mathrm{Fix}(T) \neq \emptyset$ and

$$\phi(Tx, p) \leq \phi(x, p), \quad \forall x \in C,\ p \in \mathrm{Fix}(T). \tag{37}$$

Definition 8 (see [15]). Let C be a nonempty, closed subset of a smooth Banach space E. A mapping $R : E \to C$ is called

(1) a retraction if $R^2 = R$;

(2) a sunny if $R(Rx + t(x - Rx)) = Rx$, for all $x \in E$ and $t \geq 0$.

We also know that if E is a smooth, strictly convex, and reflexive Banach space and C is nonempty, closed, and convex subset of E, then there exists a sunny generalized nonexpansive retraction R_C of E onto C if and only if $J(C)$ is closed and convex. In this case R_C is given by

$$R_C = J^{-1}\Pi_{J(C)}J. \tag{38}$$

Definition 9 (see [15]). A nonempty, closed subset C of a smooth Banach space E is said to be a *sunny generalized nonexpansive retraction* of E if there exists a sunny generalized nonexpansive R from E onto C.

Lemma 10 (see [11]). *Let C be a nonempty, closed, and subset of a smooth and strictly convex Banach space E, and let R be a retraction from E onto C. Then, the following are equivalent:*

(1) *R is sunny generalized nonexpansive;*

(2) *$\langle x - Rx, Jy - JRx\rangle \leq 0$, for all $x \in E$ and $y \in C$.*

Lemma 11 (see [11]). *Let C be a nonempty, closed, and sunny generalized nonexpansive retraction of a smooth and strictly convex Banach space E. Then, the sunny generalized nonexpansive retraction from E onto C is uniquely determined.*

Lemma 12 (see [11]). *Let C be a nonempty, closed, and subset of a smooth and strictly convex Banach space E such that there exists a sunny generalized nonexpansive retraction R from E onto C. Let $x \in E$ and $z \in C$. Then, the following hold:*

(1) *$z = Rx$ if and only if $\langle x - z, Jy - Jz\rangle \leq 0$, for all $y \in C$;*

(2) *$\phi(x, Rx) + \phi(Rx, z) \leq \phi(x, z)$.*

Lemma 13 (see [16]). *Let C be a nonempty, closed, and subset of a smooth, strictly convex, and reflexive Banach space E. Then, the following are equivalent:*

(1) *C is sunny generalized nonexpansive retraction of E;*

(2) *$J(C)$ is closed and convex.*

Remark 14. Let E be a Hilbert space. By the Lemmas 11 and 13, a sunny generalized nonexpansive retraction from E onto C reduces to a metric projection operator P from E onto C.

Lemma 15 (see [16]). *Let E be a smooth, strictly convex, and reflexive Banach space, let C be a nonempty, closed, and sunny generalized nonexpansive retraction of E, and let R be the sunny generalized nonexpansive retraction from E onto C. Let $x \in E$ and $z \in C$. Then, the following are equivalent:*

(1) $z = Rx$;

(2) $\phi(x, z) = \min_{y\in C} \phi(x, y)$.

Lemma 16 (see [4]). *Let E be a uniformly smooth and strictly convex real Banach space, and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n - y_n\| \to 0$.*

Lemma 17 (see [17]). *Let E be a uniformly smooth and strictly convex real Banach space with the Kadec-Klee property, and let C be a nonempty, closed, and convex subset of E. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in C and $p \in E$. If $x_n \to p$ and $\phi(x_n, y_n) \to 0$, then $y_n \to p$.*

Lemma 18 (see [18]). *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \le a_n + b_n, \quad \forall n \ge 1. \tag{39}$$

If $\sum_{n=0}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

Lemma 19 (see [19]). *Let E be a uniformly convex Banach space. Then, for any $r > 0$, there exists a strictly increasing, continuous, and convex function $h : [0, 2r] \to \mathbb{R}$ such that $h(0) = 0$ and*

$$\begin{aligned} \|tx + (1-t)y\|^2 \le t\|x\|^2 + (1-t)\|y\|^2 \\ - t(1-t)h(\|x-y\|), \\ \forall x, y \in B_r(z),\ t \in [0,1], \end{aligned} \tag{40}$$

where $B_r(z) = \{z \in E : \|z\| \le r\}$.

Lemma 20 (see [4]). *Let E be a smooth and uniformly convex Banach space. Then, for any $r > 0$, there exists a strictly increasing, continuous, and convex function $h : [0, 2r] \to \mathbb{R}$ such that $h(0) = 0$ and*

$$h(\|x-y\|) \le \phi(x, y), \quad \forall x, y \in B_r(z),\ t \in [0,1], \tag{41}$$

where $B_r(z) = \{z \in E : \|z\| \le r\}$.

Now, let us recall the following well-known concept and result.

Definition 21 (see [20]). Let B be a subset of a topological vector space X. A mapping $G : B \to 2^X$ is called a KKM mapping if $\operatorname{conv}\{x_1, x_2, x_3, \dots, x_m\} \subset \bigcup_{i=1}^m G(x_i)$ for $x_i \in B$ and $i = 1, 2, 3, \dots, m$, where $\operatorname{conv} A$ denotes the convex hull of the set A.

In [21], Fan gave the following famous infinite-dimensional generalization of Knaster, Kuratowski, and Mazurkiewicz's classical finite-dimensional result.

Lemma 22 (see [21]). *Let B be a subset of a Hausdorff topological vector space X, and let $G : B \to 2^X$ be a KKM mapping. If $G(x)$ is closed, for all $x \in B$ and is compact for at least one $x \in B$, then $\bigcap_{x\in B} G(x) \ne \emptyset$.*

Lemma 23 (see [14]). *Let C be a nonempty, closed, and convex subset of a smooth and strictly convex Banach space, and let $T : C \to C$ be a closed and -ϕ-nonexpansive mapping. Then,* $\mathrm{Fix}(T)$ *is a closed and convex subset of C.*

3. Existence Theorem

In this section, we prove the existence theorem of a solution for a generalized mixed equilibrium problem with a bifunction defined on the dual space of a Banach space.

Lemma 24. *Let C be a nonempty, compact, and convex subset of a uniformly smooth, strictly convex, and reflexive Banach space E, let J be the duality mapping from E into E^* such that $J(C)$ is closed and convex, let us assume that a bifunction $F : J(C) \times J(C) \to \mathbb{R}$ satisfies the following conditions (DA1)–(DA4), let C^* be a nonempty, closed, and convex subset of E^*, let $A : C^* \to E$ be an α-inverse strongly skew monotone and let $\varphi : J(C) \to \mathbb{R}$ be a convex and lower semicontinuous. Let $r > 0$ be given real number and $x \in E$ be any point. Then, there exists $z \in C$ such that*

$$\begin{aligned} F(Jz, Jy) + \langle AJz, Jy - Jz\rangle + \frac{1}{r}\langle z - x, J(y - z)\rangle \\ + \varphi(Jy) - \varphi(Jz) \ge 0, \quad \forall y \in C. \end{aligned} \tag{42}$$

Proof. Let x_0 be any point in E. For each $y \in C$, we define the mapping $G : C \to 2^E$ as follows:

$$\begin{aligned} G(y) = \{z \in C : F(Jz, Jy) + \langle AJz, Jy - Jz\rangle \\ + \frac{1}{r}\langle z - x_0, J(y - z)\rangle + \varphi(Jy) \\ -\varphi(Jz) \ge 0, \forall y \in C\}. \end{aligned} \tag{43}$$

It is easy to see that $y \in G(y)$, and hence $G(y) \ne \emptyset$.

(a) First, we will show that G is a KKM mapping.

Suppose that G is not a KKM mapping. Then, there exists a finite subset $\{y_1, y_2, y_3, \dots, y_m\}$ of C and $\alpha_i > 0$ with $\sum_{i=1}^m \alpha_i = 1$ such that $\hat{x} = \sum_{i=1}^m \alpha_i y_i \notin \bigcup_{i=1}^m G(y_i)$ for all $i = 1, 2, 3, \dots, m$.

It follows from the definition of a mapping G that

$$\begin{aligned} F(J\hat{x}, Jy_i) + \langle AJ\hat{x}, Jy_i - J\hat{x}\rangle + \frac{1}{r}\langle \hat{x} - x_0, J(y_i - \hat{x})\rangle \\ + \varphi(Jy_i) - \varphi(J\hat{x}) < 0, \quad \forall i = 1, 2, 3, \dots, m. \end{aligned} \tag{44}$$

By the assumptions of (DA1) and (DA4), we get

$$\begin{aligned} 0 = F(J\hat{x}, J\hat{x}) + \langle AJ\hat{x}, J\hat{x} - J\hat{x}\rangle \\ + \frac{1}{r}\langle \hat{x} - x_0, J(\hat{x} - \hat{x})\rangle + \varphi(J\hat{x}) - \varphi(J\hat{x}) \end{aligned}$$

$$\leq \sum_{i=1}^{m} \alpha_i \Big(F(J\hat{x}, Jy_i) + \langle AJ\hat{x}, Jy_i - J\hat{x} \rangle + \frac{1}{r} \langle \hat{x} - x_0, J(y_i - \hat{x}) \rangle + \varphi(Jy_i) - \varphi(J\hat{x}) \Big) < 0, \tag{45}$$

which is a contradiction. Thus, G is a KKM mapping on C.

(b) Next, we will show that $G(y)$ is closed, for all $y \in C$. Let $\{z_n\}$ be a sequence in $G(y)$ such that $z_n \to z_0$ as $n \to \infty$.

It then follows from $z_n \in G(y)$ that

$$F(Jz_n, Jy) + \langle AJz_n, Jy - Jz_n \rangle + \frac{1}{r} \langle z_n - x_0, J(y - z_n) \rangle + \varphi(Jy) - \varphi(Jz_n) \geq 0. \tag{46}$$

By the assumption (DA3), the continuity of J, and the lower semicontinuity of φ and $\|\cdot\|^2$, it follows from (46) that

$$\begin{aligned}
\varphi(z_0) &\leq \liminf_{n\to\infty} \varphi(z_n) \\
&\leq \liminf_{n\to\infty} \Big(F(Jz_n, Jy) + \langle AJz_n, Jy - Jz_n \rangle + \frac{1}{r} \langle z_n - x_0, J(y - z_n) \rangle + \varphi(Jy) \Big) \\
&\leq \limsup_{n\to\infty} \Big(F(Jz_n, Jy) + \langle AJz_n, Jy - Jz_n \rangle + \frac{1}{r} \langle z_n - x_0, J(y - z_n) \rangle + \varphi(Jy) \Big) \\
&= \limsup_{n\to\infty} \big(F(Jz_n, Jy) + \langle AJz_n, Jy - Jz_n \rangle + \frac{1}{r} \langle (z_n - y) + (y - x_0), J(y - z_n) \rangle + \varphi(Jy) \big) \\
&= \limsup_{n\to\infty} \Big(F(Jz_n, Jy) + \langle AJz_n, Jy - Jz_n \rangle + \frac{1}{r} \langle z_n - y, J(y - z_n) \rangle + \frac{1}{r} \langle y - x_0, J(y - z_n) \rangle + \varphi(Jy) \Big) \\
&= \limsup_{n\to\infty} \Big(F(Jz_n, Jy) + \langle AJz_n, Jy - Jz_n \rangle + \frac{1}{r} \langle y - x_0, J(y - z_n) \rangle - \frac{1}{r} \langle y - z_n, J(y - z_n) \rangle + \varphi(Jy) \Big) \\
&= \limsup_{n\to\infty} \Big(F(Jz_n, Jy) + \langle AJz_n, Jy - Jz_n \rangle + \frac{1}{r} \langle y - x_0, J(y - z_n) \rangle - \frac{1}{r} \|y - z_n\|^2 + \varphi(Jy) \Big) \\
&\leq \limsup_{n\to\infty} F(Jz_n, Jy) + \limsup_{n\to\infty} \langle AJz_n, Jy - Jz_n \rangle + \frac{1}{r} \limsup_{n\to\infty} \langle y - x_0, J(y - z_n) \rangle - \liminf_{n\to\infty} \frac{1}{r} \|y - z_n\|^2 + \varphi(Jy) \\
&\leq F(Jz_0, Jy) + \langle AJz_0, Jy - Jz_0 \rangle + \frac{1}{r} \langle y - x_0, J(y - z_0) \rangle - \frac{1}{r} \|y - z_0\|^2 + \varphi(Jy) \\
&= F(Jz_0, Jy) + \langle AJz_0, Jy - Jz_0 \rangle + \frac{1}{r} \langle y - x_0, J(y - z_0) \rangle - \frac{1}{r} \langle y - z_0, J(y - z_0) \rangle + \varphi(Jy) \\
&= F(Jz_0, Jy) + \langle AJz_0, Jy - Jz_0 \rangle + \frac{1}{r} \langle y - x_0, J(y - z_0) \rangle + \frac{1}{r} \langle z_0 - y, J(y - z_0) \rangle + \varphi(Jy) \\
&= F(Jz_0, Jy) + \langle AJz_0, Jy - Jz_0 \rangle + \frac{1}{r} \langle (y - x_0) + (z_0 - y), J(y - z_0) \rangle + \varphi(Jy) \\
&= F(Jz_0, Jy) + \langle AJz_0, Jy - Jz_0 \rangle + \frac{1}{r} \langle z_0 - x_0, J(y - z_0) \rangle + \varphi(Jy).
\end{aligned} \tag{47}$$

Now, we get

$$F(Jz_0, Jy) + \langle AJz_0, Jy - Jz_0 \rangle + \frac{1}{r} \langle z_0 - x_0, J(y - z_0) \rangle + \varphi(Jy) - \varphi(Jz_0) \geq 0. \tag{48}$$

Therefore, $z_0 \in G(y)$, and so $G(y)$ is closed, for all $y \in C$.

(c) We will show that $G(y)$ is weakly compact.

Now, we know that $G(y)$ is closed and subset of C.

Since C is compact. Therefore, $G(y)$ is compact, and then $G(y)$ is weakly compact.

By using (a), (b), and (c) and Lemma 22, we can conclude that $\bigcap_{y\in C} G(y) \neq \emptyset$.

Therefore, there exists $z \in C$ such that

$$F(Jz, Jy) + \langle AJz, Jy - Jz\rangle + \frac{1}{r}\langle z - x, J(y - z)\rangle + \varphi(Jy) - \varphi(Jz) \geq 0, \quad \forall y \in C. \tag{49}$$

□

Theorem 25. *Let C be a nonempty, closed, and convex subset of a uniformly smooth and strictly convex real Banach space E such that $J(C)$ is closed and convex, let us assume that a bifunction $F : J(C) \times J(C) \to \mathbb{R}$ satisfies the following conditions (DA1)–(DA4), let C^* be a nonempty, closed, and convex subset of E^*, let $A : C^* \to E$ be an α-inverse strongly skew monotone and let $\varphi : J(C) \to \mathbb{R}$ be a convex and lower semicontinuous. Let $r > 0$ be given real number and $x \in E$ be any point. We define a mapping $S_r : E \to C$ as follows:*

$$S_r(x) = \{z \in C : F(Jz, Jy) + \langle AJz, Jy - Jz\rangle + \frac{1}{r}\langle z - x, J(y - z)\rangle + \varphi(Jy) - \varphi(Jz) \geq 0, \ \forall y \in C\}. \tag{50}$$

Then, the following conclusion hold:

(1) *S_r is single-valued;*

(2) *$\langle S_r x - S_r y, J(S_r x - S_r y)\rangle \leq \langle x - y, J(S_r x - S_r y)\rangle$, $\forall x, y \in E$;*

(3) *$\mathrm{Fix}(S_r) = \mathrm{GMEP}(F, A, \varphi)$;*

(4) *$J(\mathrm{GMEP}(F, A, \varphi))$ is closed and convex;*

Proof. We will complete this proof by the following four items.

(1) *We will show that S_r is single-valued.*

Indeed, for any $x \in E$ and $r > 0$, let $z_1, z_2 \in S_r(x)$. Then,

$$\begin{aligned} &F(Jz_1, Jz_2) + \langle AJz_1, Jz_2 - Jz_1\rangle + \frac{1}{r}\langle z_1 - x, J(z_2 - z_1)\rangle + \varphi(Jz_2) - \varphi(Jz_1) \geq 0, \\ &F(Jz_2, Jz_1) + \langle AJz_2, Jz_1 - Jz_2\rangle + \frac{1}{r}\langle z_2 - x, J(z_1 - z_2)\rangle + \varphi(Jz_1) - \varphi(Jz_2) \geq 0. \end{aligned} \tag{51}$$

Adding the two inequalities, we have

$$\begin{aligned} 0 &\leq F(Jz_1, Jz_2) + F(Jz_2, Jz_1) \\ &\quad + \langle AJz_1, Jz_2 - Jz_1\rangle + \langle AJz_2, Jz_1 - Jz_2\rangle \\ &\quad + \frac{1}{r}\langle z_1 - x, J(z_2 - z_1)\rangle + \frac{1}{r}\langle z_2 - x, J(z_1 - z_2)\rangle \\ &\quad + \varphi(Jz_2) - \varphi(Jz_1) + \varphi(Jz_1) - \varphi(Jz_2) \\ &= F(Jz_1, Jz_2) + F(Jz_2, Jz_1) \\ &\quad + \langle AJz_1, Jz_2 - Jz_1\rangle - \langle AJz_2, Jz_2 - Jz_1\rangle \\ &\quad + \frac{1}{r}\langle z_1 - x, J(z_2 - z_1)\rangle - \frac{1}{r}\langle z_2 - x, J(z_2 - z_1)\rangle \\ &= F(Jz_1, Jz_2) + F(Jz_2, Jz_1) \\ &\quad + \langle AJz_1 - AJz_2, Jz_2 - Jz_1\rangle \\ &\quad + \frac{1}{r}\langle (z_1 - x) - (z_2 - x), J(z_2 - z_1)\rangle \\ &= F(Jz_1, Jz_2) + F(Jz_2, Jz_1) \\ &\quad + \langle AJz_1 - AJz_2, Jz_2 - Jz_1\rangle + \frac{1}{r}\langle z_1 - z_2, J(z_2 - z_1)\rangle \\ &= F(Jz_1, Jz_2) + F(Jz_2, Jz_1) \\ &\quad - \langle AJz_1 - AJz_2, Jz_1 - Jz_2\rangle + \frac{1}{r}\langle z_1 - z_2, J(z_2 - z_1)\rangle. \end{aligned} \tag{52}$$

Therefore,

$$F(Jz_1, Jz_2) + F(Jz_2, Jz_1) - \langle AJz_1 - AJz_2, Jz_1 - Jz_2\rangle + \frac{1}{r}\langle z_1 - z_2, J(z_2 - z_1)\rangle \geq 0. \tag{53}$$

From the condition (DA2), A is an α-inverse strongly skew monotone, and we have

$$\begin{aligned} 0 &\leq F(Jz_1, Jz_2) + F(Jz_2, Jz_1) \\ &\quad - \langle AJz_1 - AJz_2, Jz_1 - Jz_2\rangle + \frac{1}{r}\langle z_1 - z_2, J(z_2 - z_1)\rangle \\ &\leq -\alpha\|AJz_1 - AJz_2\|^2 + \frac{1}{r}\langle z_1 - z_2, J(z_2 - z_1)\rangle \\ &\leq \frac{1}{r}\langle z_1 - z_2, J(z_2 - z_1)\rangle. \end{aligned} \tag{54}$$

Since $r > 0$, J is monotone, E is strictly convex, and we obtain

$$z_1 = z_2. \tag{55}$$

This implies that S_r is single-valued.

(2) *We will show that $\langle S_r x - S_r y, J(S_r x - S_r y)\rangle \leq \langle x - y, J(S_r x - S_r y)\rangle$, for all $x, y \in E$.*

Indeed, for any $x, y \in C$ and $r > 0$, we have

$$\begin{aligned}
&F(JS_rx, JS_ry) + \langle AJS_rx, JS_ry - JS_rx\rangle \\
&\qquad + \frac{1}{r}\langle S_rx - x, J(S_ry - S_rx)\rangle + \varphi(JS_ry) - \varphi(JS_rx) \\
&\quad \geq 0, \\
&F(JS_ry, JS_rx) + \langle AJS_ry, JS_rx - JS_ry\rangle \\
&\qquad + \frac{1}{r}\langle S_ry - y, J(S_rx - S_ry)\rangle + \varphi(JS_rx) - \varphi(JS_ry) \\
&\quad \geq 0.
\end{aligned} \tag{56}$$

Adding the two inequalities, we have

$$\begin{aligned}
0 &\leq F(JS_rx, JS_ry) + F(JS_ry, JS_rx) \\
&\quad + \langle AJS_rx, JS_ry - JS_rx\rangle + \langle AJS_ry, JS_rx - JS_ry\rangle \\
&\quad + \frac{1}{r}\langle S_rx - x, J(S_ry - S_rx)\rangle \\
&\quad + \frac{1}{r}\langle S_ry - y, J(S_rx - S_ry)\rangle \\
&\quad + \varphi(JS_ry) - \varphi(JS_rx) + \varphi(JS_rx) - \varphi(JS_ry) \\
&= F(JS_rx, JS_ry) + F(JS_ry, JS_rx) \\
&\quad + \langle AJS_rx, JS_ry - JS_rx\rangle - \langle AJS_ry, JS_ry - JS_rx\rangle \\
&\quad + \frac{1}{r}\langle S_rx - x, J(S_ry - S_rx)\rangle \\
&\quad - \frac{1}{r}\langle S_ry - y, J(S_ry - S_rx)\rangle \\
&= F(JS_rx, JS_ry) + F(JS_ry, JS_rx) \\
&\quad + \langle AJS_rx - AJS_ry, JS_ry - JS_rx\rangle \\
&\quad + \frac{1}{r}\langle (S_rx - x) - (S_ry - y), J(S_ry - S_rx)\rangle \\
&= F(JS_rx, JS_ry) + F(JS_ry, JS_rx) \\
&\quad + \langle AJS_rx - AJS_ry, JS_ry - JS_rx\rangle \\
&\quad + \frac{1}{r}\langle (S_rx - S_ry) - (x - y), J(S_ry - S_rx)\rangle \\
&= F(JS_rx, JS_ry) + F(JS_ry, JS_rx) \\
&\quad - \langle AJS_rx - AJS_ry, JS_rx - JS_ry\rangle \\
&\quad + \frac{1}{r}\langle (S_rx - S_ry) - (x - y), J(S_ry - S_rx)\rangle .
\end{aligned} \tag{57}$$

Therefore,

$$\begin{aligned}
&F(JS_rx, JS_ry) + F(JS_ry, JS_rx) \\
&\quad - \langle AJS_rx - AJS_ry, JS_rx - JS_ry\rangle \\
&\quad + \frac{1}{r}\langle (S_rx - S_ry) - (x - y), J(S_ry - S_rx)\rangle \geq 0.
\end{aligned} \tag{58}$$

From the condition (DA2), A is an α-inverse strongly skew monotone, and we have

$$\begin{aligned}
0 &\leq F(JS_rx, JS_ry) + F(JS_ry, JS_rx) \\
&\quad - \langle AJS_rx - AJS_ry, JS_rx - JS_ry\rangle \\
&\quad + \frac{1}{r}\langle (S_rx - S_ry) - (x - y), J(S_ry - S_rx)\rangle \\
&\leq -\alpha\|AJS_rx - AJS_ry\|^2 \\
&\quad + \frac{1}{r}\langle (S_rx - S_ry) - (x - y), J(S_ry - S_rx)\rangle \\
&\leq \frac{1}{r}\langle (S_rx - S_ry) - (x - y), J(S_ry - S_rx)\rangle \\
&\leq -\frac{1}{r}\langle (S_rx - S_ry) - (x - y), J(S_rx - S_ry)\rangle .
\end{aligned} \tag{59}$$

Since $r > 0$, we have

$$\langle (S_rx - S_ry) - (x - y), J(S_rx - S_ry)\rangle \leq 0. \tag{60}$$

Therefore,

$$\langle S_rx - S_ry, J(S_rx - S_ry)\rangle - \langle x - y, J(S_rx - S_ry)\rangle \leq 0. \tag{61}$$

This implies that

$$\langle S_rx - S_ry, J(S_rx - S_ry)\rangle \leq \langle x - y, J(S_rx - S_ry)\rangle . \tag{62}$$

(3) *We will show that* $\mathrm{Fix}(S_r) = \mathrm{GMEP}(F, A, \varphi)$.

It is easy to see that

$$\begin{aligned}
z \in \mathrm{Fix}(S_r) &\Longleftrightarrow z = S_rz \\
&\Longleftrightarrow F(Jz, Jy) + \langle AJz, Jy - Jz\rangle \\
&\qquad + \frac{1}{r}\langle z - z, J(y - z)\rangle + \varphi(Jy) - \varphi(Jz) \geq 0 \\
&\Longleftrightarrow F(Jz, Jy) + \langle AJz, Jy - Jz\rangle \\
&\qquad + \varphi(Jy) - \varphi(Jz) \geq 0 \\
&\Longleftrightarrow z \in \mathrm{GMEP}(F, A, \varphi).
\end{aligned} \tag{63}$$

This implies that $\mathrm{Fix}(S_r) = \mathrm{GMEP}(F, A, \varphi)$.

(4) *We will show that* $J(\mathrm{GMEP}(F, A, \varphi))$ *is closed and convex.*

For each $y \in C$, we define the mapping $H : C \to 2^E$ as follows:

$$\begin{aligned}
H(y) = \{z \in C : F(Jz, Jy) + \langle AJz, Jy - Jz\rangle \\
+\varphi(Jy) - \varphi(Jz) \geq 0\}.
\end{aligned} \tag{64}$$

It is easy to see that $y \in H(y)$, so that $H(y) \neq \emptyset$.

Next, we will show that H is a KKM mapping.

Suppose that there exists a finite subset $\{z_1, z_2, \dots, z_m\}$ of C and $\beta_i > 0$ with $\sum_{i=1}^{m} \beta_i = 1$ such that $\widehat{z} = \sum_{i=1}^{m} \beta_i z_i \notin H(z_i)$, for all $i = 1, 2, 3, \dots, m$. Then,

$$F(J\widehat{z}, Jz_i) + \langle AJ\widehat{z}, Jz_i - J\widehat{z}\rangle + \varphi(Jz_i) - \varphi(J\widehat{z}) < 0, \quad i = 1, 2, 3, \dots, m. \tag{65}$$

It follows from (DA1) and (DA4) that

$$\begin{aligned} 0 &= F(J\widehat{z}, J\widehat{z}) + \langle AJ\widehat{z}, J\widehat{z} - J\widehat{z}\rangle + \varphi(J\widehat{z}) - \varphi(J\widehat{z}) \\ &\leq \sum_{i=1}^{m} \beta_i \left(F(J\widehat{z}, Jz_i) + \langle AJ\widehat{z}, Jz_i - J\widehat{z}\rangle + \varphi(Jz_i) - \varphi(J\widehat{z})\right) \\ &< 0. \end{aligned} \tag{66}$$

which is the contradiction. Hence, H is a KKM mapping on C.

(4.1) Next, we will show that $H(y)$ is closed, for each $y \in C$.

For any $y \in C$, let $\{z_n\}$ be any sequence in $H(y)$ such that $z_n \to z_0$ as $n \to \infty$.

Hence, $z_n - x_0 \to z - x_0$ as $n \to \infty$. Next, we will show that $z_0 \in H(y)$. Then, for each $y \in C$, we have

$$F(Jz_n, Jy) + \langle AJz_n, Jy - Jz_n\rangle + \varphi(Jy) - \varphi(Jz_n) \geq 0. \tag{67}$$

It follows from the assumption (DA3) that

$$\begin{aligned} &F(Jz_0, Jy) + \langle AJz_0, Jy - Jz_0\rangle + \varphi(Jy) - \varphi(Jz_0) \\ &\quad\geq \limsup_{n\to\infty} F(Jz_n, Jy) \\ &\qquad + \lim_{n\to\infty} \left(\langle AJz_n, Jy - Jz_n\rangle + \varphi(Jy) - \varphi(Jz_n)\right) \\ &\quad\geq 0. \end{aligned} \tag{68}$$

This implies that $z_0 \in H(y)$, and hence $H(y)$ is closed, for each $y \in C$.

Therefore, $\bigcap_{y\in C} H(y) = J(\mathrm{GMEP}(F, A, \varphi))$ is closed.

(4.2) Next, we will show that $J(\mathrm{GMEP}(F, A, \varphi))$ is convex.

Let $z_1^*, z_2^* \in J(\mathrm{GMEP}(F, A, \varphi))$; then, we have $z_1^* = Jz_1 \in J(C)$ and $z_2^* = Jz_2 \in J(C)$, where $z_1, z_2 \in C$.

For $k, t \in (0, 1)$, let $z^* = kz_1^* + (1-k)z_2^*$, and for any $y \in C$, we set $x_t^* = tJy + (1-t)z^*$.

It follows from (DA1) and (DA4) that

$$\begin{aligned} 0 &= F(x_t^*, x_t^*) + \varphi(x_t^*) - \varphi(x_t^*) \\ &\leq F(x_t^*, tJy + (1-t)z^*) \\ &\quad + \varphi(tJy + (1-t)z^*) - \varphi(x_t^*) \\ &\leq tF(x_t^*, Jy) + (1-t)F(x_t^*, z^*) \\ &\quad + t\varphi(x_t^*) + (1-t)\varphi(Jy) - \varphi(x_t^*) \\ &\leq tF(x_t^*, Jy) + (1-t)\varphi(Jy) - (1-t)\varphi(x_t^*) \\ &\leq F(x_t^*, Jy) + \varphi(Jy) - \varphi(x_t^*), \end{aligned} \tag{69}$$

$$\begin{aligned} 0 &= \langle Ax_t^*, x_t^* - x_t^*\rangle \\ &= \langle Ax_t^*, (x_t^* - Jy) + (Jy - x_t^*)\rangle \\ &= \langle Ax_t^*, x_t^* - Jy\rangle + \langle Ax_t^*, Jy - x_t^*\rangle \\ &= \langle Ax_t^*, x_t^* - Jy\rangle - \langle Ax_t^*, x_t^* - Jy\rangle \\ &\leq \langle Ax_t^*, x_t^* - Jy\rangle \\ &= \langle Ax_t^*, tJy + (1-t)z^* - Jy\rangle \\ &= \langle Ax_t^*, tJy + (1-t)z^* - (t + (1-t))Jy\rangle \\ &= \langle Ax_t^*, (1-t)(z^* - Jy)\rangle \\ &= (1-t)\langle Ax_t^*, z^* - Jy\rangle \\ &= (t-1)\langle Ax_t^*, Jy - z^*\rangle \\ &\leq \langle Ax_t^*, Jy - z^*\rangle. \end{aligned} \tag{70}$$

Adding two inequalities (69) and (70), we get

$$0 \leq F(x_t^*, Jy) + \langle Ax_t^*, Jy - z^*\rangle + \varphi(Jy) - \varphi(x_t^*). \tag{71}$$

Letting $t \downarrow 0$. It follows from (DA3) and the hemicontinuous of A that

$$F(z^*, Jy) + \langle Az^*, Jy - z^*\rangle + \varphi(Jy) - \varphi(z^*) \geq 0, \quad \forall y \in C. \tag{72}$$

Hence, $z^* \in J(\mathrm{GMEP}(F, A, \varphi))$, and thus $J(\mathrm{GMEP}(F, A, \varphi))$ is convex.

This completes the proof. □

4. Iterative Algorithm for Strong Convergence Theorem

In this section, we modify Halpern-Mann iteration to find the common solution of a generalized mixed equilibrium problem and a fixed point problem in Banach spaces.

Theorem 26. *Let E be a uniformly smooth and strictly convex real Banach space, let C be a nonempty, closed and convex subset of E such that $J(C)$ is closed, and convex of E^*, let us assume that a bifunction $F : J(C) \times J(C) \to \mathbb{R}$ satisfies the following conditions (DA1)–(DA4), let C^* be a nonempty, closed and convex subset of E^*, let $A : C^* \to E$ be an α-inverse strongly skew monotone, let $\varphi : J(C) \to \mathbb{R}$ be a convex, and lower semicontinuous, and let $T : C \to C$ be a closed and -ϕ-nonexpansive mapping. Assume that $\Omega :=$ $\mathrm{GMEP}(F, A, \varphi) \cap \mathrm{Fix}(T)$ is nonempty. Let $\{x_n\}$ be a sequence generated by*

$$x_0 \in C \textit{ chosen arbitrary},$$

$$u_n \in C \textit{ such that}$$

$$F(Ju_n, Jy) + \langle AJu_n, Jy - Ju_n\rangle + \frac{1}{r_n}\langle u_n - x_n, J(y-u_n)\rangle$$
$$+ \varphi(Jy) - \varphi(Ju_n) \geq 0, \quad \forall y \in C,$$
$$x_{n+1} = \alpha_n x_0 + (1-\alpha_n)(\beta_n x_n + (1-\beta_n)Tu_n), \quad \forall n \in \mathbb{N}, \tag{73}$$

where J is the duality mapping on E and $\{\alpha_n\}$ is a sequence in $[0,1]$, $\{\beta_n\} \subset [a,b]$, for some $0 < a < b < 1$ and $\{r_n\} \subset [c,\infty)$, for some $c > 0$ such that

$$\sum_{n=0}^{\infty}\alpha_n < \infty, \qquad \sum_{n=0}^{\infty}\beta_n < \infty, \qquad \liminf_{n\to\infty} r_n > 0. \tag{74}$$

Then, the sequence $\{R_\Omega(x_n)\}$ converges strongly to some point in Ω, where R_Ω is the sunny generalized nonexpansive retraction from E onto Ω.

Proof. We will complete this proof by the following three steps.

Step 1. We will show that the sequences $\{x_n\}$ and $\{u_n\}$ are bounded.

Let $u_n = S_{r_n}x_n$ and $y_n = \beta_n x_n + (1-\beta_n)Tu_n$, for any $n \geq 1$. Then,

$$x_{n+1} = \alpha_n x_0 + (1-\alpha_n) y_n. \tag{75}$$

From Theorem 25 and Lemma 23, we know that GMEP (F, A, φ) and $\mathrm{Fix}(T)$ are closed, and convex subset of E. Therefore, Ω is nonempty, closed and convex subset of E.

For any $p \in \Omega$, T is a closed and -ϕ-nonexpansive mapping, we compute

$$\begin{aligned}
\phi(y_n, p) &\leq \phi(\beta_n x_n + (1-\beta_n)Tu_n, p)\\
&= \|\beta_n x_n + (1-\beta_n)Tu_n\|^2\\
&\quad - 2\langle \beta_n x_n + (1-\beta_n)Tu_n, Jp\rangle + \|p\|^2\\
&= \|\beta_n x_n + (1-\beta_n)Tu_n\|^2\\
&\quad - 2\beta_n\langle x_n, Jp\rangle - 2(1-\beta_n)\langle Tu_n, Jp\rangle\\
&\quad + (\beta_n + (1-\beta_n))\|p\|^2\\
&\leq \beta_n\|x_n\|^2 + (1-\beta_n)\|Tu_n\|^2\\
&\quad - 2\beta_n\langle x_n, Jp\rangle - 2(1-\beta_n)\langle Tu_n, Jp\rangle\\
&\quad + (\beta_n + (1-\beta_n))\|p\|^2\\
&= \beta_n(\|x_n\|^2 - 2\langle x_n, Jp\rangle + \|p\|^2)\\
&\quad + (1-\beta_n)(\|Tu_n\|^2 - 2\langle Tu_n, Jp\rangle + \|p\|^2)\\
&= \beta_n\phi(x_n, p) + (1-\beta_n)\phi(Tu_n, p)\\
&\leq \beta_n\phi(x_n, p) + (1-\beta_n)\phi(u_n, p)\\
&= \beta_n\phi(x_n, p) + (1-\beta_n)\phi(S_{r_n}x_n, p)\\
&\leq \beta_n\phi(x_n, p) + (1-\beta_n)\phi(x_n, p)\\
&= \phi(x_n, p),
\end{aligned} \tag{76}$$

and we have

$$\begin{aligned}
\phi(x_{n+1}, p) &\leq \phi(\alpha_n x_0 + (1-\alpha_n)y_n, p)\\
&= \|\alpha_n x_0 + (1-\alpha_n)y_n\|^2\\
&\quad - 2\langle \alpha_n x_0 + (1-\alpha_n)y_n, Jp\rangle + \|p\|^2\\
&= \|\alpha_n x_0 + (1-\alpha_n)y_n\|^2\\
&\quad - 2\alpha_n\langle x_0, Jp\rangle - 2(1-\alpha_n)\langle y_n, Jp\rangle\\
&\quad + (\alpha_n + (1-\alpha_n))\|p\|^2\\
&\leq \alpha_n\|x_0\|^2 + (1-\alpha)\|y_n\|^2\\
&\quad - 2\alpha_n\langle x_0, Jp\rangle - 2(1-\alpha_n)\langle y_n, Jp\rangle\\
&\quad + (\alpha_n + (1-\alpha_n))\|p\|^2\\
&= \alpha_n(\|x_0\|^2 - 2\langle x_0, Jp\rangle + \|p\|^2)\\
&\quad + (1-\alpha_n)(\|y_n\|^2 - 2\langle y_n, Jp\rangle + \|p\|^2)\\
&= \alpha_n\phi(x_0, p) + (1-\alpha_n)\phi(y_n, p)\\
&\leq \alpha_n\phi(x_0, p) + (1-\alpha_n)\phi(x_n, p)\\
&\leq \alpha_n\phi(x_0, p) + \phi(x_n, p).
\end{aligned} \tag{77}$$

By virtue of $\sum_{n=0}^{\infty}\alpha_n < \infty$, it follows from Lemma 16 that $\lim_{n\to\infty}\phi(x_n, p)$ exists.

Therefore, $\{\phi(x_n, p)\}$ is bounded, and so $\{x_n\}$ is bounded. Hence, $\{u_n\}$ and $\{y_n\}$ are also bounded.

Step 2. We will show that $R_\Omega(x_n)$ is bounded.

Let $z_n = R_\Omega(x_n)$ and $p \in \Omega$. Then, $z_n \in \Omega$.

It follows from Lemma 12(2) that

$$\begin{aligned}
\phi(x_n, z_n) &= \phi(x_n, R_\Omega(x_n))\\
&\leq \phi(x_n, p) - \phi(R_\Omega(x_n), p)\\
&\leq \phi(x_n, p).
\end{aligned} \tag{78}$$

Since $\{x_n\}$ is bounded. Therefore, $\{z_n\}$ is bounded.

Hence, $R_\Omega(x_n)$ is bounded.

Step 3. We will show that $\{R_\Omega(x_n)\}$ converges strongly to some point in Ω.

From (78), we have $\phi(x_n, z_n) \leq \phi(x_n, p)$. Replacing x_n by x_0, we get $\phi(x_0, z_n) \leq \phi(x_0, p)$.

Therefore, $\{\phi(x_0, z_n)\}$ is bounded.

Now, we know that $\phi(x_{n+1}, z_n) \le \alpha_n \phi(x_0, z_n) + \phi(x_n, z_n)$. By Lemma 12(2), we get

$$\begin{aligned} \phi(x_{n+1}, z_{n+1}) &= \phi(x_{n+1}, R_\Omega(x_{n+1})) \\ &\le \phi(x_{n+1}, z_n) - \phi(R_\Omega(x_{n+1}), z_n) \\ &\le \phi(x_{n+1}, z_n) \\ &\le \alpha_n \phi(x_0, z_n) + \phi(x_n, z_n). \end{aligned} \tag{79}$$

Since $\{\phi(x_0, z_n)\}$ is bounded. There exists $M > 0$ such that $|\phi(x_0, z_n)| \le M$.

By the assumption $\sum_{n=0}^{\infty} \alpha_n < \infty$, we have

$$\sum_{n=0}^{\infty} \phi(x_0, z_n) \le M \sum_{n=0}^{\infty} \alpha_n < \infty. \tag{80}$$

It follows from Lemma 16 that $\lim_{n\to\infty} \phi(x_n, z_n)$ exists.

For any $m \in \mathbb{N}$, we get

$$\begin{gathered} \phi(x_{n+m}, p) \le \phi(x_n, p) + \sum_{j=0}^{m-1} \alpha_{n+j} \phi(x_0, p), \\ \phi(x_{n+m}, z_n) \le \phi(x_n, z_n) + \sum_{j=0}^{m-1} \alpha_{n+j} \phi(x_0, z_n). \end{gathered} \tag{81}$$

Since $z_{n+m} = R_\Omega(x_{n+m})$ and from Lemma 12(2), we have

$$\begin{aligned} &\phi(x_{n+m}, z_{n+m}) + \phi(z_{n+m}, z_n) \\ &\quad \le \phi(x_{n+m}, z_n) \le \phi(x_n, z_n) + \sum_{j=0}^{m-1} \alpha_{n+j} \phi(x_0, p). \end{aligned} \tag{82}$$

Hence,

$$\begin{aligned} \phi(z_{n+m}, z_n) \le \phi(x_n, z_n) &- \phi(x_{n+m}, z_{n+m}) \\ &+ \sum_{j=0}^{m-1} \alpha_{n+j} \phi(x_0, p). \end{aligned} \tag{83}$$

We set $r = \sup\{\|z_n\| : n \in \mathbb{N}\}$. From Lemma 20, it follows that there exists a strictly increasing, continuous, and convex function $h : [0, 2r] \to \mathbb{R}$ such that $h(0) = 0$ and

$$\begin{aligned} h(\|z_n - z_{n+m}\|) &\le \phi(z_{n+m}, z_n) \\ &\le \phi(x_n, z_n) - \phi(x_{n+m}, z_{n+m}) \\ &\quad + \sum_{j=0}^{m-1} \alpha_{n+j} \phi(x_0, z_n). \end{aligned} \tag{84}$$

Since $\{\phi(x_n, z_n)\}$ is convergent sequence, $\{\phi(x_0, z_n)\}$ is bounded, and $\sum_{n=0}^{\infty} \alpha_n < \infty$, then, we obtain

$$\lim_{n\to\infty} \|z_n - z_{n+m}\| = 0, \quad \forall m \in \mathbb{N}. \tag{85}$$

This implies that $\{z_n\}$ is a Cauchy sequence.

Note that Ω is closed.

Thus, there exists $p \in \Omega$ such that $z_n \to p$.

Therefore, the sequence $R_\Omega(x_n)$ converges strongly to some point $p \in \Omega$.

This completes the proof. □

If we set $A \equiv 0$ in Theorem 26, then Theorem 26 reduces to the following corollary which extends and improves the following result of Saewan et al. [14].

Corollary 27. *Let E be a uniformly smooth and strictly convex real Banach space, let C be a nonempty, closed, and convex subset of E such that $J(C)$ is closed and convex of E^*, let us assume that a bifunction $F : J(C) \times J(C) \to \mathbb{R}$ satisfies the following conditions (DA1)–(DA4), let C^* be a nonempty, closed, and convex subset of E^*, let $\varphi : J(C) \to \mathbb{R}$ be a convex and lower semicontinuous, and let $T : C \to C$ be a closed and -ϕ-nonexpansive mapping. Assume that $\Omega := \mathrm{MEP}(F, \varphi) \cap \mathrm{Fix}(T)$ is nonempty. Let $\{x_n\}$ be a sequence generated by*

$$\begin{gathered} x_0 \in C \text{ chosen arbitrary}, \\ u_n \in C \text{ such that} \\ F(Ju_n, Jy) + \frac{1}{r_n} \langle u_n - x_n, J(y - u_n) \rangle + \varphi(Jy) - \varphi(Ju_n) \ge 0, \\ \forall y \in C, \\ x_{n+1} = \alpha_n x_0 + (1 - \alpha_n)(\beta_n x_n + (1 - \beta_n) T u_n), \quad \forall n \in \mathbb{N}, \end{gathered} \tag{86}$$

where J is the duality mapping on E and $\{\alpha_n\}$ is a sequence in $[0, 1]$, $\{\beta_n\} \subset [a, b]$, for some $0 < a < b < 1$ and $\{r_n\} \subset [c, \infty)$, for some $c > 0$ such that

$$\sum_{n=0}^{\infty} \alpha_n < \infty, \qquad \sum_{n=0}^{\infty} \beta_n < \infty, \qquad \liminf_{n\to\infty} r_n > 0. \tag{87}$$

Then, the sequence $\{R_\Omega(x_n)\}$ converges strongly to some point in Ω, where R_Ω is the sunny generalized nonexpansive retraction from E onto Ω.

If we set $A \equiv 0$ and $\varphi \equiv 0$ in Theorem 26, then Theorem 26 reduces to the following corollary which extends and improves the following result of Chen et al. [13].

Corollary 28. *Let E be a uniformly smooth and strictly convex real Banach space, let C be a nonempty, closed, and convex subset of E such that $J(C)$ is closed and convex of E^*, let us assume that a bifunction $F : J(C) \times J(C) \to \mathbb{R}$ satisfies the following conditions (DA1)–(DA4), let C^* be a nonempty, closed, and convex subset of E^*, and let $T : C \to C$ be a closed and -ϕ-nonexpansive mapping. Assume that*

$\Omega := \mathrm{EP}(F) \cap \mathrm{Fix}(T)$ *is nonempty. Let* $\{x_n\}$ *be a sequence generated by*

$$x_0 \in C \text{ chosen arbitrary},$$
$$u_n \in C \text{ such that}$$
$$F(Ju_n, Jy) + \frac{1}{r_n}\langle u_n - x_n, J(y - u_n)\rangle \geq 0, \quad \forall y \in C,$$
$$x_{n+1} = \alpha_n x_0 + (1 - \alpha_n)(\beta_n x_n + (1 - \beta_n) T u_n), \quad \forall n \in \mathbb{N}, \tag{88}$$

where J *is the duality mapping on* E *and* $\{\alpha_n\}$ *is a sequence in* $[0, 1]$, $\{\beta_n\} \subset [a, b]$, *for some* $0 < a < b < 1$ *and* $\{r_n\} \subset [c, \infty)$, *for some* $c > 0$ *such that*

$$\sum_{n=0}^{\infty}\alpha_n < \infty, \qquad \sum_{n=0}^{\infty}\beta_n < \infty, \qquad \liminf_{n\to\infty} r_n > 0. \tag{89}$$

Then, the sequence $\{R_\Omega(x_n)\}$ *converges strongly to some point in* Ω, *where* R_Ω *is the sunny generalized nonexpansive retraction from* E *onto* Ω.

If we set $A \equiv 0$, $\varphi \equiv 0$, and $T \equiv I$ (identity mapping) in Theorem 26, then Theorem 26 reduces to the following corollary which extends and improves the following result of Chen et al. [13].

Corollary 29. *Let* E *be a uniformly smooth and strictly convex real Banach space, let* C *be a nonempty, closed, and convex subset of* E *such that* $J(C)$ *is closed and convex of* E^*, *let us assume that a bifunction* $F : J(C) \times J(C) \to \mathbb{R}$ *satisfies the following conditions (DA1)–(DA4), and let* C^* *be a nonempty, closed, and convex subset of* E^*. *Assume that* $\mathrm{EP}(F)$ *is nonempty. Let* $\{x_n\}$ *be a sequence generated by*

$$x_0 \in C \text{ chosen arbitrary},$$
$$u_n \in C \text{ such that}$$
$$F(Ju_n, Jy) + \frac{1}{r_n}\langle u_n - x_n, J(y - u_n)\rangle \geq 0, \quad \forall y \in C,$$
$$x_{n+1} = \alpha_n x_0 + (1 - \alpha_n)(\beta_n x_n + (1 - \beta_n) u_n), \quad \forall n \in \mathbb{N}, \tag{90}$$

where J *is the duality mapping on* E *and* $\{\alpha_n\}$ *is a sequence in* $[0, 1]$, $\{\beta_n\} \subset [a, b]$, *for some* $0 < a < b < 1$ *and* $\{r_n\} \subset [c, \infty)$, *for some* $c > 0$ *such that*

$$\sum_{n=0}^{\infty}\alpha_n < \infty, \qquad \sum_{n=0}^{\infty}\beta_n < \infty, \qquad \liminf_{n\to\infty} r_n > 0. \tag{91}$$

Then, the sequence $\{R_{\mathrm{EP}(F)}(x_n)\}$ *converges strongly to some point in* Ω, *where* $R_{\mathrm{EP}(F)}$ *is the sunny generalized nonexpansive retraction from* E *onto* $\mathrm{EP}(F)$.

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