Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 732069, 8 pages
http://dx.doi.org/10.1155/2013/732069

Research Article

Degree of Approximation by Hybrid Operators

Naokant Deo,[1] Hee Sun Jung,[2] and Ryozi Sakai[3]

[1] *Department of Applied Mathematics, Delhi Technological University (Formerly Delhi College of Engineering), Bawana Road, Delhi 110042, India*
[2] *Department of Mathematics Education, Sungkyunkwan University, Seoul 110-745, Republic of Korea*
[3] *Department of Mathematics, Meijo University, Nagoya 468-8502, Japan*

Correspondence should be addressed to Hee Sun Jung; hsun90@skku.edu

Received 8 May 2013; Accepted 12 July 2013

Academic Editor: Irena Rachůnková

We consider hybrid (Szász-beta) operators, which are a general sequence of integral type operators including beta function, and we give the degree of approximation by these Szász-beta-Durrmeyer operators.

1. Introduction

The Lupaş-Durrmeyer operators were introduced by Sahai and Prasad [1] who studied the asymptotic formula for simultaneous approximation, and many mathematicians have given different results for the Durrmeyer operators (see [2–6]). Now we consider here a sequence of linear positive operators, which was introduced by Gupta et al. [7] as follows. Let n and β be positive integers. For $f \in C[0,\infty)$ satisfying $\int_0^\infty f(t)/(1+t)^{n+\beta+1}dt < \infty$,

$$B_{n,\beta}[f](x) = \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^\infty b_{n+\beta,k}(t) f(t)\,dt, \quad x \in [0,\infty), \quad n+\beta+1 > 0, \tag{1}$$

where β is a positive integer,

$$s_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!}, \qquad b_{n,k}(t) = \frac{1}{B(k+1,n)}\frac{t^k}{(1+t)^{n+k+1}}, \qquad B(k+1,n) = \frac{\Gamma(k+1)\Gamma(n)}{\Gamma(k+n+1)}. \tag{2}$$

Let $0 < p \le \infty$. For a function f on $[0,\infty)$, we define the norm by

$$\|f\|_{L_p([0,\infty))} = \begin{cases} \left(\int_{[0,\infty)} |f(t)|^p dt\right)^{1/p}, & 0 < p < \infty, \\ \sup_{[0,\infty)} |f(t)|, & p = \infty. \end{cases} \tag{3}$$

Recently Jung and Sakai [8] investigated the Lupaş-Durrmeyer operators and studied the circumstances of convergence. Motivated with the idea of Jung and Sakai [8], we give the degree of approximation by Szász-Beta-Durrmeyer operators in this paper.

2. Basic Results

Lemma 1 (cf. [7]). *Let α, m, n, and r be integers with $m \ge 0$, $r \ge 1$, and $n+\alpha > m$:*

$$R_{n,m,r}(\alpha;x) := \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^\infty b_{n+\alpha,k+r}(y)(y-x)^m dy. \tag{4}$$

Then one has

(i) $R_{n,0,r}(\alpha;x) = 1$ *and* $R_{n,1,r}(\alpha;x) = ((-\alpha+1)x + r + 1)/(n+\alpha-1)$,

(ii) *for* $m \geqslant 1$

$$(n+\alpha-m-1) R_{n,m+1,r}(\alpha;x) = xR'_{n,m,r}(\alpha;x) + ((2m-\alpha+1)x+m+r+1) R_{n,m,r}(\alpha;x) + mx(x+2) R_{n,m-1,r}(x), \tag{5}$$

(iii)

$$R_{n,m,r}(\alpha;x) = O\left(\frac{1}{n^{[(m+1)/2]}}\right) g_{n,m,r}(\alpha;x), \tag{6}$$

where $g_{n,m,r}(\alpha;x)$ *is a polynomial of degree* $\leqslant m$ *such that the coefficients of* $g_{n,m,r}(\alpha;x)$ *are bounded independently of* n.

Proof. Let $R_{n,m,r}(x) := R_{n,m,r}(\alpha;x)$. Then (i)

$$R_{n,0,r}(x) = \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^{\infty} b_{n+\alpha,k+r}(y)\,dy = \sum_{k=0}^{\infty} s_{n,k}(x) = 1. \tag{7}$$

Using

$$\int_0^{\infty} x b_{n,k}(x)\,dx = \frac{k+1}{n-1}, \qquad \sum_{k=0}^{\infty} k s_{n,k}(x) = nx, \tag{8}$$

we see that

$$R_{n,1,r}(x) = \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^{\infty} b_{n+\alpha,k+r}(y)(y-x)\,dy = \sum_{k=0}^{\infty} s_{n,k}(x) \frac{k+r+1}{n+\alpha-1} - x = \frac{(-\alpha+1)x+r+1}{n+\alpha-1}. \tag{9}$$

(ii) Using $xs'_{n,k}(x) = (k-nx)s_{n,k}(x)$, we obtain

$$x\left(R'_{n,m,r}(x) + mR_{n,m-1,r}(x)\right) = \sum_{k=0}^{\infty} xs'_{n,k}(x) \int_0^{\infty} b_{n+\alpha,k+r}(y)(y-x)^m dy = \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^{\infty} (k-nx) b_{n+\alpha,k+r}(y)(y-x)^m dy. \tag{10}$$

Since we know that

$$y(1+y) b'_{n+\alpha,k+r}(y) = (k+r-(n+\alpha+1)y) b_{n+\alpha,k+r}(y),$$
$$k-nx = (k+r-(n+\alpha+1)y) - (r-(\alpha+1)x) + (n+\alpha+1)(y-x), \tag{11}$$

we have

$$(k-nx) b_{n+\alpha,k+r}(y) = y(1+y) b'_{n+\alpha,k+r}(y) - (r-(\alpha+1)x) b_{n+\alpha,k+r}(y) + (n+\alpha+1) b_{n+\alpha,k+r}(y)(y-x). \tag{12}$$

Then substituting (12) into (10), we consider the following:

$$x\left(R'_{n,m,r}(x) + mR_{n,m-1,r}(x)\right) = \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^{\infty} \big(y(1+y) b'_{n+\alpha,k+r}(y) - (r-(\alpha+1)x) b_{n+\alpha,k+r}(y) + (n+\alpha+1) b_{n+\alpha,k+r}(y)(y-x) \big) \times (y-x)^m dy := A_1 + A_2 + A_3. \tag{13}$$

Then since we see

$$\begin{aligned}\int_0^{\infty} y(1+y) b'_{n+\alpha,k+r}(y)(y-x)^m dy &= \int_0^{\infty} \left((y-x)^2 + (1+2x)(y-x) + x(1+x)\right) \times b'_{n+\alpha,k+r}(y)(y-x)^m dy \\ &= \int_0^{\infty} b'_{n+\alpha,k+r}(y)(y-x)^{m+2} dy + (1+2x)\int_0^{\infty} b'_{n+\alpha,k+r}(y)(y-x)^{m+1} dy + x(1+x)\int_0^{\infty} b'_{n+\alpha,k+r}(y)(y-x)^m dy \\ &= -(m+2)\int_0^{\infty} b_{n+\alpha,k+r}(y)(y-x)^{m+1} dy - (1+2x)(m+1)\int_0^{\infty} b_{n+\alpha,k+r}(y)(y-x)^m dy - x(1+x)m\int_0^{\infty} b_{n+\alpha,k+r}(y)(y-x)^{m-1} dy,\end{aligned} \tag{14}$$

we have

$$A_1 = \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^{\infty} y(1+y) b'_{n+\alpha,k+r}(y)(y-x)^m dy = -(m+2) R_{n,m+1,r}(x) - (1+2x)(m+1) R_{n,m,r}(x) - x(1+x) m R_{n,m-1,r}(x). \tag{15}$$

Here the last equation follows from integration by parts. Furthermore, we easily see

$$A_2 + A_3 = -(r-(\alpha+1)x)R_{n,m,r}(x) + (n+\alpha+1)R_{n,m+1,r}(x). \tag{16}$$

Therefore, we conclude

$$\begin{aligned} x\left(R'_{n,m,r}(x) + mR_{n,m-1,r}(x)\right) &= -(m+2)R_{n,m+1,r}(x) - (1+2x)(m+1)R_{n,m,r}(x) \\ &\quad - x(1+x)mR_{n,m-1,r}(x) \\ &\quad - (r-(\alpha+1)x)R_{n,m,r}(x) + (n+\alpha+1)R_{n,m+1,r}(x) \\ &= (n+\alpha-m-1)R_{n,m+1,r}(x) \\ &\quad - ((2m-\alpha+1)x+m+r+1)R_{n,m,r}(x) \\ &\quad - x(1+x)mR_{n,m-1,r}(x). \end{aligned} \tag{17}$$

Consequently, (ii) is proved.

(iii) For $m=1$, (6) holds. Let us assume (6) for $m \geqslant 1$. We note

$$R'_{n,m,r}(x) = O\left(\frac{1}{n^{[(m+1)/2]}}\right)g'_{n,m,r}(x), \quad g'_{n,m,r}(x) \in \mathscr{P}_{m-1}. \tag{18}$$

So, we have, by the assumption of induction,

$$\begin{aligned} (n+\alpha-m-1)R_{n,m+1,r}(\alpha;x) &= O\left(\frac{1}{n^{[(m+1)/2]}}\right)xg'_{n,m,r}(x) \\ &\quad + ((2m-\alpha+1)x+m+r+1)O\left(\frac{1}{n^{[(m+1)/2]}}\right) \\ &\quad \times g_{n,m,r}(x) \\ &\quad + mx(x+2)O\left(\frac{1}{n^{[m/2]}}\right)g_{n,m-1,r}(x). \end{aligned} \tag{19}$$

Here, if m is even, then

$$\begin{aligned} \left[\frac{m+1}{2}\right]+1 &= \frac{m}{2}+1 = \frac{m+2}{2} = \left[\frac{m+2}{2}\right], \\ \left[\frac{m}{2}\right]+1 &= \frac{m}{2}+1 = \frac{m+2}{2} = \left[\frac{m+2}{2}\right], \end{aligned} \tag{20}$$

and if m is odd, then

$$\begin{aligned} \left[\frac{m+1}{2}\right]+1 &= \frac{m+1}{2}+1 = \left[\frac{m+2}{2}\right], \\ \left[\frac{m}{2}\right]+1 &= \frac{m-1}{2}+1 = \frac{m+1}{2} = \left[\frac{m+2}{2}\right]. \end{aligned} \tag{21}$$

Hence we have

$$R_{n,m+1,r}(x) = O\left(\frac{1}{n^{[(m+2)/2]}}\right)g_{n,m+1,r}(x), \tag{22}$$

and here we see that $g_{n,m+1,r}(x)$ is a polynomial of degree $\leqslant m+1$ such that the coefficients of $g_{n,m+1,r}(x)$ are bounded independently of n. □

Lemma 2 (cf. [7]). *Let n, β, and r be integers with $r \geq 0$. Let $f \in C^{(r)}[0,\infty)$ satisfy for a positive integer δ*

$$\left|f^{(r)}(x)\right| \leq O(1)(x+1)^{\delta}. \tag{23}$$

Then one has, for $n+\beta-r > \delta$,

$$\lambda_{n,\beta,r}\left(B_{n,\beta}[f]\right)^{(r)}(x) = \sum_{k=0}^{\infty} s_{n,k}(x)\int_0^{\infty} b_{n+\beta-r,k+r}(y)f^{(r)}(y)\,dy, \tag{24}$$

where

$$\lambda_{n,\beta,r} := \frac{(n+\beta-1)!}{n^r(n+\beta-r-1)!}. \tag{25}$$

Proof. Using

$$b^{(r)}_{n+\beta-r,k+r}(y) = \frac{(n+\beta-1)!}{(n+\beta-r-1)!}\sum_{i=0}^{r}\binom{r}{i}(-1)^i b_{n+\beta,k+i}(y), \tag{26}$$

we have

$$\begin{aligned} \left(B_{n,\beta}[f]\right)^{(r)}(x) &= \sum_{k=0}^{\infty} s^{(r)}_{n,k}(x)\int_0^{\infty} b_{n+\beta,k}(y)f(y)\,dy \\ &= \sum_{i=0}^{r}\sum_{k=i}^{\infty}\binom{r}{i}(-1)^r(-1)^i n^r s_{n,k-i}(x) \\ &\quad \times \int_0^{\infty} b_{n+\beta,k}(y)f(y)\,dy \\ &= n^r\sum_{k=0}^{\infty} s_{n,k}(x)\int_0^{\infty}\sum_{i=0}^{r}\binom{r}{i} \\ &\quad \times (-1)^r(-1)^i b_{n+\beta,k+i}(y)f(y)\,dy \\ &= n^r\frac{(n+\beta-r-1)!}{(n+\beta-1)!}\sum_{k=0}^{\infty} s_{n,k}(x) \\ &\quad \times \int_0^{\infty}(-1)^r b^{(r)}_{n+\beta-r,k+r}(y)f(y)\,dy \\ &= \frac{1}{\lambda_{n,\beta,r}}\sum_{k=0}^{\infty} s_{n,k}(x) \\ &\quad \times \int_0^{\infty} b_{n+\beta-r,k+r}(y)f^{(r)}(y)\,dy. \end{aligned} \tag{27}$$

□

3. Main Results

Theorem 3. *Let $0 < p \le \infty$, and let δ and r be nonnegative integers. Let n and β be integers with $n + \beta - r > \delta$. Let $f \in C^{(r+1)}[0, \infty)$ satisfy*

$$\left|f^{(r)}(x)\right| \le O(1)(x+1)^{\delta}, \qquad \left|f^{(r+1)}(x)\right| \le O(1)(x+1)^{\delta+2}. \tag{28}$$

Then one has uniformly, for f and n,

$$\left|\lambda_{n,\beta,r}\left(B_{n,\beta}[f]\right)^{(r)}(x) - f^{(r)}(x)\right| \le O\left(\frac{1}{n^{1/3}}\right)(x+1)^{\delta+2}. \tag{29}$$

Proof. Let $|t-x| < \varepsilon$ and $x < \xi < t$. By the second inequality of (28),

$$\left|f^{(r)}(t) - f^{(r)}(x)\right| = |t-x|\left|f^{(r+1)}(\xi)\right| \le O(1)|t-x|(x+1)^{\delta+2}. \tag{30}$$

Let $\varepsilon = n^{-\nu}$, $0 < \nu < 1$. Then using Lemma 2 and

$$\sum_{k=0}^{\infty} s_{n,k}(x)\int_0^{\infty} b_{n+\beta-r,k+r}(y) f^{(r)}(x)\,dy = f^{(r)}(x), \tag{31}$$

we have

$$\begin{aligned}
&\left|\lambda_{n,\beta,r}\left(B_{n,\beta}[f]\right)^{(r)}(x) - f^{(r)}(x)\right| \\
&\quad = \left|\sum_{k=0}^{\infty} s_{n,k}(x)\left(\int_{|t-x|<\varepsilon} b_{n+\beta-r,k+r}(t)\left|f^{(r)}(t) - f^{(r)}(x)\right|dt \right.\right. \\
&\qquad \left.\left. + \int_{|t-x|\ge\varepsilon} b_{n+\beta-r,k+r}(t)\left|f^{(r)}(t) - f^{(r)}(x)\right|dt\right)\right| \\
&\quad = E_1 + E_2.
\end{aligned} \tag{32}$$

From (30) and Lemma 1, we have

$$\begin{aligned}
E_1 &= O(1)\left|\sum_{k=0}^{\infty} s_{n,k}(x)\int_{|t-x|<\varepsilon} b_{n+\beta-r,k+r}(t) \times |t-x|(x+1)^{\delta+2}dt\right| \\
&\le O(1)\,\varepsilon\left|R_{n,0,r}(\beta-r;x)\right|(x+1)^{\delta+2} = O(1)\,\varepsilon(x+1)^{\delta+2}.
\end{aligned} \tag{33}$$

Next, we estimate E_2. By the use of the first inequality in (28), we have

$$\begin{aligned}
E_2 &= C\left|\sum_{k=0}^{\infty} s_{n,k}(x)\int_{|t-x|\ge\varepsilon} b_{n+\beta-r,k+r}(t) \times \left(\left|f^{(r)}(t)\right| + \left|f^{(r)}(x)\right|\right)dt\right| \\
&\le C\left|\sum_{k=0}^{\infty} s_{n,k}(x)\int_{|t-x|\ge\varepsilon} b_{n+\beta-r,k+r}(t) \times \left((t+1)^{\delta} + (x+1)^{\delta}\right)dt\right|.
\end{aligned} \tag{34}$$

Now using $(t+1)^{\delta} = ((t-x)+x+1)^{\delta} = \sum_{i=0}^{\delta}\binom{\delta}{i}(t-x)^i(x+1)^{\delta-i}$ and the notation

$$\langle i\rangle = \begin{cases} 1, & i : \text{odd}, \\ 0, & i : \text{even}, \end{cases} \tag{35}$$

we have

$$\begin{aligned}
E_2 &\le C\sum_{k=0}^{\infty} s_{n,k}(x)\int_{|t-x|\ge\varepsilon} b_{n+\beta-r,k+r}(t) \times \left(\sum_{i=0}^{\delta}\binom{\delta}{i}|t-x|^i\left|\frac{t-x}{\varepsilon}\right|^{\langle i\rangle}(x+1)^{\delta-i}\right)dt \\
&\quad + C\sum_{k=0}^{\infty} s_{n,k}(x)\int_{|t-x|\ge\varepsilon} b_{n+\beta-r,k+r}(t)\left(\frac{t-x}{\varepsilon}\right)^2(x+1)^{\delta}dt \\
&= E_{21} + E_{22}.
\end{aligned} \tag{36}$$

Then, with $\varepsilon = n^{-\nu}$,

$$\begin{aligned}
E_{21} &\le C\left(\sum_{i=1}^{\delta}\binom{\delta}{i}\left|R_{n,i+\langle i\rangle,r}(\beta-r;x)\right|\left(\frac{1}{\varepsilon}\right)^{\langle i\rangle}(x+1)^{\delta-i}\right) \\
&\le C\sum_{i=1}^{\delta}\binom{\delta}{i}O\left(\frac{n^{\nu\langle i\rangle}}{n^{[(i+\langle i\rangle+1)/2]}}\right) \times \left|g_{n,i+\langle i\rangle,r}(\beta-r;x)\right|(x+1)^{\delta-i} \\
&\le O\left(\frac{1}{n^{[(i+\langle i\rangle+1)/2]-\nu\langle i\rangle}}\right)(x+1)^{\delta+1} \le O\left(\frac{1}{n^{1-\nu}}\right)(x+1)^{\delta+1}.
\end{aligned} \tag{37}$$

Here for $i \ge 1$, we get

$$\left[\frac{i+\langle i\rangle+1}{2}\right] - \nu\langle i\rangle \ge 1-\nu, \tag{38}$$

because

$$\left[\frac{i+\langle i\rangle+1}{2}\right]-\nu\langle i\rangle=\begin{cases}\dfrac{i+1}{2}-\nu, & i:\text{odd},\\ \dfrac{i}{2}-\nu, & i:\text{even},\end{cases}$$

$$\begin{aligned}E_{22}&\le C\left|R_{n,2,r}(\beta-r;x)\right|\left(\frac{1}{\varepsilon}\right)^2(x+1)^{\delta}\\&\le O\left(\frac{1}{n^{[3/2]}}\right)\left|g_{n,2,r}(\beta-r;x)\right|\left(\frac{1}{\varepsilon}\right)^2(x+1)^{\delta}\\&\le O\left(\frac{n^{2\nu}}{n^{[3/2]}}\right)\left|g_{n,2,r}(\beta-r;x)\right|(x+1)^{\delta}\\&\le O\left(\frac{1}{n^{1-2\nu}}\right)(x+1)^{\delta+2}.\end{aligned}\tag{39}$$

Finally we get

$$E_2\le O\left(\frac{1}{n^{1-2\nu}}\right)(x+1)^{\delta+2}.\tag{40}$$

From (32),

$$\begin{aligned}&\left|\lambda_{n,\beta,r}\left(B_{n,\beta}[f]\right)^{(r)}(x)-f^{(r)}(x)\right|\\&\quad\le O\left(\frac{1}{n^{\nu}}\right)(x+1)^{\delta+2}+O\left(\frac{1}{n^{1-2\nu}}\right)(x+1)^{\delta+2}.\end{aligned}\tag{41}$$

If we put $\nu=1/3$, then we get

$$\left|\lambda_{n,\beta,r}\left(B_{n,\beta}[f]\right)^{(r)}(x)-f^{(r)}(x)\right|\le O\left(\frac{1}{n^{1/3}}\right)(x+1)^{\delta+2}.\tag{42}$$

□

In the following, we let $\phi(t)=1/(1+t)$, $t\in[0,\infty)$.

Theorem 4. *Let r and γ be nonnegative integers. Let n and β be integers with $n+\beta-r>2\gamma+1$. Let $f\in C^{(r+2)}[0,\infty)$ satisfy*

$$\begin{aligned}&\left\|f^{(r+1)}(x)\phi^{2\gamma+1}(x)\right\|_{L_\infty[0,\infty)}<\infty,\\&\left\|f^{(r+2)}(x)\phi^{2\gamma}(x)\right\|_{L_\infty[0,\infty)}<\infty.\end{aligned}\tag{43}$$

Then one has uniformly, for f and n,

$$\begin{aligned}&\left|\lambda_{n,\beta,r}\left(B_{n,\beta}[f]\right)^{(r)}(x)-f^{(r)}(x)\right|\phi^{2(\gamma+1)}(x)\\&\quad\le O\left(\frac{1}{n}\right)\left(\left\|f^{(r+1)}(x)\phi^{2\gamma+1}(x)\right\|_{L_\infty[0,\infty)}\right.\\&\qquad\left.+\left\|f^{(r+2)}(x)\phi^{2\gamma}(x)\right\|_{L_\infty[0,\infty)}\right).\end{aligned}\tag{44}$$

Proof. For $f\in C^{(r+2)}[0,\infty)$, we have

$$\begin{aligned}f^{(r)}(y)&=f^{(r)}(x)+f^{(r+1)}(x)(y-x)\\&\quad+\int_x^y(y-u)f^{(r+2)}(u)\,du,\end{aligned}\tag{45}$$

$$\begin{aligned}&\left|\int_x^y(y-u)f^{(r+2)}(u)\,du\right|\\&\quad\le C\left\|f^{(r+2)}(x)\phi^{2\gamma}(x)\right\|_{L_\infty[0,\infty)}\\&\qquad\times\left((1+x)^{2\gamma}+(1+y)^{2\gamma}\right)(y-x)^2.\end{aligned}\tag{46}$$

From (45), (46), and Lemma 2, we get

$$\begin{aligned}&\lambda_{n,\beta,r}\left(B_{n,\beta}[f]\right)^{(r)}(x)\\&\quad=f^{(r)}(x)+f^{(r+1)}(x)R_{n,1,r}(\beta-r;x)\\&\qquad+\sum_{k=0}^{\infty}s_{n,k}(x)\int_0^{\infty}b_{n+\beta-r,k+r}(t)\\&\qquad\times\int_x^t(t-u)f^{(r+2)}(u)\,du\,dt,\end{aligned}\tag{47}$$

$$\begin{aligned}&\left|\sum_{k=0}^{\infty}s_{n,k}(x)\int_0^{\infty}b_{n+\beta-r,k+r}(t)\int_x^t(t-u)f^{(r+2)}(u)\,du\,dt\right|\\&\quad\le\left\|f^{(r+2)}(x)\phi^{2\gamma}(x)\right\|_{L_\infty[0,\infty)}\\&\qquad\times\left((1+x)^{2\gamma}\left|R_{n,2,r}(\beta-r;x)\right|\right.\\&\qquad\left.+\sum_{k=0}^{\infty}s_{n,k}(x)\int_0^{\infty}b_{n+\beta-r,k+r}(t)(1+t)^{2\gamma}(t-x)^2dt\right).\end{aligned}\tag{48}$$

Using $(1+t)^{2\gamma}\le C((t-x)^{2\gamma}+(1+x)^{2\gamma})$, we obtain

$$\begin{aligned}&\left|\sum_{k=0}^{\infty}s_{n,k}(x)\int_0^{\infty}b_{n+\beta-r,k+r}(t)(1+t)^{2\gamma}(t-x)^2dt\right|\\&\quad\le C\left(\left|R_{n,2\gamma+2,r}(\beta-r;x)\right|+(1+x)^{2\gamma}\left|R_{n,2,r}(\beta-r;x)\right|\right).\end{aligned}\tag{49}$$

Therefore, we have

$$\begin{aligned}&\left|\lambda_{n,\beta,r}\left(B_{n,\beta}[f]\right)^{(r)}(x)-f^{(r)}(x)\right|\phi^{2(\gamma+1)}(x)\\&\quad\le\left|f^{(r+1)}(x)\phi^{2\gamma+1}(x)\right|\left|R_{n,1,r}(\beta-r;x)\right|\phi(x)\\&\qquad+C\left\|f^{(r+2)}(x)\phi^{2\gamma}(x)\right\|_{L_\infty[0,\infty)}(1+x)^{2\gamma}\\&\qquad\times\left|R_{n,2,r}(\beta-r;x)\right|\phi^{2\gamma+2}(x)\end{aligned}$$

$$+ C\left\|f^{(r+2)}(x)\phi^{2\gamma}(x)\right\|_{L_\infty[0,\infty)} \times \left|R_{n,2(\gamma+1),r}(\beta-r;x)\right|\phi^{2(\gamma+1)}(x)$$
$$\le O\left(\frac{1}{n}\right)\left|f^{(r+1)}(x)\phi^{2\gamma+1}(x)\right|\left|g_{n,1,r}(\beta-r;x)\right|\phi(x)$$
$$+ O\left(\frac{1}{n}\right)\left\|f^{(r+2)}(x)\phi^{2\gamma}(x)\right\|_{L_\infty[0,\infty)}(1+x)^{2\gamma} \times \left|g_{n,2,r}(\beta-r;x)\right|\phi^{2(\gamma+1)}(x)$$
$$+ O\left(\frac{1}{n^{\gamma+1}}\right)\left\|f^{(r+2)}(x)\phi^{2\gamma}(x)\right\|_{L_\infty[0,\infty)} \times \left|g_{n,2(\gamma+1),r}(\beta-r;x)\right|\phi^{2(\gamma+1)}(x). \tag{50}$$

For $x \in [0,\infty)$, we have $|g_{n,1,r}(\beta-r;x)|\phi(x) \le C$, $(1+x)^{2\gamma}|g_{n,2,r}(\beta-r;x)|\phi^{2(\gamma+1)}(x) \le C$, and $|g_{n,2(\gamma+1),r}(\beta-r;x)|\phi^{2(\gamma+1)}(x) \le C$. Hence

$$\left|\lambda_{n,\beta,r}\left(B_{n,\beta}[f]\right)^{(r)}(x) - f^{(r)}(x)\right|\phi^{2(\gamma+1)}(x) \le O\left(\frac{1}{n}\right)\left(\left\|f^{(r+1)}(x)\phi^{2\gamma+1}(x)\right\|_{L_\infty[0,\infty)} + \left\|f^{(r+2)}(x)\phi^{2\gamma}(x)\right\|_{L_\infty[0,\infty)}\right). \tag{51}$$

□

Let us define the weighted modulus of smoothness by

$$\omega_k(f;\eta;t) := \sup_{0\leqslant h\leqslant t}\left\|\Delta_h^k f(\cdot)\eta(\cdot)\right\|_{L_\infty([0,\infty))}, \quad t \geqslant 0,\ k = 1, 2, \tag{52}$$

where

$$\Delta_h^1 f(x) = f(x+h) - f(x), \tag{53}$$

$$\Delta_h^2 f(x) = f(x) - 2f(x+h) + f(x+2h). \tag{54}$$

Theorem 5. *Let r and γ be nonnegative integers. Let n and β be integers with $n+\beta-r > 2\gamma+2$. Then one has, for $f \in C^r([0,\infty))$,*

$$\left\|\left(\lambda_{n,\beta,r}\left(B_{n,\beta}[f]\right)^{(r)}(x) - f^{(r)}(x)\right)\phi^{2\gamma+2}(x)\right\|_{L_\infty([0,\infty))} \leqslant C\left(\frac{1}{\sqrt{n}}\omega_1\left(f^{(r)};\phi^{2\gamma+1};\frac{1}{\sqrt{n}}\right) + \omega_2\left(f^{(r)};\phi^{2\gamma};\frac{1}{\sqrt{n}}\right)\right). \tag{55}$$

To prove Theorem 5, we need the following theorem.

Theorem 6. *Let r and γ be nonnegative integers. Let n and β be integers with $n+\beta-r > 2\gamma$. Let $f \in C^{(r)}([0,\infty))$ satisfy*

$$\left\|f^{(r)}\phi^{2\gamma}\right\|_{L_\infty([0,\infty))} < \infty. \tag{56}$$

Then one has uniformly, for n, f, and $x \in [0,\infty)$,

$$\left|\lambda_{n,\beta,r}\left(B_{n,\beta}[f]\right)^{(r)}(x)\phi^{2\gamma}(x)\right| \leqslant C\left\|f^{(r)}(x)\phi^{2\gamma}(x)\right\|_{L_\infty([0,\infty))}. \tag{57}$$

Proof. Using $(1+y)^{2\gamma} \leqslant C((1+x)^{2\gamma} + (y-x)^{2\gamma})$, we have

$$\left|\sum_{k=0}^{\infty}s_{n,k}(x)\int_0^\infty b_{n+\beta-r,k+r}(y)(1+y)^{2\gamma}dy\right| \leqslant C\left|\sum_{k=0}^{\infty}s_{n,k}(x)\int_0^\infty b_{n+\beta-r,k+r}(y) \times \left((1+x)^{2\gamma} + (y-x)^{2\gamma}\right)dy\right|$$
$$\leqslant C\left(R_{n,0,r}(\beta-r;x)\phi^{-2\gamma}(x) + R_{n,2\gamma,r}(\beta-r;x)\right). \tag{58}$$

Therefore, by Lemma 1 (6), we have

$$\left|\sum_{k=0}^{\infty}s_{n,k}(x)\int_0^\infty b_{n+\beta-r,k+r}(y)(1+y)^{2\gamma}dy\right| \leqslant C\left(\phi^{-2\gamma}(x) + O\left(\frac{1}{n^\gamma}\right)g_{n,2\gamma,r}(\beta-r;x)\right). \tag{59}$$

Since $|g_{n,2\gamma,r}(\beta-r;x)\phi^{2\gamma}(x)|$ is uniformly bounded on $[0,\infty)$, we have with Lemma 2 and (59)

$$\left|\lambda_{n,\beta,r}\left(B_{n,\beta}[f]\right)^{(r)}(x)\phi^{2\gamma}(x)\right| \leqslant \left|\sum_{k=0}^{\infty}s_{n,k}(x)\int_0^\infty b_{n+\beta-r,k+r}(y)(1+y)^{2\gamma}dy\,\phi^{2\gamma}(x)\right| \times \left\|f^{(r)}(x)\phi^{2\gamma}(x)\right\|_{L_\infty([0,\infty))}$$
$$\leqslant C\left\|f^{(r)}(x)\phi^{2\gamma}(x)\right\|_{L_\infty([0,\infty))}. \tag{60}$$

Therefore, we have the result. □

The Steklov function $[f]_h(x)$ for $f \in C([0,\infty))$ is defined as follows:

$$[f]_h(x) := \frac{4}{h^2}\int_0^{h/2}\int_0^{h/2}\left[2f(x+s+t) - f(x+2(s+t))\right]ds\,dt, \quad x \geqslant 0,\ h > 0. \tag{61}$$

Then for the Steklov function $[f]_h(x)$ with respect to $f \in C([0,\infty))$, we have the following properties.

Lemma 7 (see [8, Lemma 2.4]). *Let $f(x) \in C([0,\infty))$, and let $\eta(x)$ be a positive and nonincreasing function on $[0,\infty)$. Then*

(i) $[f]_h(x) \in C^2([0,\infty))$,

(ii)

$$\left\|\left([f]_h(x) - f(x)\right)\eta(x)\right\|_{L_\infty([0,\infty))} \leqslant \omega_2\left(f;\eta;\frac{h}{2}\right), \quad (62)$$

(iii)

$$\left\|[f]_h'(x)\eta(x)\right\|_{L_\infty([0,\infty))} \leqslant \frac{4}{h}\omega_1\left(f;\eta;\frac{h}{2}\right)\frac{\eta(x)}{\eta(x+h/2)} + \frac{1}{h}\omega_1(f;\eta;h)\frac{\eta(x)}{\eta(x+h)}, \quad (63)$$

(iv)

$$\left\|[f]_h''(x)\eta(x)\right\|_{L_\infty([0,\infty))} \leqslant \frac{4}{h^2}\left[2\omega_2\left(f;\eta;\frac{h}{2}\right) + \frac{1}{4}\omega_2(f;\eta;h)\right]. \quad (64)$$

Now, we prove Theorem 5.

Proof of Theorem 5. We know that, for $f(x) \in C^r([0,\infty))$,

$$[f]_h^{(r)}(x) = \left[f^{(r)}\right]_h(x),$$
$$[f]_h^{(r+1)}(x) = \left[f^{(r)}\right]_h'(x), \quad (65)$$
$$[f]_h^{(r+2)}(x) = \left[f^{(r)}\right]_h''(x).$$

Then first, we split it as follows:

$$\begin{aligned}&\left\|\left(\lambda_{n,\beta,r}\left(B_{n,\beta}[f]\right)^{(r)}(x) - f^{(r)}(x)\right)\phi^{2\gamma+2}(x)\right\|_{L_\infty([0,\infty))}\\ &\leqslant \left\|\lambda_{n,\beta,r}\left(B_{n,\beta}[f-[f]_h]\right)^{(r)}(x)\phi^{2\gamma+2}(x)\right\|_{L_\infty([0,\infty))}\\ &+ \left\|\left(\lambda_{n,\beta,r}\left(B_{n,\beta}[[f]_h]\right)^{(r)}(x) - [f]_h^{(r)}(x)\right)\times\phi^{2\gamma+2}(x)\right\|_{L_\infty([0,\infty))}\\ &+ \left\|\left([f]_h^{(r)}(x) - f^{(r)}(x)\right)\phi^{2\gamma+2}(x)\right\|_{L_\infty([0,\infty))}.\end{aligned} \quad (66)$$

Then for the first term, we have, using Theorem 6, (62), and (65),

$$\begin{aligned}&\left\|\lambda_{n,\beta,r}\left(B_{n,\beta}[f-[f]_h]\right)^{(r)}(x)\phi^{2\gamma+2}(x)\right\|_{L_\infty([0,\infty))}\\ &\leqslant C\left\|\left[f^{(r)}(x) - \left[f^{(r)}\right]_h(x)\right]\phi^{2\gamma+2}(x)\right\|_{L_\infty([0,\infty))}\\ &\leqslant C\omega_2\left(f^{(r)};\phi^{2\gamma+2}(x);h\right).\end{aligned} \quad (67)$$

Here, we suppose $0 < h \leqslant 1$, and then we know that

$$\frac{\phi(x)}{\phi(x+h)} \leqslant 2, \qquad \frac{\phi(x)}{\phi(x+h/2)} \leqslant 2. \quad (68)$$

For the second term, from Theorem 4, (65), (63), and (64) of Lemma 7,

$$\begin{aligned}&\left\|\left(\lambda_{n,\beta,r}\left(B_{n,\beta}[[f]_h]\right)^{(r)}(x) - [f]_h^{(r)}(x)\right)\phi^{2\gamma+2}(x)\right\|_{L_\infty([0,\infty))}\\ &\leqslant O\left(\frac{1}{n}\right)\left(\left\|[f]_h^{(r+1)}(x)\phi^{2\gamma+1}(x)\right\|_{L_\infty([0,\infty))} + \left\|[f]_h^{(r+2)}(x)\phi^{2\gamma}(x)\right\|_{L_\infty([0,\infty))}\right)\\ &\leqslant O\left(\frac{1}{n}\right)\left(\frac{1}{h}\omega_1\left(f^{(r)};\phi^{2\gamma+1};h\right) + \frac{1}{h^2}\omega_2\left(f^{(r)};\phi^{2\gamma};h\right)\right).\end{aligned} \quad (69)$$

Therefore, we have

$$\begin{aligned}&\left\|\left(\lambda_{n,\beta,r}\left(B_{n,\beta}[[f]_h]\right)^{(r)}(x) - f^{(r)}(x)\right)\phi^{2\gamma+2}(x)\right\|_{L_\infty([0,\infty))}\\ &\leqslant O\left(\frac{1}{n}\right)\left(\frac{1}{h}\omega_1\left(f^{(r)};\phi^{2\gamma+1};h\right) + \frac{1}{h^2}\omega_2\left(f^{(r)};\phi^{2\gamma};h\right)\right)\\ &+ \omega_2\left(f^{(r)};\phi^{2\gamma+2};h\right).\end{aligned} \quad (70)$$

If we let $h = 1/\sqrt{n}$, then

$$\begin{aligned}&\left\|\left(\lambda_{n,\beta,r}\left(B_{n,\beta}[[f]_h]\right)^{(r)}(x) - f^{(r)}(x)\right)\phi^{2\gamma+2}(x)\right\|_{L_\infty([0,\infty))}\\ &\leqslant C\left(\frac{1}{\sqrt{n}}\omega_1\left(f^{(r)};\phi^{2\gamma+1};\frac{1}{\sqrt{n}}\right) + \omega_2\left(f^{(r)};\phi^{2\gamma};\frac{1}{\sqrt{n}}\right)\right),\end{aligned} \quad (71)$$

because $\omega_2(f^{(r)};\phi^{2\gamma+2};1/\sqrt{n}) \leqslant \omega_2(f^{(r)};\phi^{2\gamma};1/\sqrt{n})$. □

Acknowledgment

The authors thank the referees for many kind suggestions and comments.

References

[1] A. Sahai and G. Prasad, "On simultaneous approximation by modified Lupas operators," *Journal of Approximation Theory*, vol. 45, no. 2, pp. 122–128, 1985.

[2] N. Deo, "Faster rate of convergence on Srivastava-Gupta operators," *Applied Mathematics and Computation*, vol. 218, no. 21, pp. 10486–10491, 2012.

[3] N. Deo and N. Bhardwaj, "Some approximation results for Durrmeyer operators," *Applied Mathematics and Computation*, vol. 217, no. 12, pp. 5531–5536, 2011.

[4] M. M. Derriennic, “Sur l’approximation de fonctions intégrables sur [0, 1] par des polynômes de Bernstein modifiés,” *Journal of Approximation Theory*, vol. 31, no. 4, pp. 325–343, 1981.

[5] O. Duman, M. A. Özarslan, and H. Aktuğlu, “Better error estimation for Szász-Mirakjan-beta operators,” *Journal of Computational Analysis and Applications*, vol. 10, no. 1, pp. 53–59, 2008.

[6] H. S. Kasana, G. Prasad, P. N. Agrawal, and A. Sahai, “Modified Szász operators,” in *Mathematical Analysis and Its Applications (Kuwait, 1985)*, vol. 3 of *KFAS Proceedings Series*, pp. 29–41, Pergamon, Oxford, UK, 1988.

[7] V. Gupta, G. S. Srivastava, and A. Sahai, “On simultaneous approximation by Szász-beta operators,” *Soochow Journal of Mathematics*, vol. 21, no. 1, pp. 1–11, 1995.

[8] H. S. Jung and R. Sakai, “Approximation by Lupas-type operators and Szász-Mirakyan-type operators,” *Journal of Applied Mathematics*, vol. 2012, Article ID 546784, 28 pages, 2012.