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*Research Article*

# Picard Type Iterative Scheme with Initial Iterates in Reverse Order for a Class of Nonlinear Three Point BVPs

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We consider the following class of three point boundary value problem $y''(t)+f(t,y)=0, 0<t<1, y'(0)=0, y(1)=\delta y(\eta)$, where $\delta>0, 0<\eta<1$, the source term $f(t,y)$ is Lipschitz and continuous. We use monotone iterative technique in the presence of upper and lower solutions for both well-order and reverse order cases. Under some sufficient conditions, we prove some new existence results. We use examples and figures to demonstrate that monotone iterative method can efficiently be used for computation of solutions of nonlinear BVPs.

## 1. Introduction

In recent years, multipoint boundary value problems have been extensively studied by many authors ([1–5] and the references there in). Multipoint BVPs have lots of applications in various branches of science and engineering; for example, Webb [6] studied a second-order nonlinear boundary value problem subject to some nonlocal boundary conditions, which models a thermostat, and Zou et al. [7] studied the design of a large size bridge with multipoint supports.

It is well known that one of the most important tools for dealing with existence results for nonlinear problems is the method of upper and lower solutions. The method of upper and lower solutions has a long history and some of its ideas can be traced back to Picard [8]. Later, it was extensively studied by Dragoni [9].

Recently, there have been numerous results in the presence of an upper solution $u_0$ and a lower solution $v_0$ with $u_0 \geq v_0$. But, in many cases, the upper and lower solutions may occur in the reversed order also, that is, $u_0 \leq v_0$. Cabada et al. [10] considered the monotone iterative method for the following BVP:

$$y''=f(t,y), \qquad y'(a)=0=y'(b) \tag{1}$$

with reversed ordered upper and lower solutions. So far, there have been some results in the presence of reverse ordered upper and lower solutions [10–13]. Xian et al. [14] considered the following second-order three point BVP:

$$\begin{gathered} y''(t)+f(t,y)=0, \quad 0<t<1, \\ y(0)=0, \qquad y(1)-\delta y(\eta)=0, \end{gathered} \tag{2}$$

where $f(I\times R, R)$, $I=[0,1]$, $0<\eta<1$, $0<\delta<1$. They used the fixed point index theory with non-well-ordered upper and lower solutions.

Recently, Li et al. [15] studied the existence and uniqueness of solutions of second-order three point BVP

$$\begin{gathered} y''+f(t,y)=0, \quad 0<t<1, \\ y'(0)=0, \qquad y(1)=\delta y(\eta), \quad 0<\eta<1, \ \delta>0 \end{gathered} \tag{3}$$

with upper and lower solutions in the reversed order via the monotone iterative method in Banach space.

The present work proves some new existing results for three point BVPs. Our technique is based on Picard-type iterative scheme and is quite simple and efficient from computational point of view. We believe that it can be very well adapted for this type of problem. In this paper we consider the following three point BVP:

$$\begin{gathered} y''(t)+f(t,y)=0, \quad 0<t<1, \\ y'(0)=0, \qquad y(1)=\delta y(\eta), \end{gathered} \tag{4}$$

where $f(I \times R, R)$, $I = [0,1]$, $0 < \eta < 1$, $\delta > 0$. We have allowed $\sup(\partial f/\partial y)$ to take both negative and positive values.

The paper is divided into 4 sections. In Section 2, we construct Green's function and establish maximum and anti-maximum principle. In Section 3, we generate monotone sequences by using results of Section 2 with upper and lower solutions as initial iterates ordered in one way or the other. We prove our final result of existence. In Section 4, we show that the monotone iterative scheme is a powerful technique. For that by using iterative scheme proposed in this paper we have computed the members of sequences in both cases (well-ordered and non-well-ordered case).

## 2. Preliminaries

*2.1. Construction of Green's Function.* To investigate (4), we consider the following linear three point BVP:

$$\begin{gathered} -y''(t) - \lambda y(t) = h(t), \quad 0 < t < 1, \\ y'(0) = 0, \qquad y(1) = \delta y(\eta) + b, \end{gathered} \tag{5}$$

where $h \in C(I)$ and $b$ is any constant. In this section, we construct the Green's function. We divide it into two cases.

*Case I* ($\lambda > 0$). Let us assume that

($H_0$) $0 < \lambda \leq \pi^2/4$, $\sin\sqrt{\lambda} - \delta \sin\sqrt{\lambda}\eta \geq 0$, $\delta\cos\sqrt{\lambda}\eta - \cos\sqrt{\lambda} > 0$.

It is easy to see that ($H_0$) can be satisfied.

**Lemma 1.** *Green's function for the following linear three point BVP*

$$\begin{gathered} y''(t) + \lambda y(t) = 0, \quad 0 < t < 1, \\ y'(0) = 0, \qquad y(1) = \delta y(\eta), \end{gathered} \tag{6}$$

*is given by*

$$G(t,s) = \frac{1}{\sqrt{\lambda}\left(\delta\cos\sqrt{\lambda}\eta - \cos\sqrt{\lambda}\right)} \times \begin{cases} \left[\sin\sqrt{\lambda}(1-s) \right. \\ \quad \left. +\delta\sin\sqrt{\lambda}(s-\eta)\right]\cos\sqrt{\lambda}t, & 0 \leq t \leq s \leq \eta, \\ \cos\sqrt{\lambda}s\sin\sqrt{\lambda}(1-t) \\ \quad +\delta\cos\sqrt{\lambda}s\sin\sqrt{\lambda}(t-\eta), & s \leq t,\ s \leq \eta, \\ \sin\sqrt{\lambda}(1-s)\cos\sqrt{\lambda}t, & t \leq s,\ \eta \leq s, \\ \cos\sqrt{\lambda}s\sin\sqrt{\lambda}(1-t) \\ \quad +\delta\cos\sqrt{\lambda}\eta\sin\sqrt{\lambda}(t-s), & \eta \leq s \leq t \leq 1, \end{cases} \tag{7}$$

*and if $H_0$ holds then $G(t,s) \geq 0$.*

*Proof.* See proof of Lemma 2.1 in [15]. □

**Lemma 2.** *When $\lambda > 0$, $y \in C^2(I)$ is a solution of boundary value problem* (5) *and is given by*

$$y(t) = \frac{b\cos\sqrt{\lambda}t}{\cos\sqrt{\lambda} - \delta\cos\sqrt{\lambda}\eta} - \int_0^1 G(t,s)h(s)\,ds. \tag{8}$$

*Proof.* See proof of Lemma 2.3 in [15]. □

*Remark 3.* Particulary $y \in C^2(I)$ is a solution of the boundary value problem (5) if and only if $y \in C(I)$ is a solution of the integral equation

$$y(t) = \frac{b\cos\sqrt{\lambda}t}{\cos\sqrt{\lambda} - \delta\cos\sqrt{\lambda}\eta} - \int_0^1 G(t,s)h(s)\,ds. \tag{9}$$

*Case II* ($\lambda < 0$). Assume that

($H_0'$) $\lambda < 0$, $\delta\cosh\sqrt{|\lambda|}\eta - \cosh\sqrt{|\lambda|} < 0$, $\sinh\sqrt{|\lambda|} - \delta\sinh\sqrt{|\lambda|}\eta \geq 0$.

It is easy to see that ($H_0'$) can be satisfied.

**Lemma 4.** *Green's function for the following three point BVP*

$$\begin{gathered} y''(t) + \lambda y(t) = 0, \quad 0 < t < 1, \\ y'(0) = 0, \qquad y(1) = \delta y(\eta), \end{gathered} \tag{10}$$

*for $\lambda < 0$ is given by*

$$G(t,s) = \frac{1}{D_\lambda} \begin{cases} \left[\sinh\sqrt{|\lambda|}(1-s) \right. \\ \quad \left. +\delta\sinh\sqrt{|\lambda|}(s-\eta)\right]\cosh\sqrt{|\lambda|}t, & 0 \leq t \leq s \leq \eta, \\ \cosh\sqrt{|\lambda|}s\sinh\sqrt{|\lambda|}(1-t) \\ \quad +\delta\cosh\sqrt{|\lambda|}s\sinh\sqrt{|\lambda|}(t-\eta), & s \leq t,\ s \leq \eta, \\ \sinh\sqrt{|\lambda|}(1-s)\cosh\sqrt{|\lambda|}t, & t \leq s,\ \eta \leq s, \\ \cosh\sqrt{|\lambda|}s\sinh\sqrt{|\lambda|}(1-t) \\ \quad +\delta\cosh\sqrt{|\lambda|}\eta\sin\sqrt{|\lambda|}(t-s), & \eta \leq s \leq t \leq 1, \end{cases} \tag{11}$$

*where $D_\lambda = \sqrt{|\lambda|}(\delta\cosh\sqrt{|\lambda|}\eta - \cosh\sqrt{|\lambda|})$ and if $H_0'$ holds then $G(t,s) \leq 0$.*

*Proof.* Proof is same as given in Lemma 1. □

**Lemma 5.** *When $\lambda < 0$, $y \in C^2(I)$ is a solution of boundary value problem* (5) *and is given by*

$$y(t) = \frac{b\cosh\sqrt{|\lambda|}t}{\cosh\sqrt{|\lambda|} - \delta\cosh\sqrt{|\lambda|}\eta} - \int_0^1 G(t,s)h(s)\,ds. \tag{12}$$

*Proof.* Proof is same as given in Lemma 2. □

*2.2. Maximum and Antimaximum Principle*

**Proposition 6** (antimaximum principle). *Let $0 < \lambda \le \pi^2/4$, $\sin\sqrt{\lambda} - \delta\sin\sqrt{\lambda}\eta \ge 0$, $\delta\cos\sqrt{\lambda}\eta - \cos\sqrt{\lambda} > 0$, $b \ge 0$, and $h(t) \in C[0,1]$ is such that $h(t) \ge 0$; then $y(t)$ is nonpositive on $I$.*

**Proposition 7** (maximum principle). *Let $\lambda < 0$, $\sinh\sqrt{|\lambda|} - \delta\sinh\sqrt{|\lambda|}\eta \ge 0$, $\delta\cosh\sqrt{|\lambda|}\eta - \cosh\sqrt{|\lambda|} < 0$, $b \ge 0$ and $h(t) \in C[0,1]$ is such that $h(t) \ge 0$; then $y(t)$ is nonnegative on $I$.*

## 3. Three Point Nonlinear BVP

Based on maximum and antimaximum Principle we develop theory to solve the three point nonlinear BVP and divide it into the following two subsections.

*3.1. Reverse Ordered Lower and Upper Solutions*

**Theorem 8.** *Let there exist $v_0$, $u_0$ in $C^2[0,1]$ such that $u_0 \le v_0$ and satisfies*

$$\begin{aligned} -u_0''(t) &\ge f(t,u_0), \quad 0<t<1, \\ u_0'(0) &= 0, \qquad u_0(1) \ge \delta u_0(\eta), \end{aligned} \tag{13}$$

$$\begin{aligned} -v_0''(t) &\le f(t,v_0), \quad 0<t<1, \\ v_0'(0) &= 0, \qquad v_0(1) \le \delta v_0(\eta). \end{aligned} \tag{14}$$

*If $f : D \to R$ is continuous on $D := \{(t,y) \in [0,1]\times R^2 : u_0 \le y \le v_0\}$ and there exists $M \ge 0$ such that for all $(t,y), (t,w) \in D$,*

$$y \le w \Longrightarrow f(t,w) - f(t,y) \le M(w-y), \tag{15}$$

*then the boundary value problem (5) has at least one solution in the region $D$. Further, if $\exists$ a constant $\lambda$ such that $M-\lambda \le 0$ and $(H_0)$ is satisfied, then the sequences $\{u_n\}$ generated by*

$$\begin{aligned} -u_{n+1}''(t) - \lambda u_{n+1} &= F(t,u_n), \\ u_{n+1}'(0) &= 0, \qquad u_{n+1}(1) = \delta u_{n+1}(\eta), \end{aligned} \tag{16}$$

*where $F(t,u_n) = f(t,u_n) - \lambda u_n$, with initial iterate $u_0$ converge monotonically (non-decreasing) and uniformly towards a solution $u(t)$ of (5). Similarly, using $v_0$ as an initial iterate leads to a nonincreasing sequences $\{v_n\}$ converging to a solution $v(t)$. Any solution $z(t)$ in $D$ must satisfy*

$$u(t) \le z(t) \le v(t). \tag{17}$$

*Proof.* From (13) and (16) (for $n=0$)

$$\begin{gathered} -(u_0-u_1)'' - \lambda(u_0-u_1) \ge 0, \\ (u_0-u_1)'(0) = 0, \qquad (u_0-u_1)(1) \ge (u_0-u_1)(\eta). \end{gathered} \tag{18}$$

Since $h(t) \ge 0$ and $b \ge 0$, by using Proposition 6, we have $u_0 \le u_1$.

In view of $\lambda \ge M$, from (16) we get

$$-u_{n+1}''(t) \ge -(M-\lambda)(u_{n+1}-u_n) + f(t,u_{n+1}) \tag{19}$$

and if $u_{n+1} \ge u_n$, then

$$-u_{n+1}''(t) \ge f(t,u_{n+1}). \tag{20}$$

Since $u_0 \le u_1$, then from (20) (for $n=0$) and (16) (for $n=1$), we get

$$\begin{gathered} -(u_1-u_2)'' - \lambda(u_1-u_2) \ge 0, \\ (u_1-u_2)'(0) = 0, \qquad (u_1-u_2)(1) \ge (u_1-u_2)(\eta). \end{gathered} \tag{21}$$

From Proposition 6 we have $u_1 \le u_2$.

Now from (14) and (16) (for $n=0$)

$$\begin{gathered} -(u_1-v_0)'' - \lambda(u_1-v_0) \ge 0, \\ (u_1-v_0)'(0) = 0, \qquad (u_1-v_0)(1) \ge \delta(u_1-v_0)(\eta). \end{gathered} \tag{22}$$

Thus, $u_1 \le v_0$ follows from Proposition 6.

Now assuming that $u_{n+1} \ge u_n$, $u_{n+1} \le v_0$, we show that $u_{n+2} \ge u_{n+1}$ and $u_{n+2} \le v_0$ for all $n$. From (16) (for $n+1$) and (20), we get

$$\begin{aligned} &-(u_{n+1}-u_{n+2})'' - \lambda(u_{n+1}-u_{n+2}) \ge 0, \\ &(u_{n+1}-u_{n+2})'(0) = 0, \\ &(u_{n+1}-u_{n+2})(1) \ge \delta(u_{n+1}-u_{n+2})(\eta), \end{aligned} \tag{23}$$

and hence from Proposition 6 we have $u_{n+1} \le u_{n+2}$.

From (16) (for $n+1$) and (14) we get

$$\begin{gathered} -(u_{n+2}-v_0)'' - \lambda(u_{n+2}-v_0) \ge 0, \\ (u_{n+2}-v_0)'(0) = 0, \qquad (u_{n+2}-v_0)(1) \ge \delta(u_{n+2}-v_0)(\eta). \end{gathered} \tag{24}$$

Then, from Proposition 6, $u_{n+2} \le v_0$ and hence we have

$$u_1 \le u_2 \le \cdots \le u_n \le u_{n+1} \le \cdots \le v_0, \tag{25}$$

and starting with $v_0$ it is easy to get

$$v_1 \ge v_2 \ge \cdots \ge v_n \ge v_{n+1} \ge \cdots \ge u_0. \tag{26}$$

Finally, we show that $u_n \le v_n$ for all $n$. For this, by assuming $u_n \le v_n$, we show that $u_{n+1} \le v_{n+1}$. From (16) it is easy to get

$$\begin{aligned} &-(u_{n+1}-v_{n+1})'' - \lambda(u_{n+1}-v_{n+1}) \ge 0, \\ &(u_{n+1}-v_{n+1})'(0) = 0, \\ &(u_{n+1}-v_{n+1})(1) \ge \delta(u_{n+1}-v_{n+1})(\eta). \end{aligned} \tag{27}$$

Hence, from Proposition 6, $u_{n+1} \le v_{n+1}$. Thus we have

$$\begin{aligned} v_0 &\ge v_1 \ge v_2 \ge \cdots \ge v_n \ge v_{n+1} \ge \cdots \ge u_{n+1} \\ &\ge u_n \ge \cdots \ge u_2 \ge u_1 \ge u_0. \end{aligned} \tag{28}$$

So, the sequences $u_n$ and $v_n$ are monotonically nondecreasing and nonincreasing, respectively, and are bounded by $u_0$ and $v_0$. Hence by Dini's theorem they converge uniformly. Let $u(t) = \lim_{n\to\infty} u_n(t)$ and $v(t) = \lim_{n\to\infty} v_n(t)$.

Using Lemma 2, the solution $u_n$ of (16) is given by

$$u_n = \frac{b\cos\sqrt{\lambda}t}{\cos\sqrt{\lambda} - \delta\cos\sqrt{\lambda}\eta} - \int_0^1 G(t,s)\left(f(t,u_n) - \lambda u_n\right) ds. \tag{29}$$

Then, by Lebesgue's dominated convergence theorem, taking the limit as $n$ approaches to $\infty$, we get

$$u(t) = \frac{b\cos\sqrt{\lambda}t}{\cos\sqrt{\lambda} - \delta\cos\sqrt{\lambda}\eta} - \int_0^1 G(t,s)\left(f(t,u) - \lambda u\right) ds, \tag{30}$$

which is the solution of boundary value problem (5).

Any solution $z(t)$ in $D$ can play the role of $u_0(t)$; hence, $z(t) \le v(t)$ and similarly one concludes that $z(t) \ge u(t)$. □

*3.2. Well-Ordered Lower and Upper Solutions*

**Theorem 9.** *Let there exist $v_0$, $u_0$ in $C^2[0,1]$ such that $u_0 \ge v_0$ and satisfies*

$$\begin{aligned} &-u_0''(t) \ge f(t,u_0), \quad 0<t<1,\\ &u_0'(0) = 0, \qquad u_0(1) \ge \delta u_0(\eta),\\ &-v_0''(t) \le f(t,v_0), \quad 0<t<1,\\ &v_0'(0) = 0, \qquad v_0(1) \le \delta v_0(\eta). \end{aligned} \tag{31}$$

*If $f : D_0 \to R$ is continuous on $D_0 := \{(t,y) \in [0,1]\times R^2 : v_0 \le y \le u_0\}$ and there exists $M \ge 0$ such that for all $(t,\tilde{y})$, $(t,\tilde{w}) \in D_0$*

$$\tilde{y} \le \tilde{w} \Longrightarrow f(t,\tilde{w}) - f(t,\tilde{y}) \ge -M(\tilde{w} - \tilde{y}), \tag{32}$$

*then the boundary value problem* (5) *has at least one solution in the region $D_0$. If $\exists$ a constant $\lambda < 0$, such that $\lambda + M \le 0$ and $(H_0')$ is satisfied, then the sequences $\{u_n\}$ generated by*

$$\begin{aligned} -u_{n+1}''(t) - \lambda u_{n+1} = F(t,u_n), \qquad u_{n+1}'(0) = 0,\\ u_{n+1}(1) = \delta u_{n+1}(\eta), \end{aligned} \tag{33}$$

*where $F(t,u_n) = f(t,u_n) - \lambda u_n$, with initial iterate $u_0$, converge monotonically (nonincreasing) and uniformly towards a solution $\tilde{u}(t)$ of* (5)*. Similarly, using $v_0$ as an initial iterate leads to a nondecreasing sequences $\{v_n\}$ converging to a solution $\tilde{v}(t)$. Any solution $\tilde{z}(t)$ in $D_0$ must satisfy*

$$\tilde{v}(t) \le \tilde{z}(t) \le \tilde{u}(t). \tag{34}$$

*Proof.* Proof follows from the analysis of Theorem 8. □

## 4. Numerical Illustration

To verify our results, we consider examples and show that there exists at least one value of $\lambda \in \mathbb{R}\setminus\{0\}$ such that iterative scheme generates monotone sequences which converge to solutions of nonlinear problem. Thus, these examples validate sufficient conditions derived in this paper.

*Example 10* (reverse order). Consider the boundary value problem

$$\begin{aligned} &-y''(t) = \frac{e^y}{32} - \frac{1}{64}, \quad 0<t<1,\\ &y'(0) = 0, \qquad y(1) = 2y\left(\frac{1}{3}\right). \end{aligned} \tag{35}$$

Here, $f(t,y) = (e^y/32) - (1/64)$, $\delta = 2$, $\eta = 1/3$. This problem has $v_0 = 1$ and $u_0 = -1$ as lower and upper solutions; that is, it is non-well-ordered case. The nonlinear term is Lipschitz in $y$ and continuous for all value of $y$, and Lipschitz constants are $M = e/32$. For $0.0849463 \le \lambda \le \pi^2/4$ we can see that $(H_0)$ will be true. To verify $(H_0)$, in Figure 1 we plot inequalities assumed in $(H_0)$. From Figures 2, 3, and 4, we plot members of monotone sequences $u_n$, $v_n$ for different values of $\lambda$.

*Example 11* (well order). Consider the boundary value problem

$$\begin{aligned} &-y''(t) = \frac{1}{32}\left[\frac{e^2}{4} - \frac{\sin t}{4} - 2(y(t))^3\right],\\ &y'(0) = 0, \qquad y(1) = \frac{1}{3}y\left(\frac{1}{2}\right). \end{aligned} \tag{36}$$

Here, $f(t,y) = (1/32)[(e^2/4) - (\sin t/4) - 2(y(t))^3]$, $\delta = 1/3$, $\eta = 1/2$. This problem has $v_0 = -1$ and $u_0 = 1$ as lower and upper solutions; that is, it is well-ordered case. The nonlinear term is Lipschitz in $y$ and continuous for all value of $y$, and Lipschitz constants are $M = 3/16$. For $\lambda \le -0.1875$, we can see that $(H_0')$ will be true. To verify $(H_0')$, in Figure 5 we plot inequalities assumed in $(H_0')$. From Figures 6, 7, and 8, we plot members of monotone sequences $u_n$, $v_n$ for different values of $\lambda$.

## 5. Conclusion

The monotone iterative technique coupled with upper and lower solutions is a powerful tool for computation of solutions of nonlinear three point boundary value problems. It proves the existence of solutions analytically and gives us a tool so that numerical solutions can also be computed and then some real-life problems, for example, bridge design problem, thermostat problem, and so forth, can be solved. We have plotted sequences for both $\lambda > 0$ and $\lambda < 0$. The plots are quite encouraging and will motivate researchers to explore further possibilities. Employing this technique, Mathematica/Maple/MATLAB user-friendly packages can be developed (see [16]).

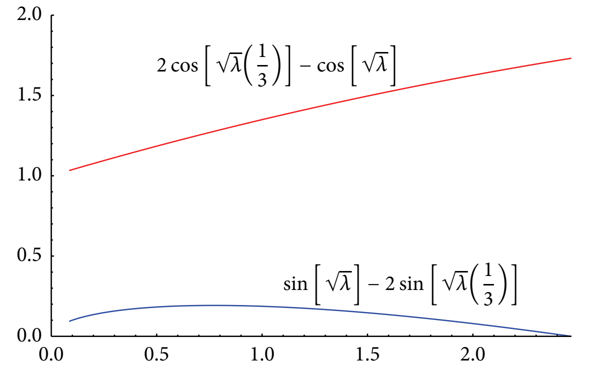


Figure 1: Plot of $\sin\sqrt{\lambda} - \delta\sin\sqrt{\lambda}\eta$ and $\delta\cos\sqrt{\lambda}\eta - \cos\sqrt{\lambda}$.

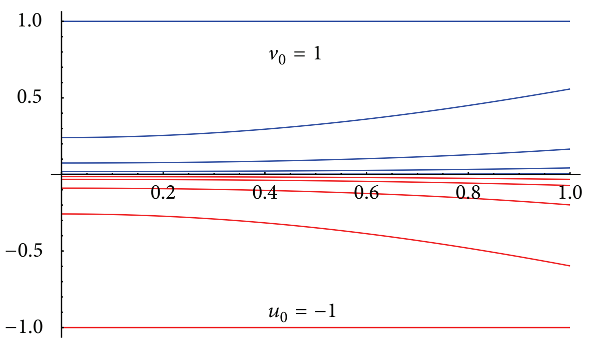


Figure 2: Plot of $u_n$ (red) and $v_n$ (blue), $n = 0, 1, 2, 3, 4$ for $\lambda = 0.9$.

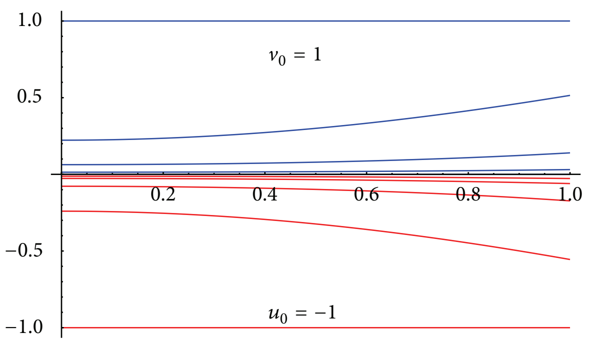


Figure 3: Plot of $u_n$ (red) and $v_n$ (blue), $n = 0, 1, 2, 3, 4$ for $\lambda = 1$.

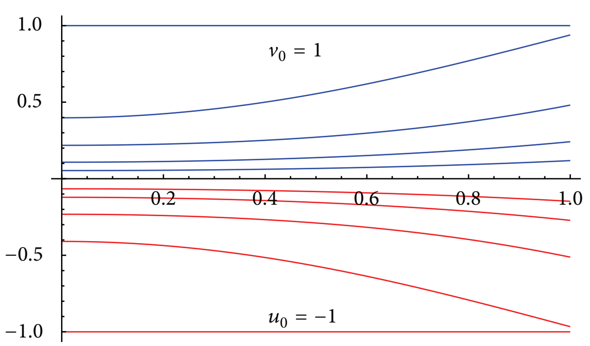


Figure 4: Plot of $u_n$ (red) and $v_n$ (blue), $n = 0, 1, 2, 3, 4$ for $\lambda = 2.3$.

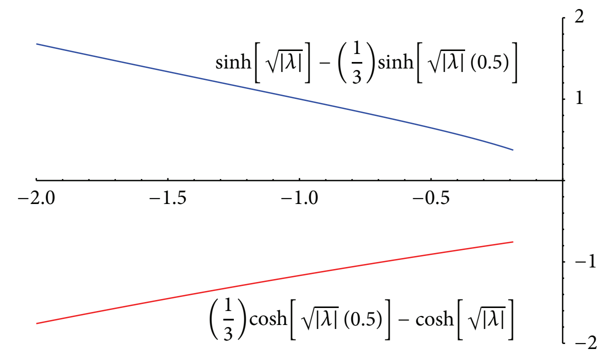


Figure 5: Plot of $\delta\cosh\sqrt{|\lambda|}\eta - \cosh\sqrt{|\lambda|}$ and $\sinh\sqrt{|\lambda|} - \delta\sinh\sqrt{|\lambda|}\eta$.

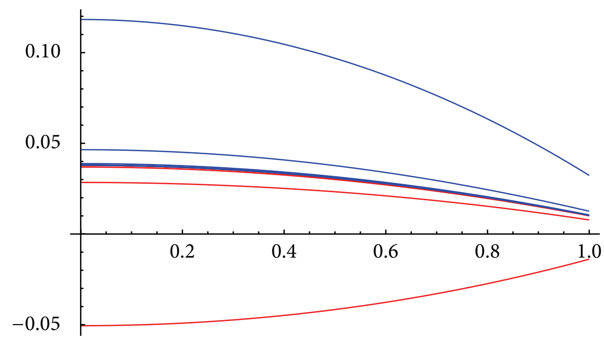

Figure 6: Plot of $u_n$ (blue) and $v_n$ (red), $n = 1, 2, 3, 4, 5$ for $\lambda = -0.2$.

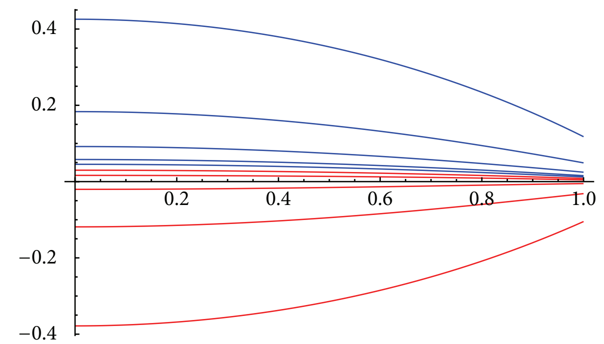

Figure 7: Plot of $u_n$ (blue) and $v_n$ (red), $n = 1, 2, 3, 4, 5$ for $\lambda = -1$.

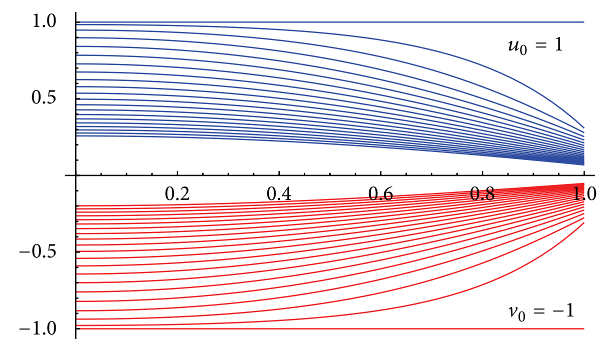


Figure 8: Plot of $u_n$ (blue) and $v_n$ (red), $n = 0, 1, 2, \ldots, 20$ for $\lambda = -20$.

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