Research Article

Exponential Stabilization for Timoshenko Beam with Distributed Delay in the Boundary Control

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We consider the exponential stabilization for Timoshenko beam with distributed delay in the boundary control. Suppose that the controller outputs are of the form $\alpha_1 u_1(t) + \beta_1 u_1(t-\tau) + \int_{-\tau}^{0} g_1(\eta) u_1(t+\eta) d\eta$ and $\alpha_2 u_2(t) + \beta_2 u_2(t-\tau) + \int_{-\tau}^{0} g_2(\eta) u_2(t+\eta) d\eta$; where $u_1(t)$ and $u_2(t)$ are the inputs of boundary controllers. In the past, most stabilization results for wave equations and Euler-Bernoulli beam with delay are required $\alpha_i > \beta_i > 0$, $i = 1, 2$. In the present paper, we will give the exponential stabilization about Timoshenko beam with distributed delay and demand to satisfy the lesser conditions for α_i, β_i, $i = 1, 2$.

1. Introduction

Since the extensive applications of Timoshenko beam in high-Tech, the stabilization problem has been a hot topic in the mathematical control theory and engineering; for instance, see [1–5] and the references therein. In many literature, the control delay problem has been neglected. Due to extensive applications of the system with delay, more and more scholars devoted to study the stabilization of the system with controller delay. It is well known that time delay caused by controller memory usually takes the form $\int_{-\infty}^{0} d\alpha(s) u(t+s)$, where $\alpha(s)$ is a bounded variation function (or matrix-valued function) and $u(t)$ is the control input. If the control is in the space $L^2_{\text{loc}}(\mathbb{R})$, then the memory controller will take the form

$$\int_{-\tau}^{0} d\alpha(s) u(t+s) = \alpha u(t) + \beta u(t-\tau) + \int_{-\tau}^{0} g(s) u(t+s) ds. \tag{1}$$

Based on this reason, Xu et al. (see [6]) studied firstly stabilization of the 1-d wave systems with delay of the form $\alpha u(t) + \beta u(t-\tau)$. They proved that the system with control delay is exponential stable if $\alpha > \beta > 0$ and unstable if $\beta > \alpha$. Nicaise and Pignotti in [7] studied the stability and instability of the wave equation with delay in boundary and internal distributed delay. Nicaise and Valein in [8] extend the 1-d wave equation to the networks of 1-d wave equations. Shang et al. in [9] studied Euler-Bernoulli beam and showed that $\beta > 0$ is not necessary, but the condition $\alpha > |\beta|$ is necessary. For the case of distributed delay, that is, $\beta > 0$ and $\int_{-\tau}^{0} |g(\eta)| d\eta \neq 0$, Nicaise and Pignotti in [10] discussed a high dimensional wave equation. Under the condition that $\alpha > \int_{-\tau}^{0} |g(\eta)| d\eta \neq 0$, they proved the velocity feedback control law also stabilizes exponentially the system.

From above we see that α, β, and $g(\eta)$ are determined by the controller. We cannot determine whether or not $\alpha > \beta > 0$ including $\alpha > |\beta|$ in practice. Under the assumption of state being measurable, Shang and Xu in [11] designed a dynamic feedback controller for cantilever Euler-Bernoulli beam that stabilizes exponentially the system for any real $|\alpha| \neq |\beta|$. Recently Han and Xu in [12] extended this result to the case of output being measurable; they showed that a state observer can realize the state reconstruction from the output of the system. Xu and Wang in [13] discussed the Timoshenko beam with boundary control delay, and they also stabilized the system by a dynamic feedback controller.

Note that the difference between [11, 13], one is a system of single input and single output, the other is a system of 2 inputs and 2 outputs. Such discussion will lead us to extend the method to a general system of multiinput and multioutput. So far, however, there is no result for any α, β, and $g(\eta)$ about Timoshenko beams. In this paper, we still consider Timoshenko beam with boundary control distributed delay. We will seek for a dynamic feedback control law that exponentially stabilizes the Timoshenko beam with distributed delay under certain conditions.

The rest is organized as follows. In Section 2, we will describe the design process of controllers, including predict system and generation of signal, and then state the main results of this paper. In Section 3, we will give the representation of the transform system. In Section 4, we will prove our first result on the stabilization of the original system. In Section 5, we will prove the second result on the exponential stabilization of the induced system. In Section 6, we conclude the paper.

2. Design of Controllers and Main Results

Let $w(x,t)$ be the displacement and $\varphi(x,t)$ the rotation angle of the beam. The motion of a cantilever beam is governed by the following partial differential equations:

$$\rho w_{tt}(x,t) - K(w_{xx} - \varphi_x)(x,t) = 0,$$
$$x \in (0,1), \ t > 0,$$
$$I_\rho \varphi_{tt}(x,t) - EI\varphi_{xx}(x,t) - K(w_x - \varphi)(x,t) = 0,$$
$$x \in (0,1), \ t > 0,$$
$$w(0,t) = \varphi(0,t) = 0, \quad t > 0, \qquad (2)$$
$$K(w_x - \varphi)(1,t) = v_1(t),$$
$$EI\varphi_x(1,t) = v_2(t),$$
$$w(x,0) = w_0(x), \qquad w_t(x,0) = w_1(x),$$
$$\varphi(x,0) = \varphi_0(x), \qquad \varphi_t(x,0) = \varphi_1(x),$$

where $v_1(t)$ and $v_2(t)$ are the control force and torque from the controllers, respectively. If the controllers have no memory, namely, $u_j(t) = v_j(t)$, $j = 1, 2$, where $u_j(t)$ are controller inputs, this model had been studied in [14]. If the controllers have memory, then the Timoshenko beam became

$$\rho w_{tt}(x,t) - K(w_{xx} - \varphi_x)(x,t) = 0,$$
$$x \in (0,1), \ t > 0,$$
$$I_\rho \varphi_{tt}(x,t) - EI\varphi_{xx}(x,t) - K(w_x - \varphi)(x,t) = 0,$$
$$x \in (0,1), \ t > 0,$$
$$w(0,t) = \varphi(0,t) = 0, \quad t > 0,$$

$$K(w_x - \varphi)(1,t) = \alpha_1 u_1(t) + \beta_1 u_1(t - \tau)$$
$$+ \int_{-\tau}^{0} g_1(\eta) u_1(t + \eta) d\eta,$$
$$EI\varphi_x(1,t) = \alpha_2 u_2(t) + \beta_2 u_2(t - \tau)$$
$$+ \int_{-\tau}^{0} g_2(\eta) u_2(t + \eta) d\eta,$$
$$w(x,0) = w_0(x), \qquad w_t(x,0) = w_1(x),$$
$$\varphi(x,0) = \varphi_0(x), \qquad \varphi_t(x,0) = \varphi_1(x),$$
$$u_1(\theta) = f_1(\theta), \quad u_2(\theta) = f_2(\theta), \quad \theta \in (-\tau, 0),$$
$$(3)$$

where τ is the delay time, $\alpha_i, \beta_i \in \mathbb{R}$ $(i = 1, 2)$ are the controller parameters, and $g_i(\eta) \in L^2[-\tau, 0]$, $j = 1, 2$, and $f_i(\theta), \theta \in (-\tau, 0)$ $(i = 1, 2)$ are bounded measurable functions that are memory values of controllers. When $g_j \equiv 0$, $j = 1, 2$, (3) is just the model in [13].

We suppose that the state of (3) is measurable; that is, $(w(x,t), \varphi(x,t), w_t(x,t), \varphi_t(x,t))$ is measurable. We introduce an auxiliary system as follows:

$$\widehat{w}_s(x,s,t) = \widehat{z}(x,s,t), \quad x \in (0,1), \ s \in (0,\tau),$$
$$\widehat{\varphi}_s(x,s,t) = \widehat{\psi}(x,s,t), \quad x \in (0,1), \ s \in (0,\tau),$$
$$\widehat{z}_s(x,s,t) = \frac{K}{\rho}(\widehat{w}_{xx}(x,s,t) - \widehat{\varphi}_x(x,s,t)),$$
$$\widehat{\psi}_s(x,s,t) = \frac{EI}{I_\rho}\widehat{\varphi}_{xx}(x,s,t) + \frac{K}{I_\rho}(\widehat{w}_x(x,s,t) - \widehat{\varphi}(x,s,t)),$$
$$\widehat{w}(0,s,t) = \widehat{\varphi}(0,s,t) = 0, \quad s \in (0,\tau),$$
$$K(\widehat{w}_x - \widehat{\varphi})(1,s,t) = \beta_1 u_1(t + s - \tau)$$
$$+ \int_{-\tau}^{-s} g_1(\eta) u_1(t + s + \eta) d\eta,$$
$$s \in (0,\tau),$$
$$EI\widehat{\varphi}_x(1,s,t) = \beta_2 u_2(t + s - \tau)$$
$$+ \int_{-\tau}^{-s} g_2(\eta) u_2(t + s + \eta) d\eta,$$
$$s \in (0,\tau), \ t > 0,$$
$$\widehat{w}(x,0,t) = w(x,t), \qquad \widehat{z}(x,0,t) = w_t(x,t),$$
$$x \in (0,1),$$
$$\widehat{\varphi}(x,0,t) = \varphi(x,t), \qquad \widehat{\psi}_t(x,0,t) = \varphi_t(x,t),$$
$$x \in (0,1), t > 0.$$
$$(4)$$

Equation (4) is a partial state predictor.

Denote the state of (4) at the moment $s = \tau$ by

$$(p_1(x,t), p_2(x,t), q_1(x,t), q_2(x,t))$$
$$= (\widehat{w}(x,\tau,t), \widehat{\varphi}(x,\tau,t), \widehat{z}(x,\tau,t), \widehat{\psi}(x,\tau,t)). \tag{5}$$

Using (3) we can verify that the functions group $(p_1(x,t), p_2(x,t), q_1(x,t), q_2(x,t))$ satisfy the following partial differential equations:

$$p_{1,t}(x,t) = q_1(x,t) + a_1(x)u_1(t) + a_2(x)u_2(t),$$
$$x \in (0,1), \quad t > 0,$$

$$p_{2,t}(x,t) = q_2(x,t) + a_3(x)u_1(t) + a_4(x)u_2(t),$$
$$x \in (0,1), \quad t > 0,$$

$$q_{1,t}(x,t) = \frac{K}{\rho}(p_{1,xx} - p_{2,x})(x,t) + b_1(x)u_1(t)$$
$$+ b_2(x)u_2(t),$$

$$q_{2,t}(x,t) = \frac{EI}{I_\rho}p_{2,xx}(x,t) + \frac{K}{I_\rho}(p_{1,x} - p_2)(x,t)$$
$$+ b_3(x)u_1(t) + b_4(x)u_2(t),$$

$$p_1(0,t) = p_2(0,t) = q_1(0,t) = q_2(0,t) = 0,$$
$$t > 0,$$

$$K(p_{1,x} - p_2)(1,t) = \beta_1 u_1(t), \quad t > 0,$$

$$EIp_{2,x}(1,t) = \beta_2 u_2(t), \quad t > 0,$$

$$p_1(x,0) = E_1(w_0, \varphi_0, w_1, \varphi_1)(x) - \int_{-\tau}^{0} a_1(x,s)f_1(s)\,ds$$
$$- \int_{-\tau}^{0} a_2(x,s)f_2(s)\,ds,$$

$$p_2(x,0) = E_2(w_0, \varphi_0, w_1, \varphi_1)(x) - \int_{-\tau}^{0} a_3(x,s)f_1(s)\,ds$$
$$- \int_{-\tau}^{0} a_4(x,s)f_2(s)\,ds,$$

$$q_1(x,0) = E_3(w_0, \varphi_0, w_1, \varphi_1)(x) + \int_{-\tau}^{0} b_1(x,s)f_1(s)\,ds$$
$$+ \int_{-\tau}^{0} b_2(x,s)f_2(s)\,ds,$$

$$q_2(x,0) = E_4(w_0, \varphi_0, w_1, \varphi_1)(x) + \int_{-\tau}^{0} b_3(x,s)f_1(s)\,ds$$
$$+ \int_{-\tau}^{0} b_4(x,s)f_2(s)\,ds, \tag{6}$$

where $a_i(x,r), b_i(x,r), a_i(x), b_i(x)$ $(i = 1, 2, 3, 4)$ are measurable function and E_i $(i = 1, 2, 3, 4)$ are bounded linear operators on $[H^1[0,1] \times L^2[0,1]]^2$; they are determined later.

Equation (6) is a system without delay, but the controls appear in the system interior and boundary. First we consider the stabilization problem of (6). Let us consider the energy functional of (6)

$$E(t) = \frac{1}{2}\left\| (p_1, p_2) \right\|^2_{V_K^1(0,1) \times V_{EI}^1(0,1)}$$

$$+ \frac{1}{2}\left\| (q_1, q_2) \right\|^2_{L_\rho^2(0,1) \times L_{I_\rho}^2(0,1)}$$

$$= \frac{1}{2}\int_0^1 K|p_{1,x}(x,t) - p_2(x,t)|^2\,dx$$

$$+ \frac{1}{2}\int_0^1 EI|p_{2,x}(x,t)|^2\,dx \tag{7}$$

$$+ \frac{1}{2}\int_0^1 \rho|q_1(x,t)|^2\,dx$$

$$+ \frac{1}{2}\int_0^1 I_\rho|q_2(x,t)|^2\,dx.$$

A direct calculation gives

$$\frac{dE(t)}{dt}$$

$$= u_1(t)\left[\beta_1 q_1(1,t)\right.$$

$$+ \int_0^1 K(p_{1,x} - p_2)\left[a_1'(x) - a_3(x)\right]dx$$

$$+ \int_0^1 EIp_{2,x}(x,t)a_3'(x)\,dx$$

$$+ \int_0^1 \rho q_1(x,t)b_1(x)\,dx$$

$$+ \left.\int_0^1 I_\rho q_2(x,t)b_3(x)\,dx\right] \tag{8}$$

$$+ u_2(t)\left[\beta_2 q_2(1,t) + \int_0^1 K(p_{1,x} - p_2)\right.$$

$$\times \left[a_2'(x) - a_4(x)\right]dx$$

$$+ \int_0^1 EIp_{2,x}(x,t)a_4'(x)\,dx$$

$$+ \int_0^1 \rho q_1(x,t)b_2(x)\,dx$$

$$+ \left.\int_0^1 I_\rho q_2(x,t)b_4(x)\,dx\right].$$

Set

$$U_1(p_1, p_2, q_1, q_2)$$
$$= \beta_1 q_1(1,t)$$
$$+ \int_0^1 K(p_{1,x}(x,t) - p_2(x,t))\left(a_1'(x) - a_3(x)\right)dx$$

$$+ \int_0^1 \rho q_1(x,t) b_1(x) dx + \int_0^1 EI p_{2,x}(x,t) a_3'(x) dx$$

$$+ \int_0^1 I_\rho q_2(x,t) b_3(x) dx;$$

$$U_2(p_1, p_2, q_1, q_2)$$

$$= \beta_2 q_2(1,t)$$

$$+ \int_0^1 K(p_{1,x}(x,t) - p_2(x,t))(a_2'(x) - a_4(x)) dx$$

$$+ \int_0^1 \rho q_1(x,t) b_2(x) dx + \int_0^1 EI p_{2,x}(x,t) a_4'(x) dx$$

$$+ \int_0^1 I_\rho q_2(x,t) b_4(x) dx. \tag{9}$$

We take the feedback control law as

$$u_1(t) = -U_1(p_1, p_2, q_1, q_2),$$
$$u_2(t) = -U_2(p_1, p_2, q_1, q_2). \tag{10}$$

Then, the closed loop system associated with (6) is

$$p_{1,t}(x,t) = q_1(x,t) - a_1(x) U_1(p_1, p_2, q_1, q_2)$$
$$- a_2(x) U_2(p_1, p_2, q_1, q_2),$$
$$x \in (0,1), \ t > 0,$$

$$p_{2,t}(x,t) = q_2(x,t) - a_3(x) U_1(p_1, p_2, q_1, q_2)$$
$$- a_4(x) U_2(p_1, p_2, q_1, q_2),$$
$$x \in (0,1), \ t > 0,$$

$$q_{1,t}(x,t) = \frac{K}{\rho}(p_{1,xx} - p_{2,x})(x,t) - b_1(x) U_1$$
$$\times (p_1, p_2, q_1, q_2) - b_2(x) U_2(p_1, p_2, q_1, q_2),$$

$$q_{2,t}(x,t) = \frac{EI}{I_\rho} p_{2,xx}(x,t) + \frac{K}{I_\rho}(p_{1,x} - p_2)(x,t)$$
$$- b_3(x) U_1(p_1, p_2, q_1, q_2)$$
$$- b_4(x) U_2(p_1, p_2, q_1, q_2),$$

$$p_1(0,t) = p_2(0,t) = q_1(0,t) = q_2(0,t) = 0, \quad t > 0,$$

$$K(p_{1,x} - p_2)(1,t) = -\beta_1 U_1(p_1, p_2, q_1, q_2), \quad t > 0,$$

$$EI p_{2,xx}(1,t) = -\beta_2 U_2(p_1, p_2, q_1, q_2), \quad t > 0,$$

$$p_1(x,0) = E_1(w_0, \varphi_0, w_1, \varphi_1)(x) - \int_{-\tau}^0 a_1(x,s) f_1(s) ds$$

$$- \int_{-\tau}^0 a_2(x,s) f_1(s) ds,$$

$$p_2(x,0) = E_2(w_0, \varphi_0, w_1, \varphi_1)(x) - \int_{-\tau}^0 a_3(x,s) f_1(s) ds$$

$$- \int_{-\tau}^0 a_4(x,s) f_1(s) ds,$$

$$q_1(x,0) = E_3(w_0, \varphi_0, w_1, \varphi_1)(x) + \int_{-\tau}^0 b_1(x,s) f_1(s) ds$$

$$+ \int_{-\tau}^0 b_2(x,s) f_1(s) ds,$$

$$q_2(x,0) = E_4(w_0, \varphi_0, w_1, \varphi_1)(x) + \int_{-\tau}^0 b_3(x,s) f_1(s) ds$$

$$+ \int_{-\tau}^0 b_4(x,s) f_1(s) ds. \tag{11}$$

We estimate the error of the system (3) with control (10) and the system (11).

Let $(w(x,t), \varphi(x,t), w_t(x,t), \varphi_t(x,t))$ be the solution to (3) with control signals (10) and let function group $(p_1(x,t), p_2(x,t), q_1(x,t), q_2(x,t))$ be the solution to (11). Set $W(x,t) = (w(x,t), \varphi(x,t))$ and $W_t(x,t) = (w_t(x,t), \varphi_t(x,t))$, and set $P(x,t) = (p_1(x,t), p_2(x,t))$ and $Q(x,t) = (q_1(x,t), q_2(x,t))$.

To discuss the stability $(W(x,t), W_t(x,t))$, we consider the error both solutions in the energy space

$$\|P(\cdot,t) - W(\cdot, t+\tau)\|^2_{V_K^1(0,1) \times V_{EI}^1(0,1)}$$
$$+ \|Q(\cdot,t) - W_t(\cdot, t+\tau)\|^2_{L_\rho^2(0,1) \times L_{I_\rho}^2(0,1)}. \tag{12}$$

In this paper, we will prove the following results.

Theorem 1. *Let* $(W(x,t), W_t(x,t))$ *be the solution to* (3) *with controls* (10) *and let* $(P(x,t), Q(x,t))$ *be the solution to the closed-loop system* (11). *If the system* (11) *is asymptotically (exponentially) stable, then the system* (3) *also is asymptotically (exponentially) stable.*

Theorem 2. *Suppose that* $K/\rho \neq EI/I_\rho$. *Let* $\mu_n, n \in \mathbb{N}$ *be the eigenvalues of the free system (the system* (2) *without controls). Set*

$$\xi_n^{(1)} = \int_{-\tau}^0 g_1(\eta) e^{-i\sqrt{\mu_n}(\tau+\eta)} d\eta,$$
$$\xi_n^{(2)} = \int_{-\tau}^0 g_2(\eta) e^{-i\sqrt{\mu_n}(\tau+\eta)} d\eta. \tag{13}$$

Then the following assertions are true:

(1) *when*

$$\inf_n \left| \frac{\beta_1}{\rho} + \alpha_1 e^{-i\tau\sqrt{\mu_n}} + \xi_n^{(1)} \right| > 0,$$
$$\inf_n \left| \frac{\beta_2}{I_\rho} + \alpha_2 e^{-i\tau\sqrt{\mu_n}} + \xi_n^{(2)} \right| > 0, \tag{14}$$

the system (11) *is exponentially stable;*

(2) *if for all $n \in \mathbb{N}$,*

$$\left| \frac{\beta_1}{\rho} + \alpha_1 e^{-i\tau\sqrt{\mu_n}} + \xi_n^{(1)} \right| > 0,$$

$$\left| \frac{\beta_2}{I_\rho} + \alpha_2 e^{-i\tau\sqrt{\mu_n}} + \xi_n^{(2)} \right| > 0 \tag{15}$$

but

$$\inf_n \left| \frac{\beta_1}{\rho} + \alpha_1 e^{-i\tau\sqrt{\mu_n}} + \xi_n^{(1)} \right| = 0,$$

$$\inf_n \left| \frac{\beta_2}{I_\rho} + \alpha_2 e^{-i\tau\sqrt{\mu_n}} + \xi_n^{(2)} \right| = 0, \tag{16}$$

then the system (11) *is asymptotically stable.*

In the following sections, we will prove our results. In Section 3, we will determine functions $a_i(x), b_i(x)$, $a_i(x,s), b_i(x,s)$ ($i = 1, 2, 3, 4$). In Section 4, we will prove Theorem 1. In Section 5, we pay our attention to the proof of Theorem 2.

3. Representation of the System (6)

In this section, we will obtain the expressions for the functions $a_i(x), b_i(x), a_i(x,s), b_i(x,s)$ ($i = 1, 2, 3, 4$) appearing in system (6) using (3) and (4).

We begin with introducing two useful lemmas.

Lemma 3 (see [13]). *Define the differential operator in $L^2_\rho(0,1) \times L^2_{I_\rho}(0,1)$ as follows:*

$$\mathscr{L}(w, \varphi) = \left(-\frac{K}{\rho}\left(w''(x) - \varphi'(x)\right), -\frac{EI}{I_\rho}\varphi''(x) \right.$$
$$\left. -\frac{K}{I_\rho}\left(w'(x) - \varphi(x)\right) \right)^T, \tag{17}$$

with domain

$$\mathscr{D}(\mathscr{L})$$
$$= \left\{ \begin{matrix} (w(x), \varphi(x)) \in H^2(0,1) \\ \times H^2(0,1) \end{matrix} \;\middle|\; \begin{matrix} w(0) = \varphi(0) = 0, \\ K\left(w'(1) - \varphi(1)\right) = 0, \\ EI\varphi'(1) = 0 \end{matrix} \right\}. \tag{18}$$

Then $\mathscr{L}$ is a positive define operator with compact resolvent in $L^2_\rho(0,1) \times L^2_{I_\rho}(0,1)$; its eigenvalues are

$$0 < \mu_1 < \mu_2 < \cdots < \mu_n < \cdots \tag{19}$$

and the eigenfunctions $\Phi_n(x) = (w_n(x), \varphi_n(x))^T$ corresponding to μ_n are real functions and form a normalized orthogonal basis for $L^2_\rho(0,1) \times L^2_{I_\rho}(0,1)$.

Lemma 4 (see [13]). *Let $\Phi_n(x) = (w_n(x), \varphi_n(x))$ be the normalized eigenfunction corresponding to the eigenvalue μ_n of $\mathscr{L}$. Then it holds that*

$$\int_0^1 K\left|\omega_n'(x) - \varphi_n(x)\right|^2 dx + \int_0^1 EI\left|\varphi_n'(x)\right|^2 dx = \mu_n,$$

$$0 < \inf_n \left\{ \rho|w_n(1)|^2 + \left|I_\rho\varphi_n(1)\right|^2 \right\} \tag{20}$$

$$\le \sup_n \left\{ \rho|w_n(1)|^2 + \left|I_\rho\varphi_n(1)\right|^2 \right\} < \infty.$$

Now let us return to (3). We write the equation in (3) into the vector form

$$\begin{pmatrix} w_{tt}(x,t) \\ \varphi_{tt}(x,t) \end{pmatrix} - \begin{pmatrix} \dfrac{K}{\rho}\partial_{xx} & -\dfrac{K}{\rho}\partial_x \\ \dfrac{K}{I_\rho}\partial_x & \dfrac{EI}{I_\rho}\partial_{xx} - \dfrac{K}{I_\rho} \end{pmatrix} \begin{pmatrix} w(x,t) \\ \varphi(x,t) \end{pmatrix} = 0 \tag{21}$$

and the boundary conditions are $\begin{pmatrix} w(0,t) \\ \varphi(0,t) \end{pmatrix} = 0$, and

$$\begin{pmatrix} K\partial_x & -K \\ 0 & EI\partial_x \end{pmatrix} \begin{pmatrix} w(x,t) \\ \varphi(x,t) \end{pmatrix}_{x=1}$$

$$= \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} + \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix} \begin{pmatrix} u_1(t-\tau) \\ u_2(t-\tau) \end{pmatrix} \tag{22}$$

$$+ \int_{-\tau}^0 \begin{pmatrix} g_1(\eta) & 0 \\ 0 & g_2(\eta) \end{pmatrix} \begin{pmatrix} u_1(t+\eta) \\ u_2(t+\eta) \end{pmatrix} d\eta.$$

The initial datum are

$$\begin{pmatrix} w(x,0) \\ \varphi(x,0) \end{pmatrix} = \begin{pmatrix} w_0(x) \\ \varphi_0(x) \end{pmatrix}, \qquad \begin{pmatrix} w_t(x,0) \\ \varphi_t(x,0) \end{pmatrix} = \begin{pmatrix} w_1(x) \\ \varphi_1(x) \end{pmatrix}. \tag{23}$$

Set $W(x,t) = (w(x,t), \varphi(x,t))^T$ and $U(t) = (u_1(t), u_2(t))^T$. Define 2×2 matrices

$$\Lambda_1 = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}, \qquad \Lambda_2 = \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix},$$

$$\Lambda_3(\eta) = \begin{pmatrix} g_1(\eta) & 0 \\ 0 & g_2(\eta) \end{pmatrix} \tag{24}$$

and define an operator B from $\mathbb{R}^2$ to $H^{-1}(0,1) \times H^{-1}(0,1)$, where $H^{-1}(0,1) = (V_\omega^1(0,1))^*$ is dual space,

$$B = \begin{pmatrix} \delta(x-1) & 0 \\ 0 & \delta(x-1) \end{pmatrix}, \tag{25}$$

and define an operator Γ_N from $H^2(0,1) \times H^2(0,1)$ to $\mathbb{R}^2$ by

$$\Gamma_N W = \begin{pmatrix} K\left(w'(1) - \varphi(1)\right) \\ EI\varphi'(1) \end{pmatrix}, \tag{26}$$

where $W(x) = (w(x), \varphi(x))^T$.

With help of these notations, we can rewrite (3) into

$$W_{tt}(x,t) + \mathscr{L}W(x,t)$$

$$= B\left(\Lambda_1 U(t) + \Lambda_2 U(t-\tau) + \int_{-\tau}^0 \Lambda_3(\eta) U(t+\eta) d\eta\right),$$

$$t > 0,$$

$$W(0,t) = 0, \qquad \Gamma_N W(1,t) = 0,$$

$$W(x,0) = W_0(x) = (w_0(x), \varphi_0(x))^T,$$

$$W_t(x,0) = W_1(x) = (w_1(x), \varphi_1(x))^T,$$

$$(27)$$

and (4) into

$$\widehat{W}_s(x,s,t) = \widehat{V}(x,s,t),$$

$$\widehat{V}_s(x,s,t) + \mathscr{L}\widehat{W}(x,s,t) = B\Lambda_2 U(t+s-\tau)$$

$$+ B\int_{-\tau}^{-s} \Lambda_3(\eta) U(t+s+\eta) d\eta,$$

$$\widehat{W}(0,s,t) = 0, \qquad \Gamma_N \widehat{W}(1,s,t) = 0,$$

$$\widehat{W}(x,0,t) = W(x,t),$$

$$\widehat{V}(x,0,t) = W_t(x,t),$$

$$(28)$$

where $\widehat{W}(x,s,t) = (\widehat{w}(x,s,t), \widehat{\varphi}(x,s,t))^T$, $\widehat{V}(x,s,t) = (\widehat{z}(x,s,t), \widehat{\psi}(x,s,t))^T$.

We define two families of the bounded linear operators on $L^2_\rho(0,1) \times L^2_{I_\rho}(0,1)$ by

$$\text{Cos}(t\mathscr{L})F = \sum_{n=1}^\infty \cos\sqrt{\mu_n}t (F,\Phi_n)_{L^2_\rho \times L^2_{I_\rho}} \Phi_n,$$

$$\text{Sin}(t\mathscr{L})F = \sum_{n=1}^\infty \frac{\sin\sqrt{\mu_n}t}{\sqrt{\mu_n}} (F,\Phi_n)_{L^2_\rho \times L^2_{I_\rho}} \Phi_n.$$

$$(29)$$

Clearly, the following equalities hold, for any $t \in \mathbb{R}$,

$$\text{Sin}(t\mathscr{L}) = \int_0^t \text{Cos}(t\mathscr{L}) dt,$$

$$\frac{d}{dt}(\text{Cos}(t\mathscr{L})) = -\mathscr{L}\text{Sin}(t\mathscr{L}).$$

$$(30)$$

It is easy to know that the vector-valued function

$$W(x,t) = \text{Cos}(t\mathscr{L})W_0 + \text{Sin}(t\mathscr{L})W_1$$

$$+ \int_0^t \text{Sin}((t-s)\mathscr{L}) B[\Lambda_1 U(s) + \Lambda_2 U(s-\tau)] ds$$

$$+ \int_0^t \text{Sin}((t-s)\mathscr{L}) B \int_{-\tau}^0 \Lambda_3(\eta) U(s+\eta) d\eta ds$$

$$(31)$$

is differentiable with respect to t and

$$W_t(x,t) = -\mathscr{L}\text{Sin}(t\mathscr{L})W_0 + \text{Cos}(t\mathscr{L})W_1$$

$$+ \int_0^t \text{Cos}((t-s)\mathscr{L}) B[\Lambda_1 U(s) + \Lambda_2 U(s-\tau)] ds$$

$$+ \int_0^t \text{Cos}((t-s)\mathscr{L}) B \int_{-\tau}^0 \Lambda_3(\eta) U(s+\eta) d\eta ds.$$

$$(32)$$

Further, $W(x,t)$ satisfies (27).

Similarly, we know the vector-valued function

$$\widehat{W}(x,s,t)$$

$$= \text{Cos}(s\mathscr{L})W(\cdot,t) + \text{Sin}(s\mathscr{L})W_t(\cdot,t)$$

$$+ \int_0^s \text{Sin}((s-r)\mathscr{L}) B\Lambda_2 U(t+r-\tau) dr$$

$$+ \int_0^s \text{Sin}((s-r)\mathscr{L}) B \int_{-\tau}^{-r} \Lambda_3(\eta) U(t+r+\eta) d\eta dr,$$

$$(33)$$

$$\widehat{V}(x,s,t)$$

$$= -\mathscr{L}\text{Sin}(s\mathscr{L})W(\cdot,t) + \text{Cos}(s\mathscr{L})W_t(\cdot,t)$$

$$+ \int_0^s \text{Cos}((s-r)\mathscr{L}) B\Lambda_2 U(t+r-\tau) dr$$

$$+ \int_0^s \text{Cos}((s-r)\mathscr{L}) B \int_{-\tau}^{-r} \Lambda_3(\eta) U(t+r+\eta) d\eta dr.$$

$$(34)$$

satisfy (28).

Set

$$P(x,t) = \widehat{W}(x,\tau,t), \qquad Q(x,t) = \widehat{V}(x,\tau,t).$$ $$(35)$$

Then we have

$$\begin{pmatrix} P(x,t) \\ Q(x,t) \end{pmatrix} = \begin{pmatrix} \text{Cos}(\tau\mathscr{L}) & \text{Sin}(\tau\mathscr{L}) \\ -\mathscr{L}\text{Sin}(\tau\mathscr{L}) & \text{Cos}(\tau\mathscr{L}) \end{pmatrix} \begin{pmatrix} W(x,t) \\ W_t(x,t) \end{pmatrix}$$

$$+ \int_0^\tau \begin{pmatrix} \text{Sin}((\tau-r)\mathscr{L}) \\ \text{Cos}((\tau-r)\mathscr{L}) \end{pmatrix} B\Lambda_2 U(t+r-\tau) dr$$

$$+ \int_0^\tau \begin{pmatrix} \text{Sin}((\tau-r)\mathscr{L}) \\ \text{Cos}((\tau-r)\mathscr{L}) \end{pmatrix}$$

$$\times B \int_{-\tau}^{-r} \Lambda_3(\eta) U(t+r+\eta) d\eta dr.$$

$$(36)$$

Thus,

$$\begin{pmatrix} P_t(x,t) \\ Q_t(x,t) \end{pmatrix} = \begin{pmatrix} \text{Cos}(\tau\mathscr{L}) & \text{Sin}(\tau\mathscr{L}) \\ -\mathscr{L}\text{Sin}(\tau\mathscr{L}) & \text{Cos}(\tau\mathscr{L}) \end{pmatrix} \begin{pmatrix} W_t(x,t) \\ W_{tt}(x,t) \end{pmatrix}$$

$$+ \int_0^\tau \begin{pmatrix} \text{Cos}((\tau-r)\mathscr{L}) \\ -\mathscr{L}\text{Sin}((\tau-r)\mathscr{L}) \end{pmatrix}$$

$$\times B\Lambda_2 U(t+r-\tau) dr$$

$$+ \begin{pmatrix} 0 \\ B\Lambda_2 U(t) \end{pmatrix} - \begin{pmatrix} \text{Sin}(\tau\mathcal{L}) \\ \text{Cos}(\tau\mathcal{L}) \end{pmatrix} B\Lambda_2 U(t-\tau)$$

$$- \begin{pmatrix} \text{Sin}(\tau\mathcal{L}) \\ \text{Cos}(\tau\mathcal{L}) \end{pmatrix} B \int_{-\tau}^{0} \Lambda_3(\eta) U(t+\eta) d\eta$$

$$+ \int_{0}^{\tau} \begin{pmatrix} \text{Cos}((\tau-r)\mathcal{L}) \\ -\mathcal{L}\,\text{Sin}((\tau-r)\mathcal{L}) \end{pmatrix} B$$

$$\times \int_{-\tau}^{-r} \Lambda_3(\eta) U(t+r+\eta) d\eta dr$$

$$+ \int_{-\tau}^{0} \begin{pmatrix} \text{Sin}((\tau+\eta)\mathcal{L}) \\ \text{Cos}((\tau+\eta)\mathcal{L}) \end{pmatrix} B\Lambda_3(\eta) U(t) d\eta.$$

(37)

Note that

$$\begin{pmatrix} W_t(x,t) \\ W_{tt}(x,t) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & I \\ -\mathcal{L} & 0 \end{pmatrix} \begin{pmatrix} W(x,t) \\ W_t(x,t) \end{pmatrix}$$

$$+ \begin{pmatrix} 0 \\ B\left(\Lambda_1 U(t) + \Lambda_2 U(t-\tau) + \int_{-\tau}^{0} \Lambda_3(\eta) U(t+\eta) d\eta\right) \end{pmatrix}.$$

(38)

So it holds that

$$\begin{pmatrix} P_t(x,t) \\ Q_t(x,t) \end{pmatrix} = \begin{pmatrix} 0 & I \\ -\mathcal{L} & 0 \end{pmatrix} \begin{pmatrix} P(x,t) \\ Q(x,t) \end{pmatrix}$$

$$+ \begin{pmatrix} \text{Sin}(\tau\mathcal{L}) B\Lambda_1 U(t) + \int_{-\tau}^{0} \text{Sin}((\tau+\eta)\mathcal{L}) B\Lambda_3(\eta) U(t) d\eta \\ \text{Cos}(\tau\mathcal{L}) B\Lambda_1 U(t) + \int_{-\tau}^{0} \text{Cos}((\tau+\eta)\mathcal{L}) B\Lambda_3(\eta) U(t) d\eta + B\Lambda_2 U(t) \end{pmatrix}.$$

(39)

Therefore, we have equations

$$P_t(x,t) = Q(x,t) + \text{Sin}(\tau\mathcal{L}) B\Lambda_1 U(t)$$

$$+ \int_{-\tau}^{0} \text{Sin}((\tau+\eta)\mathcal{L}) B\Lambda_3(\eta) U(t) d\eta,$$

$$Q_t(x,t) = -\mathcal{L}P(x,t) + \text{Cos}(\tau\mathcal{L}) B\Lambda_1 U(t)$$

$$+ \int_{-\tau}^{0} \text{Cos}((\tau+\eta)\mathcal{L}) B\Lambda_3(\eta) U(t) d\eta,$$

$$P(0,t) = Q(0,t) = 0, \qquad \Gamma_N P(\cdot,t) = \Lambda_2 U(t)$$

(40)

and initial conditions

$$P(x,0) = \text{Cos}(\tau\mathcal{L}) W_0 + \text{Sin}(\tau\mathcal{L}) W_1$$

$$- \int_{-\tau}^{0} \text{Sin}(s\mathcal{L}) B\Lambda_2 f(s) ds$$

$$+ \int_{-\tau}^{0} \int_{-\tau}^{s} \text{Sin}((\tau-s+\eta)\mathcal{L})$$

$$\times B\Lambda_3(\eta) f(s) d\eta ds,$$

$$Q(x,0) = -\mathcal{L}\,\text{Sin}(\tau\mathcal{L}) W_0 + \text{Cos}(\tau\mathcal{L}) W_1$$

$$+ \int_{-\tau}^{0} \text{Cos}(s\mathcal{L}) B\Lambda_2 f(s) ds$$

$$+ \int_{-\tau}^{0} \int_{-\tau}^{s} \text{Cos}((\tau-s+\eta)\mathcal{L}) B\Lambda_3(\eta) f(s) d\eta ds,$$

(41)

(42)

where $f(s) = (f_1(s), f_2(s))^T$.

Since all entries of B are meaningful as linear functional on $H^1(0,1)$, so for any $Z = (z_1, z_2) \in \mathbb{R}^2$ and $\Phi_n(x) \in H^1(0,1) \times H^1(0,1)$,

$$(BZ, \Phi_n)_{L_\rho^2(0,1) \times L_{I_\rho}^2(0,1)}$$

$$= z_1 \int_0^1 \rho\delta(x-1) w_n(x) dx + z_2 \int_0^1 I_\rho\delta(x-1) \varphi_n(x) dx$$

$$= \rho z_1 w_n(1) + I_\rho z_2 \varphi_n(1) = [z_1, z_2] [\rho w_n(1), \varphi_n(1)]^T.$$

(43)

Therefore, we have the following results.

Theorem 5. *Let $\{\mu_n; n \in \mathbb{N}\}$ be the list of all eigenvalues of $\mathcal{L}$. Then the functions that appear in (6) are*

$$a_1(x) = \rho \sum_{n=1}^{\infty} w_n(1) w_n(x)$$

$$\times \left(\alpha_1 \frac{\sin \tau\sqrt{\mu_n}}{\sqrt{\mu_n}} \right.$$

$$\left. + \int_{-\tau}^{0} g_1(\eta) \frac{\sin(\tau+\eta)\sqrt{\mu_n}}{\sqrt{\mu_n}} d\eta \right),$$

$$a_2(x) = I_\rho \sum_{n=1}^{\infty} \varphi_n(1) w_n(x)$$

$$\times \left(\alpha_2 \frac{\sin \tau\sqrt{\mu_n}}{\sqrt{\mu_n}} \right.$$

$$\left. + \int_{-\tau}^{0} g_2(\eta) \frac{\sin(\tau+\eta)\sqrt{\mu_n}}{\sqrt{\mu_n}} d\eta \right),$$

$$a_3(x) = \rho \sum_{n=1}^{\infty} w_n(1) \varphi_n(x)$$
$$\times \left(\alpha_1 \frac{\sin \tau \sqrt{\mu_n}}{\sqrt{\mu_n}} \right.$$
$$\left. + \int_{-\tau}^{0} g_1(\eta) \frac{\sin (\tau + \eta) \sqrt{\mu_n}}{\sqrt{\mu_n}} d\eta \right),$$

$$a_4(x) = I_\rho \sum_{n=1}^{\infty} \varphi_n(1) \varphi_n(x)$$
$$\times \left(\alpha_2 \frac{\sin \tau \sqrt{\mu_n}}{\sqrt{\mu_n}} \right.$$
$$\left. + \int_{-\tau}^{0} g_2(\eta) \frac{\sin (\tau + \eta) \sqrt{\mu_n}}{\sqrt{\mu_n}} d\eta \right),$$

$$b_1(x) = \rho \sum_{n=1}^{\infty} w_n(1) w_n(x)$$
$$\times \left(\alpha_1 \cos \tau \sqrt{\mu_n} \right.$$
$$\left. + \int_{-\tau}^{0} g_1(\eta) \cos (\tau + \eta) \sqrt{\mu_n} d\eta \right),$$

$$b_2(x) = I_\rho \sum_{n=1}^{\infty} \varphi_n(1) w_n(x)$$
$$\times \left(\alpha_2 \cos \tau \sqrt{\mu_n} \right.$$
$$\left. + \int_{-\tau}^{0} g_2(\eta) \cos (\tau + \eta) \sqrt{\mu_n} d\eta \right),$$

$$b_3(x) = \rho \sum_{n=1}^{\infty} w_n(1) \varphi_n(x)$$
$$\times \left(\alpha_1 \cos \tau \sqrt{\mu_n} \right.$$
$$\left. + \int_{-\tau}^{0} g_1(\eta) \cos (\tau + \eta) \sqrt{\mu_n} d\eta \right),$$

$$b_4(x) = I_\rho \sum_{n=1}^{\infty} \varphi_n(1) \varphi_n(x)$$
$$\times \left(\alpha_2 \cos \tau \sqrt{\mu_n} \right.$$
$$\left. + \int_{-\tau}^{0} g_2(\eta) \cos (\tau + \eta) \sqrt{\mu_n} d\eta \right),$$

$$a_1(x, s) = \rho \sum_{n=1}^{\infty} w_n(1) w_n(x)$$
$$\times \left(\beta_1 \frac{\sin s \sqrt{\mu_n}}{\sqrt{\mu_n}} \right.$$
$$\left. - \int_{-\tau}^{s} g_1(\eta) \frac{\sin (\tau - s + \eta) \sqrt{\mu_n}}{\sqrt{\mu_n}} d\eta \right),$$

$$a_2(x, s) = I_\rho \sum_{n=1}^{\infty} \varphi_n(1) w_n(x)$$
$$\times \left(\beta_2 \frac{\sin s \sqrt{\mu_n}}{\sqrt{\mu_n}} \right.$$
$$\left. - \int_{-\tau}^{s} g_2(\eta) \frac{\sin (\tau - s + \eta) \sqrt{\mu_n}}{\sqrt{\mu_n}} d\eta \right),$$

$$a_3(x, s) = \rho \sum_{n=1}^{\infty} w_n(1) \varphi_n(x)$$
$$\times \left(\beta_1 \frac{\sin s \sqrt{\mu_n}}{\sqrt{\mu_n}} \right.$$
$$\left. - \int_{-\tau}^{s} g_1(\eta) \frac{\sin (\tau - s + \eta) \sqrt{\mu_n}}{\sqrt{\mu_n}} d\eta \right),$$

$$a_4(x, s) = I_\rho \sum_{n=1}^{\infty} \varphi_n(1) \varphi_n(x)$$
$$\times \left(\beta_2 \frac{\sin s \sqrt{\mu_n}}{\sqrt{\mu_n}} \right.$$
$$\left. - \int_{-\tau}^{s} g_2(\eta) \frac{\sin (\tau - s + \eta) \sqrt{\mu_n}}{\sqrt{\mu_n}} d\eta \right),$$

$$b_1(x, s) = \rho \sum_{n=1}^{\infty} w_n(1) w_n(x)$$
$$\times \left(\beta_1 \cos s \sqrt{\mu_n} \right.$$
$$\left. + \int_{-\tau}^{s} g_1(\eta) \cos (\tau - s + \eta) \sqrt{\mu_n} d\eta \right),$$

$$b_2(x, s) = I_\rho \sum_{n=1}^{\infty} \varphi_n(1) w_n(x)$$
$$\times \left(\beta_2 \cos s \sqrt{\mu_n} \right.$$
$$\left. + \int_{-\tau}^{s} g_2(\eta) \cos (\tau - s + \eta) \sqrt{\mu_n} d\eta \right),$$

$$b_3(x, s) = \rho \sum_{n=1}^{\infty} w_n(1) \varphi_n(x)$$
$$\times \left(\beta_1 \cos s \sqrt{\mu_n} \right.$$
$$\left. + \int_{-\tau}^{s} g_1(\eta) \cos (\tau - s + \eta) \sqrt{\mu_n} d\eta \right),$$

$$b_4(x, s) = I_\rho \sum_{n=1}^{\infty} \varphi_n(1) \varphi_n(x)$$
$$\times \left(\beta_2 \cos s \sqrt{\mu_n} \right.$$
$$\left. + \int_{-\tau}^{s} g_2(\eta) \cos (\tau - s + \eta) \sqrt{\mu_n} d\eta \right),$$

$$(44)$$

and the linear operators are

$$E_1\left(\omega_0, \varphi_0, \omega_1, \varphi_1\right)(x)$$
$$= \sum_{n=1}^{\infty}\left[\cos\tau\sqrt{\mu_n}\left(W_0, \Phi_n\right) + \frac{\sin\tau\sqrt{\mu_n}}{\sqrt{\mu_n}}\left(W_1, \Phi_n\right)\right]w_n(x),$$

$$E_2\left(\omega_0, \varphi_0, \omega_1, \varphi_1\right)(x)$$
$$= \sum_{n=1}^{\infty}\left[\cos\tau\sqrt{\mu_n}\left(W_0, \Phi_n\right)\right.$$
$$\left. + \frac{\sin\tau\sqrt{\mu_n}}{\sqrt{\mu_n}}\left(W_1, \Phi_n\right)\right]\varphi_n(x),$$

$$E_3\left(\omega_0, \varphi_0, \omega_1, \varphi_1\right)(x)$$
$$= \sum_{n=1}^{\infty}\left[-\sqrt{\mu_n}\sin\tau\sqrt{\mu_n}\left(W_0, \Phi_n\right)\right.$$
$$\left. + \cos\tau\sqrt{\mu_n}\left(W_1, \Phi_n\right)\right]w_n(x),$$

$$E_4\left(\omega_0, \varphi_0, \omega_1, \varphi_1\right)(x)$$
$$= \sum_{n=1}^{\infty}\left[-\sqrt{\mu_n}\sin\tau\sqrt{\mu_n}\left(W_0, \Phi_n\right)\right.$$
$$\left. + \cos\tau\sqrt{\mu_n}\left(W_1, \Phi_n\right)\right]\varphi_n(x),$$

$$\tag{45}$$

4. The Proof of Theorem 1

In this section, we will prove Theorem 1. Here we mainly estimate the error:

$$\|P(\cdot, t) - W(\cdot, t + \tau)\|^2_{V_K^1(0,1)\times V_{EI}^1(0,1)}$$
$$+ \|Q(\cdot, t) - W_t(\cdot, t + \tau)\|^2_{L_\rho^2(0,1)\times L_{I_\rho}^2(0,1)}. \tag{46}$$

According to the calculation in Section 3, we have

$$P(x, t) - W(x, t + \tau)$$
$$= -\int_0^\tau \mathrm{Sin}\left((\tau - r)\mathscr{L}\right)B\Lambda_1 U(t + r)\,dr$$
$$- \int_0^\tau \mathrm{Sin}\left((\tau - r)\mathscr{L}\right)B\int_{-\tau}^0 \Lambda_3(\eta)U(t + r + \eta)\,d\eta\,dr, \tag{47}$$

$$Q(x, t) - W_t(x, t + \tau)$$
$$= -\int_0^\tau \mathrm{Cos}\left((\tau - r)\mathscr{L}\right)B\Lambda_1 U(t + r)\,dr$$
$$- \int_0^\tau \mathrm{Cos}\left((\tau - r)\mathscr{L}\right)B\int_{-\tau}^0 \Lambda_3(\eta)U(t + r + \eta)\,d\eta\,dr. \tag{48}$$

So,

$$\|P(\cdot, t) - W(\cdot, t + \tau)\|^2_{V_K^1(0,1)\times V_{EI}^1(0,1)}$$
$$+ \|Q(\cdot, t) - W_t(\cdot, t + \tau)\|^2_{L_\rho^2(0,1)\times L_{I_\rho}^2(0,1)}$$

$$\leq 4(\alpha_1\rho)^2 \sum_{n=1}^{\infty}|\omega_n(1)|^2$$
$$\times \left|\int_0^\tau \sin\sqrt{\mu_n}(\tau - r)u_1(r + t)\,dr\right|^2$$
$$+ 4\left(\alpha_2 I_\rho\right)^2 \sum_{n=1}^{\infty}|\varphi_n(1)|^2$$
$$\times \left|\int_0^\tau \sin\sqrt{\mu_n}(\tau - r)u_2(r + t)\,dr\right|^2$$
$$+ 4\rho^2 \sum_{n=1}^{\infty}|\omega_n(1)|^2$$
$$\times \left|\int_0^\tau \sin\sqrt{\mu_n}(\tau - r)\left(\int_{-\tau}^0 g_1(\eta)u_1(r + t + \eta)\,d\eta\right)dr\right|^2$$
$$+ 4I_\rho^2 \sum_{n=1}^{\infty}|\varphi_n(1)|^2$$
$$\times \left|\int_0^\tau \sin\sqrt{\mu_n}(\tau - r)\left(\int_{-\tau}^0 g_2(\eta)u_2(r + t + \eta)\,d\eta\right)dr\right|^2$$
$$+ 4(\alpha_1\rho)^2 \sum_{n=1}^{\infty}|\omega_n(1)|^2$$
$$\times \left|\int_0^\tau \cos\sqrt{\mu_n}(\tau - r)u_1(r + t)\,dr\right|^2$$
$$+ 4\left(\alpha_2 I_\rho\right)^2 \sum_{n=1}^{\infty}|\varphi_n(1)|^2$$
$$\times \left|\int_0^\tau \cos\sqrt{\mu_n}(\tau - r)u_2(r + t)\,dr\right|^2$$
$$+ 4\rho^2 \sum_{n=1}^{\infty}|\omega_n(1)|^2$$
$$\times \left|\int_0^\tau \cos\sqrt{\mu_n}(\tau - r)\left(\int_{-\tau}^0 g_1(\eta)u_1(r + t + \eta)\,d\eta\right)dr\right|^2$$
$$+ 4I_\rho^2 \sum_{n=1}^{\infty}|\varphi_n(1)|^2$$
$$\times \left|\int_0^\tau \cos\sqrt{\mu_n}(\tau - r)\left(\int_{-\tau}^0 g_2(\eta)u_2(r + t + \eta)\,d\eta\right)dr\right|^2. \tag{49}$$

Note that $\{\cos \sqrt{\mu_n} t, \sin \sqrt{\mu_n} t; n \in \mathbb{N}\}$ is a Riesz basis sequence for $L^2[0, \tau]$. Thus, there exist positive constants M_i $(i = 1, 2, 3, 4)$ such that

$$
\|P(\cdot, t) - W(\cdot, t + \tau)\|^2_{V^1_K(0,1) \times V^1_{EI}(0,1)}
$$

$$
+ \|Q(\cdot, t) - W_t(\cdot, t + \tau)\|^2_{L^2_\rho(0,1) \times L^2_{I_\rho}(0,1)}
$$

$$
\leq M_1^2 \int_0^\tau |u_1(r + t)|^2 dr + M_2^2 \int_0^\tau |u_2(r + t)|^2 dr
$$

$$
+ M_3^2 \int_0^\tau \left| \int_{-\tau}^0 g_1(\eta) u_1(t + r + \eta) d\eta \right|^2 dr
$$

$$
+ M_4^2 \int_0^\tau \left| \int_{-\tau}^0 g_2(\eta) u_2(t + r + \eta) d\eta \right|^2 dr
$$

$$
\leq M_1^2 \int_0^\tau |u_1(r + t)|^2 dr + M_2^2 \int_0^\tau |u_2(r + t)|^2 dr
$$

$$
+ M_3^2 \tau^2 \int_{-\tau}^0 |g_1(\eta)|^2 d\eta \int_{t-\tau}^{t+\tau} |u_1(s)|^2 ds
$$

$$
+ M_4^2 \tau^2 \int_{-\tau}^0 |g_2(\eta)|^2 d\eta \int_{t-\tau}^{t+\tau} |u_2(s)|^2 ds.
$$

$$(50)$$

Let $(P(x, t), Q(x, t))$ be the solution to (11), and $E(t)$ be its energy functional; then we have

$$
E(t) = \frac{1}{2} \|(P(x, t), Q(x, t))\|^2_{\mathcal{H}},
$$

$$
\frac{dE(t)}{dt} = -U_1^2(P, Q)(t) - U_2^2(P, Q)(t).
$$

$$(51)$$

Therefore, we have

$$
\int_0^\tau \|U(t + r)\|^2_{\mathbb{R}^2} dr
$$

$$
= \int_t^{t+\tau} \left(U_1^2(P, Q)(r) + U_2^2(P, Q)(r) \right) dr
$$

$$
= E(t) - E(t + \tau),
$$

$$
\int_0^\tau \|U(t - r)\|^2_{\mathbb{R}^2} dr = E(t - \tau) - E(t).
$$

$$(52)$$

So, we can get

$$
\|P(\cdot, t) - W(\cdot, t + \tau)\|^2_{V^1_K(0,1) \times V^1_{EI}(0,1)}
$$

$$
+ \|Q(\cdot, t) - W_t(\cdot, t + \tau)\|^2_{L^2_\rho(0,1) \times L^2_{I_\rho}(0,1)}
$$

$$
\leq \max_{i=1,2} \{M_i^2\} [E(t) - E(t + \tau)]
$$

$$
+ \max_{j=3,4} \{M_j^2\} \tau^2 \int_{-\tau}^0 \left(|g_1(\eta)|^2 + |g_2(\eta)|^2 \right) d\eta
$$

$$
\times [E(t - \tau) - E(t + \tau)].
$$

$$(53)$$

If $(P(x, t), Q(x, t))$ is exponential stable, there exists a positive constant $\varepsilon > 0$ such that $E(t) \leq E(0)e^{-\varepsilon t}$. We can obtain the following result from above:

$$
\|P(\cdot, t) - W(\cdot, t + \tau)\|^2_{V^1_K(0,1) \times V^1_{EI}(0,1)}
$$

$$
+ \|Q(\cdot, t) - W_t(\cdot, t + \tau)\|^2_{L^2_\rho(0,1) \times L^2_{I_\rho}(0,1)} \leq M e^{-\varepsilon(t-\tau)},
$$

$$(54)$$

where M is a positive constant. So $(W(x, t), W_t(x, t))$ also decays exponentially.

5. The Proof of Theorem 2

In this section, we will discuss the stability of system (11). At first we consider L^2 well posed of the system (6). For the sake of simplicity, we use the vector form of (6); that is,

$$
P_t(x, t) = Q(x, t) + \mathrm{Sin}(\tau \mathscr{L}) B\Lambda_1 U(t)
$$

$$
+ \int_{-\tau}^0 \mathrm{Sin}((\tau + \eta) \mathscr{L}) B\Lambda_3(\eta) U(t) d\eta,
$$

$$
Q_t(x, t) = -\mathscr{L}P(x, t) + \mathrm{Cos}(\tau \mathscr{L}) B\Lambda_1 U(t)
$$

$$
+ \int_{-\tau}^0 \mathrm{Cos}((\tau + \eta) \mathscr{L}) B\Lambda_3(\eta) U(t) d\eta,
$$

$$
\Gamma_N P(\cdot, t) = \Lambda_2 U(t),
$$

$$
P(0, t) = Q(0, t) = 0,
$$

$$
P(x, 0) = P_0(x), \qquad Q(x, 0) = Q_0(x).
$$

$$(55)$$

The observation system corresponding to (55) is

$$
W_t(x, t) = V(x, t), \quad x \in (0, 1), \ t > 0,
$$

$$
V_t(x, t) = -\mathscr{L}W(x, t), \quad x \in (0, 1), \ t > 0,
$$

$$
W(x, 0) = W_0,
$$

$$
V(x, 0) = V_0,
$$

$$
y_1(t) = \beta_1 z(1, t) + \int_0^1 K(w_x(x, t) - \varphi(x, t))
$$

$$
\times (a_1'(x) - a_3(x)) dx
$$

$$
+ \int_0^1 EI\varphi_x(x, t) a_3'(x) dx
$$

$$
+ \int_0^1 \rho z(x, t) b_1(x) dx + \int_0^1 I_\rho \psi(x, t) b_3(x) dx;
$$

$$
y_2(t) = \beta_2 \psi(1, t) + \int_0^1 K(w_x(x, t) - \varphi(x, t))
$$

$$
\times (a_2'(x) - a_4(x)) dx
$$

$$
+ \int_0^1 EI\varphi_x(x, t) a_4'(x) dx
$$

$$
+ \int_0^1 \rho z(x, t) b_2(x) dx + \int_0^1 I_\rho \psi(x, t) b_4(x) dx,
$$

$$(56)$$

where $W(x,t) = (w(x,t), \varphi(x,t))^T$ and $V(x,t) = (z(x,t), \psi(x,t))^T$.

We can write the observation as

$$y_1(t) = \beta_1 z(1,t) + \left\langle W(\cdot,t), (a_1, a_3)^T \right\rangle_{V_k^1 \times V_{EI}^1}$$
$$+ \left\langle V(\cdot,t), (b_1, b_3)^T \right\rangle_{L_\rho^2 \times L_{I_\rho}^2},$$
$$y_2(t) = \beta_1 \psi(1,t) + \left\langle W(\cdot,t), (a_2, a_4)^T \right\rangle_{V_k^1 \times V_{EI}^1}$$
$$+ \left\langle V(\cdot,t), (b_2, b_4)^T \right\rangle_{L_\rho^2 \times L_{I_\rho}^2}. \tag{57}$$

Since

$$\begin{pmatrix} a_1(x) & a_2(x) \\ a_3(x) & a_4(x) \end{pmatrix}$$
$$= \mathrm{Sin}(\tau\mathscr{L}) B\Lambda_1$$
$$+ \int_{-\tau}^0 \mathrm{Sin}((\tau + \eta)\mathscr{L}) B\Lambda_3(\eta)\, d\eta,$$
$$\begin{pmatrix} b_1(x) & b_2(x) \\ b_3(x) & b_4(x) \end{pmatrix}$$
$$= \mathrm{Cos}(\tau\mathscr{L}) B\Lambda_1$$
$$+ \int_{-\tau}^0 \mathrm{Cos}((\tau + \eta)\mathscr{L}) B\Lambda_3(\eta)\, d\eta,$$
$$Y(t) = C(W,V) = \Lambda_2 V(1,t)$$
$$+ \left[\Lambda_1 \mathscr{L}^{1/2} \mathrm{Sin}(\tau\mathscr{L}) B \right.$$
$$+ \left. \mathscr{L}^{1/2} \int_{-\tau}^0 \mathrm{Sin}((\tau + \eta)\mathscr{L}) B\Lambda_3(\eta) d\eta \right]^* \mathscr{L}^{1/2} W(t)$$
$$+ \left[\Lambda_1 \mathrm{Cos}(\tau\mathscr{L}) B \right.$$
$$+ \left. \int_{-\tau}^0 \mathrm{Cos}((\tau + \eta)\mathscr{L}) B\Lambda_3(\eta) d\eta \right]^* V(t). \tag{58}$$

Taking the Laplace transform for above equation leads to, for any $\Re\lambda > 0$,

$$\lambda P(x) = Q(x) + \mathrm{Sin}(\tau\mathscr{L}) B\Lambda_1 U(\lambda)$$
$$+ \int_{-\tau}^0 \mathrm{Sin}((\tau + \eta)\mathscr{L}) B\Lambda_3(\eta) U(\lambda)\, d\eta,$$
$$x \in (0,1),$$
$$\lambda Q(x) = -\mathscr{L}P(x) + \mathrm{Cos}(\tau\mathscr{L}) B\Lambda_1 U(\lambda)$$
$$+ \int_{-\tau}^0 \mathrm{Cos}((\tau + \eta)\mathscr{L}) B\Lambda_3(\eta) U(\lambda)\, d\eta,$$
$$\Gamma_N P(\cdot,t) = \Lambda_2 U(\lambda),$$
$$P(0,t) = Q(0,t) = 0,$$

$$Y(\lambda) = \Lambda_2 Q(1)$$
$$+ \left[\Lambda_1 \mathscr{L}^{1/2} \mathrm{Sin}(\tau\mathscr{L}) B \right.$$
$$+ \left. \mathscr{L}^{1/2} \int_{-\tau}^0 \mathrm{Sin}((\tau + \eta)\mathscr{L}) B\Lambda_3(\eta) d\eta \right]^* \mathscr{L}^{1/2} P(\cdot)$$
$$+ \left[\Lambda_1 \mathrm{Cos}(\tau\mathscr{L}) B \right.$$
$$+ \left. \int_{-\tau}^0 \mathrm{Cos}((\tau + \eta)\mathscr{L}) B\Lambda_3(\eta) d\eta \right]^* Q(\cdot). \tag{59}$$

We have the following results by solving (59):

$$(\lambda^2 + \mathscr{L}) p(x)$$
$$= \left[\lambda \left(\mathrm{Sin}(\tau\mathscr{L}) B\Lambda_1 \right.\right.$$
$$+ \left.\left. \int_{-\tau}^0 \mathrm{Sin}((\tau + \eta)\mathscr{L}) B\Lambda_3(\eta) d\eta \right) \right.$$
$$+ \mathrm{Cos}(\tau\mathscr{L}) B\Lambda_1$$
$$+ \int_{-\tau}^0 \mathrm{Cos}((\tau + \eta)\mathscr{L}) B\Lambda_3(\eta) d\eta$$
$$\left. + B\Lambda_2 \right] U(\lambda),$$
$$Y(\lambda) = \left[\lambda \left(\mathrm{Sin}(\tau\mathscr{L}) B\Lambda_1 \right.\right.$$
$$+ \left.\left. \int_{-\tau}^0 \mathrm{Sin}((\tau + \eta)\mathscr{L}) B\Lambda_3(\eta) d\eta \right) \right.$$
$$+ \mathrm{Cos}(\tau\mathscr{L}) B\Lambda_1 + \int_{-\tau}^0 \mathrm{Cos}((\tau + \eta)\mathscr{L}) B\Lambda_3(\eta) d\eta$$
$$\left. + B\Lambda_2 \right]^* P. \tag{60}$$

So we can get

$$Y(\lambda) = \left[\lambda \left(\mathrm{sin}(\tau\mathscr{L}) B\Lambda_1 \right.\right.$$
$$+ \left.\left. \int_{-\tau}^0 \mathrm{Sin}((\tau + \eta)\mathscr{L}) B\Lambda_3(\eta) d\eta \right) \right.$$
$$+ \mathrm{Cos}(\tau\mathscr{L}) B\Lambda_1 + \int_{-\tau}^0 \mathrm{Cos}((\tau + \eta)\mathscr{L}) B\Lambda_3(\eta) d\eta$$
$$\left. + B\Lambda_2 \right]^* (\lambda^2 + \mathscr{L})^{-1}$$

$$\times \left[\lambda \left(\operatorname{Sin} (\tau \mathscr{L}) B\Lambda_1 \right. \right.$$

$$+ \int_{-\tau}^{0} \operatorname{Sin} ((\tau + \eta) \mathscr{L}) B\Lambda_3 (\eta) d\eta \right)$$

$$+ \operatorname{Cos} (\tau \mathscr{L}) B\Lambda_1$$

$$\left. + \int_{-\tau}^{0} \operatorname{Cos} ((\tau+\eta) \mathscr{L}) B\Lambda_3 (\eta) d\eta + B\Lambda_2 \right] U(\lambda),$$

$$(61)$$

and hence the transform matrix is

$$H(\lambda) = \left[\lambda \left(\operatorname{Sin} (\tau \mathscr{L}) B\Lambda_1 \right. \right.$$

$$+ \int_{-\tau}^{0} \operatorname{Sin} ((\tau + \eta) \mathscr{L}) B\Lambda_3 (\eta) d\eta \right)$$

$$+ \operatorname{Cos} (\tau \mathscr{L}) B\Lambda_1$$

$$+ \int_{-\tau}^{0} \operatorname{Cos} ((\tau + \eta) \mathscr{L}) B\Lambda_3 (\eta) d\eta$$

$$\left. + B\Lambda_2 \right]^{*} \left(\lambda^2 + \mathscr{L} \right)^{-1}$$

$$\times \left[\lambda \left(\operatorname{Sin} (\tau \mathscr{L}) B\Lambda_1 \right. \right.$$

$$+ \int_{-\tau}^{0} \operatorname{Sin} ((\tau + \eta) \mathscr{L}) B\Lambda_3 (\eta) d\eta \right)$$

$$+ \operatorname{Cos} (\tau \mathscr{L}) B\Lambda_1$$

$$\left. + \int_{-\tau}^{0} \operatorname{Cos} ((\tau + \eta) \mathscr{L}) B\Lambda_3 (\eta) d\eta + B\Lambda_2 \right].$$

$$(62)$$

For any $Z = (z_1, z_2) \in \mathbb{C}^2$, we can get

$$(H(\lambda) Z, Z)_{\mathbb{C}^2}$$

$$= \sum_{n=1}^{\infty} \frac{1}{\lambda^2 + \mu_n}$$

$$\times \left| \left(\left[\lambda \left(\operatorname{Sin} (\tau \mathscr{L}) B\Lambda_1 \right. \right. \right. \right.$$

$$+ \int_{-\tau}^{0} \operatorname{Sin} ((\tau + \eta) \mathscr{L}) B\Lambda_3 (\eta) d\eta \right)$$

$$+ \operatorname{Cos} (\tau \mathscr{L}) B\Lambda_1 + \int_{-\tau}^{0} \operatorname{Cos} ((\tau + \eta) \mathscr{L})$$

$$\left. \left. \times B\Lambda_3 (\eta) d\eta + B\Lambda_2 \right] Z, \Phi_n \right)_{L_\rho^2 \times L_{I_\rho}^2} \right|^2.$$

$$(63)$$

We can easily get

$$(B\Lambda_2 Z, \Phi_n)_{L_\rho^2 \times L_{I_\rho}^2} = \beta_1 z_1 \rho w_n (1) + \beta_2 z_2 I_\rho \varphi_n (1),$$

$$(\operatorname{Sin}(\tau \mathscr{L}) B\Lambda_1 Z, \Phi_n)_{L_\rho^2 \times L_{I_\rho}^2}$$

$$= \frac{\operatorname{Sin} (\tau \sqrt{\mu_n})}{\sqrt{\mu_n}} \left[\alpha_1 z_1 \rho w_n (1) + \alpha_2 z_2 I_\rho \varphi_n (1) \right],$$

$$(\operatorname{Cos}(\tau \mathscr{L}) B\Lambda_1 Z, \Phi_n)_{L_\rho^2 \times L_{I_\rho}^2}$$

$$= \operatorname{Cos} (\tau \sqrt{\mu_n}) \left[\alpha_1 z_1 \rho w_n (1) + \alpha_2 z_2 I_\rho \varphi_n (1) \right],$$

$$\left(\int_{-\tau}^{0} \operatorname{Sin}(\tau + \eta) \mathscr{L} B\Lambda_3 (\eta) d\eta Z, \Phi_n \right)_{L_\rho^2 \times L_{I_\rho}^2}$$

$$= \int_{-\tau}^{0} \frac{\operatorname{Sin} ((\tau + \eta) \sqrt{\mu_n})}{\sqrt{\mu_n}}$$

$$\times \left[g_1 (\eta) z_1 \rho w_n (1) + g_2 (\eta) z_2 I_\rho \varphi_n (1) \right] d\eta,$$

$$\left(\int_{-\tau}^{0} \operatorname{Cos}(\tau + \eta) \mathscr{L} B\Lambda_3 (\eta) d\eta Z, \Phi_n \right)_{L_\rho^2 \times L_{I_\rho}^2}$$

$$= \int_{-\tau}^{0} \operatorname{Cos} ((\tau + \eta) \sqrt{\mu_n})$$

$$\times \left[g_1 (\eta) z_1 \rho w_n (1) + g_2 (\eta) z_2 I_\rho \varphi_n (1) \right] d\eta.$$

$$(64)$$

Thus, we have

$$\left| \left(\left[\lambda \left(\operatorname{Sin} (\tau \mathscr{L}) B\Lambda_1 + \int_{-\tau}^{0} \operatorname{Sin} ((\tau + \eta) \mathscr{L}) B\Lambda_3 (\eta) d\eta \right) \right. \right. \right.$$

$$+ \operatorname{Cos} (\tau \mathscr{L}) B\Lambda_1 + \int_{-\tau}^{0} \operatorname{Cos} ((\tau + \eta) \mathscr{L}) B\Lambda_3 (\eta) d\eta$$

$$\left. \left. + B\Lambda_2 \right] Z, \Phi_n \right)_{L_\rho^2 \times L_{I_\rho}^2} \right|^2$$

$$\leq F \left| \frac{|\lambda|^2}{\mu_n} + 2 \right| \left[\rho |\omega_n (1)|^2 + I_\rho |\varphi_n (1)|^2 \right] \|Z\|_{\mathbb{C}^2}^2, \quad (65)$$

where F is a positive constant dependent on $\alpha_i, \beta_i, g_i(\eta), i = 1, 2$. Therefore, we have the following result:

$$\|H(\lambda)\| \leq F \sum_{n=1}^{\infty} \frac{1}{|\lambda|^2 + \mu_n} \left| \frac{|\lambda|^2}{\mu_n} + 2 \right|$$

$$\times \left[\rho |\omega_n (1)|^2 + I_\rho |\varphi_n (1)|^2 \right]. \quad (66)$$

From Lemma 4, we have

$$\sup_{\Re \lambda > \delta > 0} \|H(\lambda)\| < \infty. \quad (67)$$

Hence the system (6) is L_{loc}^2 well posed (see, [15]).

Next, we consider the exact observability of the system (6).

Lemma 6 (see [16]). *Let $\mathcal{H}$ be a separable Hilbert space, and let $\mathcal{L}$ be a unbounded positive definite operator. Assume that $\mathcal{L}$ satisfies the following conditions:*

(1) *$\mathcal{L}$ has compact resolvent and its spectrum is $\sigma(\mathcal{L}) = \{\mu_n; n \in \mathbb{N}\}$;*

(2) *the spectra of $\mathcal{L}$ satisfy the separable condition*

$$\inf_{n \neq m} |\sqrt{\mu_n} - \sqrt{\mu_m}| = \delta > 0; \tag{68}$$

(3) *the corresponding eigenvectors $\{\Phi_n; n \in \mathbb{N}\}$ with $\|\Phi_n\|_{\mathcal{H}} = 1$ form a normalized orthogonal basis for $\mathcal{H}$.*

Let $\mathbb{Y}$ be a Hilbert space. Assume that $C : D(\mathcal{L}) \to \mathbb{Y}$ is an admissible observation operator for $\mathcal{L}$. Then the following system:

$$Z_{tt} + \mathcal{L}Z(t) = 0, \qquad Z(0) = Z_0,$$
$$Z_t(0) = Z_1, \qquad Y(t) = C(Z, Z_t) \tag{69}$$

is exactly observable in finite time in the energy space $D(\mathcal{L}^{1/2}) \times \mathcal{H}$ if and only if

$$\inf_{n \in \mathbb{N}} \left\| C\left(\frac{\Phi_n}{i\sqrt{\mu_n}}, \Phi_n \right) \right\|_{\mathbb{Y}} > 0. \tag{70}$$

Now we apply Lemma 6 to the system (55). We can easily know that the condition (68) is fulfilled when $c_1 = \sqrt{K/\rho} \neq c_2 = \sqrt{EI/I_\rho}$ (see Remark 2.1 in [8]).

For $\Phi_n(x) = (w_n(x), \varphi_n(x))^T$, we have

$$C\left(\frac{\Phi_n}{i\sqrt{\mu_n}}, \Phi_n \right)$$

$$= \Lambda_2 \Phi_n(1)$$

$$+ \left[\Lambda_1 \mathcal{L}^{1/2} \mathrm{Sin}(\tau\mathcal{L}) B \right.$$

$$+ \mathcal{L}^{1/2} \int_{-\tau}^0 \mathrm{Sin}((\tau+\eta)\mathcal{L})B\Lambda_3(\eta)d\eta \Big]^* \mathcal{L}^{1/2} \frac{\Phi_n}{i\sqrt{\mu_n}}$$

$$+ \left[\Lambda_1 \mathrm{Cos}(\tau\mathcal{L}) B \right.$$

$$+ \int_{-\tau}^0 \mathrm{Cos}((\tau+\eta)\mathcal{L})B\Lambda_3(\eta)d\eta \Big]^* \Phi_n. \tag{71}$$

For any $Z = (z_1, z_2) \in \mathbb{C}^2$, we have

$$\left(\Lambda_1 \left(\mathcal{L}^{1/2} \mathrm{Sin}(\tau\mathcal{L}) B \right)^* \mathcal{L}^{1/2} \frac{\Phi_n}{i\sqrt{\mu_n}}, Z \right)_{\mathbb{C}^2}$$

$$= \left(\mathcal{L}^{1/2} \frac{\Phi_n}{i\sqrt{\mu_n}}, \mathcal{L}^{1/2} \mathrm{Sin}(\tau\mathcal{L}) B\Lambda_1 Z \right)_{L^2_\rho \times L^2_{I_\rho}} \tag{72}$$

$$= -i\sqrt{\mu_n} (\Phi_n, \mathrm{Sin}(\tau\mathcal{L}) B\Lambda_1 Z)_{L^2_\rho \times L^2_{I_\rho}}$$

$$= -i\sin(\sqrt{\mu_n}\tau) \overline{[\alpha_1 z_1 \rho w_n(1) + \alpha_2 z_2 I_\rho \varphi_n(1)]}.$$

Similarity, we have

$$\left(\left(\mathcal{L}^{1/2} \int_{-\tau}^0 \mathrm{Sin}((\tau+\eta)\mathcal{L}) B\Lambda_3(\eta)d\eta \right)^* \mathcal{L}^{1/2} \frac{\Phi_n}{i\sqrt{\mu_n}}, Z \right)_{\mathbb{C}^2}$$

$$= -i\int_{-\tau}^0 \sin((\tau+\eta)\sqrt{\mu_n})$$

$$\times \overline{[g_1(\eta) z_1 \rho w_n(1) + g_2(\eta) z_2 I_\rho \varphi_n(1)]}d\eta,$$

$$(\Lambda_1(\mathrm{Cos}(\tau\mathcal{L})B)^* \Phi_n, Z)_{\mathbb{C}^2}$$

$$= \cos(\sqrt{\mu_n}\tau) \overline{[\alpha_1 z_1 \rho w_n(1) + \alpha_2 z_2 I_\rho \varphi_n(1)]},$$

$$\left(\left(\int_{-\tau}^0 \mathrm{Cos}((\tau+\eta)\mathcal{L}) B\Lambda_3(\eta)d\eta \right)^* \Phi_n, Z \right)_{\mathbb{C}^2}$$

$$= \int_{-\tau}^0 \cos((\tau+\eta)\sqrt{\mu_n})$$

$$\times \overline{[g_1(\eta) z_1 \rho w_n(1) + g_2(\eta) z_2 I_\rho \varphi_n(1)]}d\eta. \tag{73}$$

Thus it holds that

$$C\left(\frac{\Phi_n}{i\sqrt{\mu_n}}, \Phi_n \right)$$

$$= \begin{pmatrix} \left(\frac{\beta_1}{\rho} + \alpha_1 e^{-i\tau\sqrt{\mu_n}} + \int_{-\tau}^0 g_1(\eta) e^{-i((\tau+\eta)\sqrt{\mu_n})}d\eta \right) \rho w_n(1) \\ \left(\frac{\beta_2}{I_\rho} + \alpha_2 e^{-i\tau\sqrt{\mu_n}} + \int_{-\tau}^0 g_2(\eta) e^{-i((\tau+\eta)\sqrt{\mu_n})}d\eta \right) I_\rho \varphi_n(1) \end{pmatrix}. \tag{74}$$

Set

$$\xi_n^{(1)} = \int_{-\tau}^0 g_1(\eta) e^{-i\sqrt{\mu_n}(\tau+\eta)}d\eta,$$
$$\xi_n^{(2)} = \int_{-\tau}^0 g_2(\eta) e^{-i\sqrt{\mu_n}(\tau+\eta)}d\eta. \tag{75}$$

Then

$$\left\| C\left(\frac{\Phi_n}{i\sqrt{\mu_n}}, \Phi_n \right) \right\|_{\mathbb{C}^2}^2 = \left| \frac{\beta_1}{\rho} + \alpha_1 e^{-i\tau\sqrt{\mu_n}} + \xi_n^{(1)} \right|^2 |\rho w_n(1)|^2$$

$$+ \left| \frac{\beta_2}{I_\rho} + \alpha_2 e^{-i\tau\sqrt{\mu_n}} + \xi_n^{(2)} \right|^2 |I_\rho \varphi_n(1)|^2. \tag{76}$$

Obviously, when

$$A_1 = \inf_n \left| \frac{\beta_1}{\rho} + \alpha_1 e^{-i\tau\sqrt{\mu_n}} + \xi_n^{(1)} \right| > 0,$$
$$A_2 = \inf_n \left| \frac{\beta_2}{I_\rho} + \alpha_2 e^{-i\tau\sqrt{\mu_n}} + \xi_n^{(2)} \right| > 0, \tag{77}$$

we have

$$\left\| C\left(\frac{\Phi_n}{i\sqrt{\mu_n}}, \Phi_n\right) \right\|_{\mathbb{C}^2}^2$$

$$\geq A_1^2 \left|\rho w_n(1)\right|^2 + A_2^2 \left|I_\rho \varphi_n(1)\right|^2 \qquad (78)$$

$$\geq \min\{A_1, A_2\} \left(\left|\rho w_n(1)\right|^2 + \left|I_\rho \varphi_n(1)\right|^2 \right);$$

using Lemma 4,

$$\inf_n \left\| C\left(\frac{\Phi_n}{i\sqrt{\mu_n}}, \Phi_n\right) \right\|_{\mathbb{C}^2}^2 > 0. \qquad (79)$$

According to Lemma 6, the system (56) is exactly observable in finite time, and hence the closed-loop system (11) is exponentially stable.

If for all $n \in \mathbb{N}$,

$$\left| \frac{\beta_1}{\rho} + \alpha_1 e^{-i\tau\sqrt{\mu_n}} + \xi_n^{(1)} \right| > 0,$$

$$\left| \frac{\beta_2}{I_\rho} + \alpha_2 e^{-i\tau\sqrt{\mu_n}} + \xi_n^{(2)} \right| > 0, \qquad (80)$$

we can see that in this case, there is no eigenvalue of system (11) on the imaginary axis. Moreover, if the conditions

$$\inf_n \left| \frac{\beta_1}{\rho} + \alpha_1 e^{-i\tau\sqrt{\mu_n}} + \xi_n^{(1)} \right| = 0,$$

$$\inf_n \left| \frac{\beta_2}{I_\rho} + \alpha_2 e^{-i\tau\sqrt{\mu_n}} + \xi_n^{(2)} \right| = 0 \qquad (81)$$

hold, then the imaginary axis is an asymptote of the eigenvalues of the system (11). Therefore, the stability theorem [17] asserts then that the system (11) is asymptotically stable. Therefore, we get the result of the Theorem 2.

6. Conclusion

In this paper, we designed a new controller for a Timoshenko beam with distributed delay in the boundary that stabilizes exponentially the system. In the design process of new controllers, there are main steps: (1) to translate the delay system into a system without delay; (2) for the undelay system, we used the collocated feedback law to obtain the control signals; (3) using the obtained control signals, act on the delay system. This control strategy can be regarded as extension form of [15]. In the stability analysis, the key trick is to use the exact observability of the dual system in finite time to obtain the exponential stability of the closed-loop system.

In the proof of main result, the condition $K/\rho \neq EI/I_\rho$ is used to ensure the separability of the spectrum (see, the condition (2)) in Lemma 6. In the statement of our result

(Theorem 2), the conditions are stronger than the practice; in fact,

$$\left\| C\left(\frac{\Phi_n}{i\sqrt{\mu_n}}, \Phi_n\right) \right\|_{\mathbb{C}^2}^2 = \left| \frac{\beta_1}{\rho} + \alpha_1 e^{-i\tau\sqrt{\mu_n}} + \xi_n^{(1)} \right|^2 \left|\rho w_n(1)\right|^2$$

$$+ \left| \frac{\beta_2}{I_\rho} + \alpha_2 e^{-i\tau\sqrt{\mu_n}} + \xi_n^{(2)} \right|^2 \left|I_\rho \varphi_n(1)\right|^2; \qquad (82)$$

one only needs to request

$$\inf_n \left\| C\left(\frac{\Phi_n}{i\sqrt{\mu_n}}, \Phi_n\right) \right\|_{\mathbb{C}^2}^2 > 0, \qquad (83)$$

so, the conditions

$$\inf_n \left| \frac{\beta_1}{\rho} + \alpha_1 e^{-i\tau\sqrt{\mu_n}} + \xi_n^{(1)} \right| > 0,$$

$$\inf_n \left| \frac{\beta_2}{I_\rho} + \alpha_2 e^{-i\tau\sqrt{\mu_n}} + \xi_n^{(2)} \right| > 0, \qquad (84)$$

are sufficient, but not necessary. Since

$$\xi_n^{(1)} = \int_{-\tau}^0 g_1(\eta) e^{i\sqrt{\mu_n}(\tau-\eta)} d\eta,$$

$$\xi_n^{(2)} = \int_{-\tau}^0 g_2(\eta) e^{i\sqrt{\mu_n}(\tau-\eta)} d\eta, \qquad (85)$$

so $\lim_{n\to\infty} \xi_n^{(j)} = 0$, $j = 1, 2$. Therefore, when

$$\left| \frac{\beta_1}{\rho} + \alpha_1 e^{-i\tau\sqrt{\mu_n}} + \xi_n^{(1)} \right| > 0,$$

$$\left| \frac{\beta_2}{I_\rho} + \alpha_2 e^{-i\tau\sqrt{\mu_n}} + \xi_n^{(2)} \right| > 0 \qquad (86)$$

that means that there is no eigenvalue on the imaginary axis, and

$$\inf_n \left| \frac{\beta_1}{\rho} + \alpha_1 e^{-i\tau\sqrt{\mu_n}} \right| > 0,$$

$$\inf_n \left| \frac{\beta_2}{I_\rho} + \alpha_2 e^{-i\tau\sqrt{\mu_n}} \right| > 0, \qquad (87)$$

we have

$$\inf_n \left\| C\left(\frac{\Phi_n}{i\sqrt{\mu_n}}, \Phi_n\right) \right\|_{\mathbb{C}^2}^2 > 0. \qquad (88)$$

Clearly, when $\beta_1 \neq \rho\alpha_1$ and $\beta_2 \neq I_\rho\alpha_2$, we also have

$$\inf_n \left\| C\left(\frac{\Phi_n}{i\sqrt{\mu_n}}, \Phi_n\right) \right\|_{\mathbb{C}^2}^2 > 0. \qquad (89)$$

Therefore, the conditions in Theorem 2 are easily verified.

The control method proposed in this paper can be used to the system of output availed system by using the Luenberger observer. Also we have noted that the method is only fitting the continuous model; for the model of data-driven system (e.g., see, [18]), it might fail. So we need to study the corresponding control strategy for the data-driven system.

Acknowledgment

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